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THE SET OF COMMON FIXED POINTS OF A ONE-PARAMETER CONTINUOUS SEMIGROUP OF NONEXPANSIVE MAPPINGS IS $F\big(\frac{1}{2}T(1)+\frac{1}{2}T(\sqrt{2})\big)$ IN STRICTLY CONVEX BANACH SPACES

Tomonari Suzuki

Abstract. In this paper, we prove the following. Let E be a strictly convex Banach space. Let $\{T(t) : t \geq 0\}$ be a one-parameter strongly continuous semigroup of nonexpansive mappings on a subset C of E. Then

$$\bigcap_{t\geq 0} F\big(T(t)\big) = F\left(\frac{1}{2}T(1)+\frac{1}{2}T(\sqrt{2})\right)$$

holds, where $F\big(T(t)\big)$ is the set of fixed points of $T(t)$ for each $t \geq 0$.

1. Introduction

Throughout this paper we denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ the set of all positive integers, all integers, all rational numbers and all real numbers, respectively.

Let C be a subset of a Banach space E. A mapping T on C is called a *nonexpansive mapping* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T. We know that $F(T)$ is nonempty in the case that E is uniformly convex and C is bounded closed and convex; see [4, 5, 10, 15]. A family of mappings $\{T(t) : t \geq 0\}$ is called a *one-parameter strongly continuous semigroup* of nonexpansive mappings on C (*nonexpansive semigroup*, for short) if the following are satisfied:

(sg 1) For each $t \geq 0$, $T(t)$ is a nonexpansive mapping on C;

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(sg 2) $T(s+t) = T(s) \circ T(t)$ for all $s, t \geq 0$;

(sg 3) for each $x \in C$, the mapping $t \mapsto T(t)x$ from $[0, \infty)$ into C is strongly continuous.

We know that $\bigcap_{t \geq 0} F\big(T(t)\big)$ is nonempty in the case when C is weakly compact convex and every nonexpansive mapping on a closed convex subset of C has a fixed point; see Bruck [8]. The author in [27] proved the following.

Theorem 1 ([27]). *Let $\{T(t) : t \geq 0\}$ be a one-parameter strongly continuous semigroup of nonexpansive mappings on a subset C of a Banach space E. Let α and β be positive real numbers satisfying $\alpha/\beta \notin \mathbb{Q}$. Then*

$$\bigcap_{t \geq 0} F\big(T(t)\big) = F\big(T(\alpha)\big) \cap F\big(T(\beta)\big)$$

holds.

Using this theorem, the author has proved many convergence theorems for non-expansive semigroups. For example, the following is proved in [26].

Theorem 2 ([26]). *Let C be a compact convex subset of a Banach space E and let $\{T(t) : t \geq 0\}$ be a one-parameter strongly continuous semigroup of nonexpansive mappings on C. Let α and β be positive real numbers satisfying $\alpha/\beta \notin \mathbb{Q}$. Let $\lambda \in (0, 1)$, and let $\{\theta_n\}$ be a sequence in $[0, 1]$ satisfying*

$$\liminf_{n \to \infty} \theta_n = 0, \quad \limsup_{n \to \infty} \theta_n > 0, \quad \text{and} \quad \lim_{n \to \infty} (\theta_{n+1} - \theta_n) = 0.$$

Define a sequence $\{x_n\}$ in C by $x_1 \in C$ and

$$x_{n+1} = (1 - \theta_n)\lambda T(\alpha)x_n + \theta_n \lambda T(\beta)x_n + (1 - \lambda)x_n$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T(t) : t \geq 0\}$.

The following proposition is a corollary of Bruck's result in [7].

Proposition 1 (Bruck [7]). *Let C be a subset of a strictly convex Banach space E. Let S and T be nonexpansive mappings from C into E with $F(S) \cap F(T) \neq \varnothing$. Then for each $\lambda \in (0, 1)$,*

$$F(S) \cap F(T) = F\big(\lambda S + (1 - \lambda)T\big)$$

holds, where $\lambda S + (1 - \lambda)T$ is a mapping from C into E defined by $\big(\lambda S + (1 - \lambda)T\big)x = \lambda Sx + (1 - \lambda)Tx$ for $x \in C$.

In Proposition 1, the assumption of $F(S) \cap F(T) \neq \varnothing$ is essential because we know the following example.

Example 1. Let E be a Banach space and fix $v \in E$ with $v \neq 0$. Define nonexpansive mappings S and T on E by

$$Sx = x + v \quad \text{and} \quad Tx = x - v$$

for all $x \in C$. Then

$$F(S) = F(T) = \varnothing \quad \text{and} \quad F\left(\frac{1}{2}S + \frac{1}{2}T\right) = E$$

holds.

