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Research Article

Stability of Matrix Polytopes with a Dominant Vertex and Implications for System Dynamics

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The paper considers the class of matrix polytopes with a dominant vertex and the class of uncertain dynamical systems defined in discrete time and continuous time, respectively, by such polytopes. We analyze the standard concept of stability in the sense of Schur—abbreviated as SS (resp., Hurwitz—abbreviated as HS), and we develop a general framework for the investigation of the diagonal stability relative to an arbitrary Hölder p-norm, $1 \leq p \leq \infty$, abbreviated as SDS_p (resp., HDS_p). Our framework incorporates, as the particular case with $p = 2$, the known condition of quadratic stability satisfied by a diagonal positive-definite matrix, i.e. SDS_2 (resp., HDS_2) means that the standard inequality of Stein (resp., Lyapunov) associated with all matrices of the polytope has a common diagonal solution. For the considered class of matrix polytopes, we prove the equivalence between SS and SDS_p (resp., HS and HDS_p), $1 \leq p \leq \infty$ (fact which is not true for matrix polytopes with arbitrary structures). We show that the dominant vertex provides all the information needed for testing these stability properties and for computing the corresponding robustness indices. From the dynamical point of view, if an uncertain system is defined by a polytope with a dominant vertex, then the standard asymptotic stability ensures supplementary properties for the state-space trajectories, which refer to special types of Lyapunov functions and contractive invariant sets (characterized through vector p-norms weighted by diagonal positive-definite matrices). The applicability of the main results is illustrated by two numerical examples that cover both discrete- and continuous-time cases for the class of uncertain dynamics studied in our paper.

1. Introduction

1.1. Research Context and Objective. Consider the matrix polytope

$$\mathscr{A} = \left\{ \mathbf{A} \in \mathbb{R}^{n\times n} \mid \mathbf{A} = \sum_{k=1}^{K} \gamma_k \mathbf{A}_k,\ \gamma_k \geq 0,\ \sum_{k=1}^{K} \gamma_k = 1 \right\}, \quad (1)$$

where $\{\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_K\}$ is a finite set of real $n \times n$ matrices. The Schur (resp., Hurwitz) stability—abbreviated as SS (resp., HS)—has been investigated for matrix polytope (1) starting with the 80s, by papers 4such as [1–13]. Research was strongly motivated by the dynamics analysis of linear systems with model uncertainties (which inherently occur due to incomplete or approximate information on process parameters). Description (1) is also referred to as a "polytopic matrix", and from the modeling-power point of view, it incorporates the class of "interval matrices," defined by hyperrectangles in $\mathbb{R}^{n\times n}$, i.e.

$$\mathscr{A}^I = \left\{ \mathbf{A} \in \mathbb{R}^{n\times n} \mid \mathbf{A}_0 - \mathbf{R} \leq \mathbf{A} \leq \mathbf{A}_0 + \mathbf{R} \right\}, \qquad \mathbf{A}_0, \mathbf{R} \in \mathbb{R}^{n\times n},\ \ \mathbf{R} \geq 0, \quad (2)$$

where the inequalities have a componentwise meaning. Significant results on Schur and Hurwitz stability of interval matrices have been reported in [10, 14–24].

Also starting with the 80s the linear algebra literature developed studies on *a stronger type of matrix stability*, called "diagonal stability"; pioneering works such as [25, 26] should be mentioned. In accordance with the monograph [27], a square matrix is Schur (resp., Hurwitz) diagonally stable if the Stein (resp., Lyapunov), inequality associated with that matrix has diagonal positive-definite solutions. As a natural expansion, our work [28] introduced the Stein (resp., Lyapunov), inequalities relative to a Hölder p-norm, $1 \le p \le \infty$, and generalized the aforementioned diagonal stability concept to "Schur (resp., Hurwitz) diagonal stability relative to a Hölder p-norm"—abbreviated as SDS_p (resp., HDS_p). For $p = 2$, the framework proposed by [28] coincides with the classic approach presented by [27].

The SDS_p (resp., HDS_p), $1 \le p \le \infty$, has been recently explored by our papers [29, 30] for interval matrices and for arbitrary polytopic matrices, respectively. It is worth saying that the monograph [27] addressed the standard case of diagonal stability (i.e. SDS_2 and HDS_2 in our nomenclature) for interval matrices.

During the last decade, diagonal stability ensured a visible research potential for systems and control engineering, mainly related to the simpler form of the Lyapunov function candidates, as outlined by works such as [27–29, 31–35]. These works use the same terminology "diagonal stability" in the sense of a system property that is induced by the original matrix property discussed in the previous paragraphs.

Our current paper focuses on the stability of a class of matrix polytopes of form (1) called "with a dominant vertex"—the concept is to be rigorously introduced by Definition 3 in the next section. We also study the dynamics of the polytopic systems associated with this class of matrix polytopes, described by the following equations:

(i) in the discrete-time case,

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t), \quad \mathbf{A} \in \mathscr{A},\ \mathbf{x}(t_0) = \mathbf{x}_0,\ t, t_0 \in \mathbb{Z}_+,\ t \ge t_0, \tag{3-S}$$

(ii) in the continuous-time case,

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t), \quad \mathbf{A} \in \mathscr{A},\ \mathbf{x}(t_0) = \mathbf{x}_0,\ t, t_0 \in \mathbb{R}_+,\ t \ge t_0. \tag{3-H}$$

In both models (3-S) and (3-H), the entries of matrix $\mathbf{A}$ are considered fixed (not time varying); they are uncertain in the sense that their values are incompletely known but surely satisfy the condition $\mathbf{A} \in \mathscr{A}$. In other words, a single matrix $\mathbf{A}$ is used for modeling a certain evolution of the process, whereas for modeling two different evolutions (taking place separately) two distinct matrices $\mathbf{A}_1 \ne \mathbf{A}_2$, $\mathbf{A}_1, \mathbf{A}_2 \in \mathscr{A}$ may be needed.