In this paper, as motivated by above, we consider the following problem: Let $\{T(t) : t \geq 0\}$ be a one-parameter strongly continuous semigroup of nonexpansive mappings, let α and β be positive real numbers and let $\lambda \in (0,1)$. We may not assume $F\big(T(\alpha)\big) \cap F\big(T(\beta)\big) \neq \varnothing$. Then does

$$F\big(T(\alpha)\big) \cap F\big(T(\beta)\big) = F\big(\lambda T(\alpha) + (1-\lambda)T(\beta)\big)$$

hold? Our answer is positive.

Our problem is meaningful as follows: Finding a common fixed point of two mappings is much easier than that for infinite families of mappings. However, as Theorem 2, that for two mappings is still difficult. That for single mappings is much easier than that for two mappings. See [1-3, 6, 9, 11-14, 16-26, 29-36] and others.

2. Preliminaries

We recall that a Banach space E is called *strictly convex* if $\|x+y\|/2 < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. We first prove the following lemma.

Lemma 1. *Let E be a strictly convex Banach space. Let $v, x, y, z \in E$ such that $v \neq 0$, $y = z + \beta v$, and*

$$\|x - z\| = \|x - y\| + \|y - z\| = \alpha\|v\|$$

for some $\alpha > 0$, $\beta > 0$. Then $x = z + \alpha v$.

Proof. From $\|(x-y)+(y-z)\| = \|x-y\| + \|y-z\|$ and the strict convexity of E, we have that $\{x-y, y-z\}$ is linearly dependent. Since $y - z = \beta v \neq 0$, we can write $x - y = \gamma(y - z)$ for some $\gamma \geq 0$. Since

$$x - z = y + \gamma(y-z) - z = (1+\gamma)\beta v,$$

we have

$$(1+\gamma)\beta\|v\| = \|x - z\| = \alpha\|v\|$$

and hence $(1+\gamma)\beta=\alpha$. Thus, we obtain $x-z=\alpha v$. This completes the proof. ∎

The following lemmas play important roles in the proof of our main result.

Lemma 2. *Let α and β be positive real numbers with $\alpha/\beta \notin \mathbb{Q}$. Define four nondecreasing sequences $\{i_n\}$, $\{j_n\}$, $\{k_n\}$, and $\{\ell_n\}$ in $\mathbb{N} \cup \{0\}$ as follows:*

(*i*) $i_1 = k_1 = 1$ *and* $j_1 = \ell_1 = 0$*;*

(*ii*) *in the case of* $i_n\alpha - j_n\beta > k_n\beta - \ell_n\alpha$,

$$i_{n+1} = i_n + \ell_n, \quad j_{n+1} = j_n + k_n, \quad k_{n+1} = k_n \quad \textit{and} \quad \ell_{n+1} = \ell_n$$

and in the case of $i_n\alpha - j_n\beta < k_n\beta - \ell_n\alpha$,

$$i_{n+1} = i_n, \quad j_{n+1} = j_n, \quad k_{n+1} = j_n + k_n \quad \textit{and} \quad \ell_{n+1} = i_n + \ell_n$$

for all $n \in \mathbb{N}$.

Then the following hold:

(1) $\{i_n\alpha - j_n\beta\}$ *and* $\{k_n\beta - \ell_n\alpha\}$ *are nonincreasing sequences in* $(0,\infty)$ *and converge to* 0*; and*

(2) *limits of the four sequences* $\{i_n\}$, $\{j_n\}$, $\{k_n\}$, *and* $\{\ell_n\}$ *are* ∞.

Proof. We note that $i_n\alpha - j_n\beta \neq k_n\beta - \ell_n\alpha$ by the assumption of $\alpha/\beta \notin \mathbb{Q}$. So, we can define four sequences $\{i_n\}$, $\{j_n\}$, $\{k_n\}$, and $\{\ell_n\}$ in $\mathbb{N}\cup\{0\}$. We shall prove $i_n\alpha - j_n\beta > 0$ and $k_n\beta - \ell_n\alpha > 0$ by induction. We have

$$i_1\alpha - j_1\beta = \alpha > 0 \quad \text{and} \quad k_1\beta - \ell_1\alpha = \beta > 0.$$

We assume that $i_n\alpha - j_n\beta > 0$ and $k_n\beta - \ell_n\alpha > 0$ for some $n \in \mathbb{N}$. In the case of $i_n\alpha - j_n\beta > k_n\beta - \ell_n\alpha$, from

$$\begin{aligned} i_{n+1}\alpha - j_{n+1}\beta &= (i_n + \ell_n)\alpha - (j_n + k_n)\beta \\ &= (i_n\alpha - j_n\beta) - (k_n\beta - \ell_n\alpha), \end{aligned}$$

we have

$$0 < i_{n+1}\alpha - j_{n+1}\beta < i_n\alpha - j_n\beta.$$

By the definition of the sequences, we have

$$k_{n+1}\beta - \ell_{n+1}\alpha = k_n\beta - \ell_n\alpha > 0.$$

In the case of $i_n\alpha - j_n\beta < k_n\beta - \ell_n\alpha$, from

$$\begin{aligned} k_{n+1}\beta - \ell_{n+1}\alpha &= (j_n + k_n)\beta - (i_n + \ell_n)\alpha \\ &= (k_n\beta - \ell_n\alpha) - (i_n\alpha - j_n\beta), \end{aligned}$$

we have
$$0 < k_{n+1}\beta - \ell_{n+1}\alpha < k_n\beta - \ell_n\alpha.$$
We also have
$$i_{n+1}\alpha - j_{n+1}\beta = i_n\alpha - j_n\beta > 0.$$
By induction, we have $i_n\alpha - j_n\beta > 0$ and $k_n\beta - \ell_n\alpha > 0$ for $n \in \mathbb{N}$. Also, we have shown the nonincreasingness. That is, $\{i_n\alpha - j_n\beta\}$ and $\{k_n\beta - \ell_n\alpha\}$ are nonincreasing sequences in $(0, \infty)$. So, these two sequences have the limits. We put
$$s = \lim_{n\to\infty}(i_n\alpha - j_n\beta) \quad \text{and} \quad t = \lim_{n\to\infty}(k_n\beta - \ell_n\alpha).$$
We shall prove $s = t = 0$. We first assume that $s < t$. Then we can choose $m \in \mathbb{N}$ such that
$$s \leq i_m\alpha - j_m\beta < t \leq k_m\beta - \ell_m\alpha.$$
For $n \in \mathbb{N}$ with $n \geq m$, since $i_n\alpha - j_n\beta < k_n\beta - \ell_n\alpha$, we have $i_{n+1} = i_n$, $j_{n+1} = j_n$, $k_{n+1} = j_n + k_n$ and $\ell_{n+1} = i_n + \ell_n$. Thus,
$$i_n\alpha - j_n\beta = i_m\alpha - j_m\beta$$
for all $n \in \mathbb{N}$ with $n \geq m$, and hence
$$0 < i_m\alpha - j_m\beta = s.$$
For $n \in \mathbb{N}$ with $n > m$, we have
$$\begin{aligned} k_n\beta - \ell_n\alpha &= (j_{n-1} + k_{n-1})\beta - (i_{n-1} + \ell_{n-1})\alpha \\ &= k_{n-1}\beta - \ell_{n-1}\alpha - s \\ &= k_m\beta - \ell_m\alpha - (n - m)s. \end{aligned}$$
Since $s > 0$, we have $t = \lim_n(k_n\beta - \ell_n\alpha) = -\infty$. This is a contradiction. Hence, we obtain $s \geq t$. Similarly we can prove $s \leq t$. Therefore $s = t$. We next assume that $t > 0$. Then we can choose $m \in \mathbb{N}$ satisfying
$$t \leq i_m\alpha - j_m\beta < 2t \quad \text{and} \quad t \leq k_m\beta - \ell_m\alpha < 2t.$$
By the above argument, either of the following holds:
$$i_{m+1}\alpha - j_{m+1}\beta = (i_m\alpha - j_m\beta) - (k_m\beta - \ell_m\alpha) < 2t - t = t$$
or
$$k_{m+1}\beta - \ell_{m+1}\alpha = (k_m\beta - \ell_m\alpha) - (i_m\alpha - j_m\beta) < 2t - t = t.$$
This is a contradiction. Hence, we obtain $s = t = 0$. Therefore we have (1). Since $\alpha/\beta \notin \mathbb{Q}$, (2) follows from (1). ■

Lemma 3. *Let α and β be positive real numbers with $\alpha \neq \beta$ and $\alpha/\beta \in \mathbb{Q}$. Define four nondecreasing sequences $\{i_n\}$, $\{j_n\}$, $\{k_n\}$, and $\{\ell_n\}$ in $\mathbb{N} \cup \{0\}$ as follows:*

(i) $i_1 = k_1 = 1$ *and* $j_1 = \ell_1 = 0$*;*

(ii) *in the case of* $i_n\alpha - j_n\beta = k_n\beta - \ell_n\alpha$,

$$i_{n+1} = i_n, \quad j_{n+1} = j_n, \quad k_{n+1} = k_n \quad \textit{and} \quad \ell_{n+1} = \ell_n,$$

in the case of $i_n\alpha - j_n\beta > k_n\beta - \ell_n\alpha$,

$$i_{n+1} = i_n + \ell_n, \quad j_{n+1} = j_n + k_n, \quad k_{n+1} = k_n \quad \textit{and} \quad \ell_{n+1} = \ell_n$$

and in the case of $i_n\alpha - j_n\beta < k_n\beta - \ell_n\alpha$,

$$i_{n+1} = i_n, \quad j_{n+1} = j_n, \quad k_{n+1} = j_n + k_n \quad \textit{and} \quad \ell_{n+1} = i_n + \ell_n$$

for $n \in \mathbb{N}$*;*

Then the following hold:

(1) $\{i_n\alpha - j_n\beta\}$ *and* $\{k_n\beta - \ell_n\alpha\}$ *are nonincreasing sequences in* $(0, \infty)$*; and*

(2) *there exists* $n_0 \in \mathbb{N}$ *such that* $i_{n_0}\alpha - j_{n_0}\beta = k_{n_0}\beta - \ell_{n_0}\alpha$.

Proof. As in the proof of Lemma 2, we can prove (1). Since $\alpha/\beta \in \mathbb{Q}$, there exist $p, q \in \mathbb{N}$ such that $\alpha/\beta = p/q$. Put $\gamma = \beta/q$. Then $\alpha = p\gamma$ and $\beta = q\gamma$. Hence

$$i_n\alpha - j_n\beta = (i_n p - j_n q)\gamma \quad \text{and} \quad k_n\beta - \ell_n\alpha = (k_n q - \ell_n p)\gamma$$

for $n \in \mathbb{N}$. Under the assumption that (2) does not hold, the proof of Lemma 2 shows that the sequences $\{i_n\alpha - j_n\beta\}$ and $\{k_n\beta - \ell_n\alpha\}$ converge to 0. This contradicts the fact that $i_n p - j_n q,\ k_n q - \ell_n p \in \mathbb{N}$ for all $n \in \mathbb{N}$. Therefore (2) holds. ∎

3. Main Results

In this section, we prove our main results.

Theorem 3. *Let E be a strictly convex Banach space. Let $\{T(t) : t \geq 0\}$ be a one-parameter strongly continuous semigroup of nonexpansive mappings on a subset C of E. Let α and β be different positive real numbers. Then*

$$F\big(T(\alpha)\big) \cap F\big(T(\beta)\big) = \{z \in C : \lambda T(\alpha)z + (1-\lambda)T(\beta)z = z\}$$

holds for every $\lambda \in (0,1)$.

We first prove this theorem in the case of $\alpha/\beta \notin \mathbb{Q}$. We will next pass to the case of $\alpha/\beta \in \mathbb{Q}$. Before each proof, we consider a concrete case in order to give the idea of the proof.

Set $\alpha = 1$ and $\beta = \sqrt{2}$. Then $\alpha/\beta \notin \mathbb{Q}$. We fix $z \in C$ with

$$\lambda T(1)z + (1-\lambda)T(\sqrt{2})z = z$$

for some $\lambda \in (0,1)$. We put

$$v = T(1)z - T(\sqrt{2})z \quad \text{and} \quad d = \|v\|$$

and we assume $d > 0$. We note that

$$T(1)z = z + (1-\lambda)v \quad \text{and} \quad T(\sqrt{2})z = z - \lambda v.$$

Since $T(1)$ and $T(\sqrt{2}-1)$ are nonexpansive, we have

$$\begin{aligned} d &= \|T(1)z - T(\sqrt{2})z\| = \|T(1)z - T(1)\circ T(\sqrt{2}-1)z\| \\ &\leq \|T(\sqrt{2}-1)z - z\| \end{aligned}$$

and

$$\|T(\sqrt{2})z - T(\sqrt{2}-1)z\| \leq \|T(1)z - z\| = (1-\lambda)d.$$

Since $\|T(\sqrt{2})z - z\| = \lambda d$, we have

$$\begin{aligned} d &\leq \|T(\sqrt{2}-1)z - z\| \\ &\leq \|T(\sqrt{2}-1)z - T(\sqrt{2})z\| + \|T(\sqrt{2})z - z\| \\ &\leq (1-\lambda)d + \lambda d = d \end{aligned}$$

and hence

$$\|T(\sqrt{2}-1)z - z\| = \|T(\sqrt{2}-1)z - T(\sqrt{2})z\| + \|T(\sqrt{2})z - z\| = d.$$

By Lemma 1, we have

$$T(\sqrt{2}-1)z = z - v.$$

We also have

$$(2-\lambda)d = \|T(1)z - T(\sqrt{2}-1)z\| \leq \|T(2-\sqrt{2})z - z\|$$

and

$$\|T(1)z - T(2-\sqrt{2})z\| \leq \|T(\sqrt{2}-1)z - z\| = d.$$

Since $\|T(1)z - z\| = (1-\lambda)d$, we have

$$\begin{aligned}(2-\lambda)d &\le \|T(2-\sqrt{2})z - z\| \\ &\le \|T(2-\sqrt{2})z - T(1)z\| + \|T(1)z - z\| \\ &\le (2-\lambda)d\end{aligned}$$

and hence

$$\|T(2-\sqrt{2})z - z\| = \|T(2-\sqrt{2})z - T(1)z\| + \|T(1)z - z\| = (2-\lambda)d.$$

By Lemma 1, we have

$$T(2-\sqrt{2})z = z + (2-\lambda)v.$$

We have

$$(3-\lambda)d = \|T(2-\sqrt{2})z - T(\sqrt{2}-1)z\| \le \|T(3-2\sqrt{2})z - z\|$$

and

$$\|T(2-\sqrt{2})z - T(3-2\sqrt{2})z\| \le \|T(\sqrt{2}-1)z - z\| = d.$$

Since $\|T(2-\sqrt{2})z - z\| = (2-\lambda)d$, we have

$$\begin{aligned}(3-\lambda)d &\le \|T(3-2\sqrt{2})z - z\| \\ &\le \|T(3-2\sqrt{2})z - T(2-\sqrt{2})z\| + \|T(2-\sqrt{2})z - z\| \\ &\le (3-\lambda)d\end{aligned}$$

and hence

$$\begin{aligned}\|T(3-2\sqrt{2})z - z\| &= \|T(3-2\sqrt{2})z - T(2-\sqrt{2})z\| + \|T(2-\sqrt{2})z - z\| \\ &= (3-\lambda)d.\end{aligned}$$

By Lemma 1, we have

$$T(3-2\sqrt{2})z = z + (3-\lambda)v.$$

Continuing this process, we obtain the following.

$$\begin{aligned}
T(.414213562373)z &\approx T(\sqrt{2}-1)z = z - v \\
T(.585786437627)z &\approx T(2-\sqrt{2})z = z + (2-\lambda)v \\
T(.171572875254)z &\approx T(3-2\sqrt{2})z = z + (3-\lambda)v \\
T(.242640687119)z &\approx T(3\sqrt{2}-4)z = z - (4-\lambda)v \\
T(.071067811865)z &\approx T(5\sqrt{2}-7)z = z - (7-2\lambda)v \\
T(.100505063388)z &\approx T(10-7\sqrt{2})z = z + (10-3\lambda)v \\
T(.029437251523)z &\approx T(17-12\sqrt{2})z = z + (17-5\lambda)v
\end{aligned}$$

$$
\begin{aligned}
&T(.041630560343)z \approx T(17\sqrt{2}-24)z = z-(24-7\lambda)v\\
&T(.012193308820)z \approx T(29\sqrt{2}-41)z = z-(41-12\lambda)v\\
&T(.017243942703)z \approx T(58-41\sqrt{2})z = z+(58-17\lambda)v\\
&T(.005050633883)z \approx T(99-70\sqrt{2})z = z+(99-29\lambda)v\\
&T(.007142674936)z \approx T(99\sqrt{2}-140)z = z-(140-41\lambda)v\\
&T(.002092041053)z \approx T(169\sqrt{2}-239)z = z-(239-70\lambda)v\\
&T(.002958592830)z \approx T(338-239\sqrt{2})z = z+(338-99\lambda)v\\
&T(.000866551777)z \approx T(577-408\sqrt{2})z = z+(577-169\lambda)v\\
&T(.001225489276)z \approx T(577\sqrt{2}-816)z = z-(816-239\lambda)v\\
&T(.000358937499)z \approx T(985\sqrt{2}-1393)z = z-(1393-408\lambda)v\\
&T(.000507614279)z \approx T(1970-1393\sqrt{2})z = z+(1970-577\lambda)v\\
&T(.000148676780)z \approx T(3363-2378\sqrt{2})z = z+(3363-985\lambda)v\\
&T(.000210260719)z \approx T(3363\sqrt{2}-4756)z = z-(4756-1393\lambda)v
\end{aligned}
$$

This contradicts to the compactness of $\{T(t)z : t \in [0,1]\}$. Therefore we obtain $d = 0$.

Now, we give the proof.

Proof of Theorem 3 in the case of $\alpha/\beta \notin \mathbb{Q}$. We define four nondecreasing sequences $\{i_n\}$, $\{j_n\}$, $\{k_n\}$, and $\{\ell_n\}$ in $\mathbb{N} \cup \{0\}$ as in Lemma 2. Now, we fix $z \in C$ with

$$\lambda T(\alpha)z + (1-\lambda)T(\beta)z = z$$

for some $\lambda \in (0,1)$. We note that

$$\{T(t)z : t \in [0,1]\}$$

is compact and hence bounded. We put

$$v = T(\alpha)z - T(\beta)z \quad \text{and} \quad d = \|v\|$$

and we assume $d > 0$. We shall prove

$$T(i_n\alpha - j_n\beta)z = z + (1-\lambda)\, i_n\, v + \lambda\, j_n\, v, \tag{1}$$

and

$$T(k_n\beta - \ell_n\alpha)z = z - \lambda\, k_n\, v - (1-\lambda)\, \ell_n\, v \tag{2}$$

by induction. It is obvious that

$$T(i_1\alpha - j_1\beta)z = T(\alpha)z = z + (1-\lambda)\ v = z + (1-\lambda)\ i_1\ v + \lambda\ j_1\ v,$$

and

$$T(k_1\beta - \ell_1\alpha)z = T(\beta)z = z - \lambda\ v = z - \lambda\ k_1\ v - (1-\lambda)\ \ell_1\ v.$$

We assume that (1) and (2) hold for some $n \in \mathbb{N}$. We note that

$$\|T(i_n\alpha - j_n\beta)z - z\| = (1-\lambda)\ i_n\ d + \lambda\ j_n\ d,$$

$$\|T(k_n\beta - \ell_n\alpha)z - z\| = \lambda\ k_n\ d + (1-\lambda)\ \ell_n\ d,$$

and

$$\|T(i_n\alpha - j_n\beta)z - T(k_n\beta - \ell_n\alpha)z\| = (1-\lambda)\ (i_n + \ell_n)\ d + \lambda\ (j_n + k_n)\ d.$$

In the case of

$$i_n\alpha - j_n\beta > k_n\beta - \ell_n\alpha,$$

we have

$$\begin{aligned}
&\|T(i_n\alpha - j_n\beta)z - T(k_n\beta - \ell_n\alpha)z\| \\
&\le \big\|T\big((i_n\alpha - j_n\beta) - (k_n\beta - \ell_n\alpha)\big)z - z\big\| \\
&= \|T(i_{n+1}\alpha - j_{n+1}\beta)z - z\|
\end{aligned}$$

and

$$\begin{aligned}
&\|T(i_n\alpha - j_n\beta)z - T(i_{n+1}\alpha - j_{n+1}\beta)z\| \\
&= \big\|T(i_n\alpha - j_n\beta)z - T\big((i_n\alpha - j_n\beta) - (k_n\beta - \ell_n\alpha)\big)z\big\| \\
&\le \|T(k_n\beta - \ell_n\alpha)z - z\|.
\end{aligned}$$

Since

$$\begin{aligned}
&(1-\lambda)\ (i_n + \ell_n)\ d + \lambda\ (j_n + k_n)\ d \\
&\le \|T(i_{n+1}\alpha - j_{n+1}\beta)z - z\| \\
&\le \|T(i_{n+1}\alpha - j_{n+1}\beta)z - T(i_n\alpha - j_n\beta)z\| + \|T(i_n\alpha - j_n\beta)z - z\| \\
&= \|T(i_{n+1}\alpha - j_{n+1}\beta)z - T(i_n\alpha - j_n\beta)z\| + (1-\lambda)\ i_n\ d + \lambda\ j_n\ d \\
&\le \lambda\ k_n\ d + (1-\lambda)\ \ell_n\ d + (1-\lambda)\ i_n\ d + \lambda\ j_n\ d\,,
\end{aligned}$$

we have

$$
\begin{aligned}
&\|T(i_{n+1}\alpha - j_{n+1}\beta)z - z\| \\
&= \|T(i_{n+1}\alpha - j_{n+1}\beta)z - T(i_n\alpha - j_n\beta)z\| + \|T(i_n\alpha - j_n\beta)z - z\| \\
&= (1-\lambda)\,(i_n + \ell_n)\,d + \lambda\,(j_n + k_n)\,d.
\end{aligned}
$$

By Lemma 1, we obtain

$$
\begin{aligned}
T(i_{n+1}\alpha - j_{n+1}\beta)z &= z + (1-\lambda)\,(i_n + k_n)\,v + \lambda\,(j_n + \ell_n)\,v \\
&= z + (1-\lambda)\,i_{n+1}\,v + \lambda\,j_{n+1}\,v.
\end{aligned} \tag{3}
$$

Since $k_n = k_{n+1}$ and $\ell_n = \ell_{n+1}$, we have

$$
T(k_{n+1}\beta - \ell_{n+1}\alpha)z = z - \lambda\,k_{n+1}\,v - (1-\lambda)\,\ell_{n+1}\,v. \tag{4}
$$

In the case of $i_n\alpha - j_n\beta < k_n\beta - \ell_n\alpha$, we can similarly prove (3) and (4). So, by induction, (1) and (2) hold for all $n \in \mathbb{N}$. Since limits of $\{i_n\}$, $\{j_n\}$, $\{k_n\}$, and $\{\ell_n\}$ are ∞, $\{T(t)z : t \in [0, \alpha+\beta]\}$ is unbounded subset. This is a contradiction. So, we have $d = 0$. This implies

$$
T(\alpha)z = T(\beta)z = z.
$$

This completes the proof. ∎

We shall state the idea of the proof in the case of $\alpha/\beta \in \mathbb{Q}$. Set

$$
\alpha = \frac{5\pi}{6} \quad \text{and} \quad \beta = \frac{4\pi}{7}.
$$

Then $\alpha/\beta = 35/24 \in \mathbb{Q}$. Put $\gamma = \beta/24$. Then we have

$$
\alpha = 35\gamma \quad \text{and} \quad \beta = 24\gamma.
$$

We fix $z \in C$ with

$$
\lambda T(5\pi/6)z + (1-\lambda)T(4\pi/7)z = z
$$

for some $\lambda \in (0,1)$. We put

$$
v = T(5\pi/6)z - T(4\pi/7)z \quad \text{and} \quad d = \|v\|
$$

and we assume $d > 0$. Then we have the following.

$$
\begin{aligned}
T(11\gamma)z &= T(\alpha - \beta)z = z + v \\
T(13\gamma)z &= T(2\beta - \alpha)z = z - (1+\lambda)v \\
T(2\gamma)z &= T(3\beta - 2\alpha)z = z - (2+\lambda)v \\
T(9\gamma)z &= T(3\alpha - 4\beta)z = z + (3+\lambda)v
\end{aligned}
$$

$$\begin{aligned}
T(7\gamma)z &= T(5\alpha - 7\beta)z = z + (5 + 2\lambda)v \\
T(5\gamma)z &= T(7\alpha - 10\beta)z = z + (7 + 3\lambda)v \\
T(3\gamma)z &= T(9\alpha - 13\beta)z = z + (9 + 4\lambda)v \\
T(\gamma)z &= T(11\alpha - 16\beta)z = z + (11 + 5\lambda)v
\end{aligned}$$

Since $T(\gamma)$ is nonexpansive, we have

$$(13 + 6\lambda)d = \|T(\gamma)z - T(2\gamma)z\| \leq \|T(\gamma)z - z\| = (11 + 5\lambda)d.$$

This is a contradiction. Therefore we obtain $d = 0$.

Now, we give the proof.

Proof of Theorem 3 in the case of $\alpha/\beta \in \mathbb{Q}$. We define four nondecreasing sequences $\{i_n\}$, $\{j_n\}$, $\{k_n\}$, and $\{\ell_n\}$ in $\mathbb{N} \cup \{0\}$ as in Lemma 3. We put

$$m = \min\{n \in \mathbb{N} : i_n\alpha - j_n\beta = k_n\beta - \ell_n\alpha\} - 1.$$

We note that either

$$i_m\alpha - j_m\beta = 2(k_m\beta - \ell_m\alpha) \quad \text{or} \quad k_m\beta - \ell_m\alpha = 2(i_m\alpha - j_m\beta)$$

holds. Now, we fix $z \in C$ with

$$\lambda T(\alpha)z + (1 - \lambda)T(\beta)z = z$$

for some $\lambda \in (0, 1)$. We put

$$v = T(\alpha)z - T(\beta)z \quad \text{and} \quad d = \|v\|$$

and we assume $d > 0$. Then from the proof of the case of $\alpha/\beta \notin \mathbb{Q}$, we know that

$$T(i_m\alpha - j_m\beta)z = z + (1 - \lambda)\, i_m\, v + \lambda\, j_m\, v,$$

and

$$T(k_m\beta - \ell_m\alpha)z = z - \lambda\, k_m\, v - (1 - \lambda)\, \ell_m\, v.$$

In the case of

$$i_m\alpha - j_m\beta > k_m\beta - \ell_m\alpha,$$

we have

$$i_m\alpha - j_m\beta = 2\ (k_m\beta - \ell_m\alpha).$$

So, we obtain

$$\begin{aligned}
&(1-\lambda)\ (i_m+\ell_m)\ d + \lambda\ (j_m+k_m)\ d \\
&= \|T(i_m\alpha - j_m\beta)z - T(k_m\beta - \ell_m\alpha)z\| \\
&= \big\|T\big(2(k_m\beta - \ell_m\alpha)\big)z - T(k_m\beta - \ell_m\alpha)z\big\| \\
&\leq \|T(k_m\beta - \ell_m\alpha)z - z\| \\
&= \lambda\ k_m\ d + (1-\lambda)\ \ell_m\ d.
\end{aligned}$$

This is a contradiction. In the case of

$$i_m\alpha - j_m\beta < k_m\beta - \ell_m\alpha,$$

we have

$$k_m\beta - \ell_m\alpha = 2\ (i_m\alpha - j_m\beta).$$

So, we obtain

$$\begin{aligned}
&(1-\lambda)\ (i_m+\ell_m)\ d + \lambda\ (j_m+k_m)\ d \\
&= \|T(i_m\alpha - j_m\beta)z - T(k_m\beta - \ell_m\alpha)z\| \\
&\leq \|T(i_m\alpha - j_m\beta)z - z\| \\
&= (1-\lambda)\ i_m\ d + \lambda\ j_m\ d.
\end{aligned}$$

This is also a contradiction. So, we have $d = 0$. This implies

$$T(\alpha)z = T(\beta)z = z.$$

This completes the proof. ∎

Theorem 4. *Let E be a strictly convex Banach space. Let $\{T(t) : t \geq 0\}$ be a one-parameter strongly continuous semigroup of nonexpansive mappings on a subset C of E. Let α and β be positive real numbers satisfying $\alpha/\beta \notin \mathbb{Q}$. Then*

$$\bigcap_{t\geq 0} F\big(T(t)\big) = \{z \in C : \lambda T(\alpha)z + (1-\lambda)T(\beta)z = z\}$$

holds for every $\lambda \in (0,1)$.

Proof. Fix $\lambda \in (0,1)$. It is obvious

$$\bigcap_{t\geq 0} F\big(T(t)\big) \subset \{z \in C : \lambda T(\alpha)z + (1-\lambda)T(\beta)z = z\}.$$

We assume that $z \in C$ satisfies

$$\lambda T(\alpha)z + (1-\lambda)T(\beta)z = z.$$

Then from Theorem 3, we have $T(\alpha)z = T(\beta)z = z$. So, from Theorem 1, z is a common fixed point of $\{T(t) : t \geq 0\}$. This completes the proof. ∎

As a direct consequence, we obtain the following.

Corollary 1. *Let E be a strictly convex Banach space. Let $\{T(t) : t \geq 0\}$ be a one-parameter strongly continuous semigroup of nonexpansive mappings on a subset C of E. Then*

$$\bigcap_{t\geq 0} F\big(T(t)\big) = F\left(\frac{1}{2}T(1) + \frac{1}{2}T(\sqrt{2})\right)$$

holds.

4. Two-parameter Semigroups

In this section, we discuss two-parameter nonexpansive semigroups. A family of mappings $\{T(p) : p \in [0,\infty)^2\}$ is called a *two-parameter nonexpansive semigroup* on C if the following are satisfied:

(sg 1) For each $p \in [0,\infty)^2$, $T(p)$ is a nonexpansive mapping on C.

(sg 2) $T(p+q) = T(p) \circ T(q)$ for all $p, q \in [0,\infty)^2$;

(sg 3) for each $x \in C$, the mapping $p \mapsto T(p)x$ from $[0,\infty)^2$ into C is continuous.

The author proved in [28] the following theorem, which is the natural generalization of Theorem 1.

Theorem 5 ([28]). *Let $\{T(p) : t \in [0,\infty)^2\}$ be a two-parameter nonexpansive semigroup on a subset C of a Banach space E. Let $p_1, p_2 \in [0,\infty)^2$ such that $\{p_1, p_2\}$ is linearly independent in the usual sense. Let $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $\{1, \alpha_1, \alpha_2\}$ is linearly independent over $\mathbb{Q}$, that is, $\nu_0 + \nu_1\alpha_1 + \nu_2\alpha_2 = 0$ implies $\nu_0 = \nu_1 = \nu_2 = 0$ for $\nu_0, \nu_1, \nu_2 \in \mathbb{Z}$. Suppose $p_0 = \alpha_1 p_1 + \alpha_2 p_2 \in [0,\infty)^2$. Then*

$$\bigcap_{p\in[0,\infty)^2} F\big(T(p)\big) = F\big(T(p_0)\big) \cap F\big(T(p_1)\big) \cap F\big(T(p_2)\big)$$

holds.

It is a natural problem whether or not the conclusion which is similar to Theorem 4 holds. Of course, under the assumption of $\bigcap_{p\in[0,\infty)^2} F\big(T(p)\big) \neq \varnothing$, it holds. However, our answer of this problem is negative.

Example 2 ([28]). Put $E = C = \mathbb{R}$, $e_1 = (1,0) \in \mathbb{R}^2$ and $e_2 = (0,1) \in \mathbb{R}^2$. Define a two-parameter nonexpansive semigroup $\{T(p) : p \in [0,\infty)^2\}$ on C by

$$T(\lambda_1 e_1 + \lambda_2 e_2)x = x + \lambda_1 - \lambda_2$$

for $\lambda_1, \lambda_2 \in [0,\infty)$ and $x \in E$. Define a nonexpansive mapping S on C by

$$Sx = \frac{\sqrt{2}+\sqrt{3}+1}{6}T(\sqrt{2}e_1 + \sqrt{3}e_2)x + \frac{3-\sqrt{2}}{6}T(e_1)x + \frac{2-\sqrt{3}}{6}T(e_2)x$$

for $x \in C$. Then

$$\bigcap_{p\in[0,\infty)^2} F\big(T(p)\big) = \varnothing \subsetneqq C = F(S)$$

holds.

Proof. We note that $F\big(T(e_1)\big) = \varnothing$ and $Sx = x$ for all $x \in C$. ■

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Tomonari Suzuki
Department of Mathematics,
Kyushu Institute of Technology,
Sensuicho, Tobata,
Kitakyushu 804-8550,
Japan
E-mail: suzuki-t@mns.kyutech.ac.jp