1.2. Paper Structure. For a matrix polytope (1) with a dominant vertex, we prove that SS (resp., HS) is equivalent to SDS_p (resp., HDS_p), $1 \le p \le \infty$, unlike the case of an arbitrary matrix polytope (e.g., [30]) where (i) SDS_p (resp., HDS_p) is more conservative than SS (resp., HS) and (ii) results on SDS_{p_1} and SDS_{p_2} (resp., HDS_{p_1} and HDS_{p_2}) may be different for $p_1 \ne p_2$, $1 \le p_1, p_2 \le \infty$. These aspects are discussed by Section 2 of our work. Section 3 analyzes the implications of Section 2 for the dynamics of a polytopic system (3-S), respectively (3-H), defined by a matrix polytope with a dominant vertex. We show that the asymptotic stability of such a system is equivalent to the existence of Lyapunov functions and contractive invariant sets expressed in terms of any Hölder p-norm, by using an appropriate weighting matrix of diagonal form (whose positive entries depend on the chosen norm). The utility of our main results is illustrated in Section 4 by numerical examples, covering Schur (resp., Hurwitz) stability for matrix polytopes with a dominant vertex, as well as the implications for the dynamics of discrete-time (resp., continuous-time), polytopic systems.

Throughout the text, in equation numbering we use the extension S (resp., H), for referring to Schur (resp., Hurwitz) stability and/or to discrete-time (resp., continuous-time), dynamics—as in the above equation (3-S) (resp., (3-H)). The extensions (S) and (H) play the same role for the labels of definitions and theorems.

To ensure the fluent presentation of our results, their proofs are given in the Appendix.

1.3. Notations and Nomenclature. Let $\mathbf{x} = [x_1 \cdots x_n]^T$, $\mathbf{y} = [y_1 \cdots y_n]^T \in \mathbb{R}^n$ be vectors.

(i) $\|\mathbf{x}\|_p$ is the Hölder vector p-norm defined by $\|\mathbf{x}\|_p = [|x_1|^p + \cdots + |x_n|^p]^{1/p}$ for $1 \le p < \infty$ and by $\|\mathbf{x}\|_\infty = \max_{1 \le i \le n} |x_i|$ for $p = \infty$.

(ii) "$\mathbf{x} \le \mathbf{y}$", "$\mathbf{x} < \mathbf{y}$" mean componentwise inequalities, i.e. $x_i \le y_i$, $x_i < y_i$, $i = 1, \ldots, n$.

Let $\mathbf{M} = [m_{ij}]$, $\mathbf{Q} = [q_{ij}] \in \mathbb{R}^{n \times n}$ be square matrices.

(iii) $\|\mathbf{M}\|_p$ is the matrix norm induced by the vector p-norm through $\|\mathbf{M}\|_p = \sup_{x \ne 0}(\|\mathbf{M}\mathbf{x}\|_p / \|\mathbf{x}\|_p) = \max_{\|\mathbf{x}\|_p = 1} \|\mathbf{M}\mathbf{x}\|_p$.

(iv) $\mu_p(\mathbf{M}) = \lim_{h \downarrow 0} h^{-1}(\|I + h\mathbf{M}\|_p - 1)$ is the matrix measure [36, page 41], based on the matrix norm $\|\ \|_p$.

(v) If $\mathbf{D} = \operatorname{diag}\{d_1, \ldots, d_n\}$, $d_i > 0$, $i = 1, \ldots, n$, then the following regions of the complex plane $G_j^c(\mathbf{D}^{-1}\mathbf{M}\mathbf{D}) = \{z \in \mathbb{C} \mid |z - m_{jj}| \le \sum_{i=1, i \ne j}^n ((d_j/d_i)|m_{ij}|)\}$, $j = 1, \ldots, n$, are called the generalized Gershgorin's disks of $\mathbf{M}$ defined with $\mathbf{D}$ for columns.

(vi) If $\mathbf{D} = \operatorname{diag}\{d_1, \ldots, d_n\}$, $d_i > 0$, $i = 1, \ldots, n$, then the following regions of the complex plane $G_i^r(\mathbf{D}^{-1}\mathbf{M}\mathbf{D}) = \{z \in \mathbb{C} \mid |z - m_{ii}| \le \sum_{j=1, j \ne i}^n ((d_j/d_i)|m_{ij}|)\}$, $i = 1, \ldots, n$, are called the generalized Gershgorin's disks of $\mathbf{M}$ defined with $\mathbf{D}$ for rows.

(vii) $\sigma(\mathbf{M}) = \{z \in \mathbb{C} \mid \det(z\mathbf{I} - \mathbf{M}) = 0\}$ is the spectrum of $\mathbf{M}$, and $\lambda_i(\mathbf{M}) \in \sigma(\mathbf{M})$, $i = 1, \ldots, n$, are the eigenvalues of $\mathbf{M}$.

(viii) If $\sigma(\mathbf{M}) \subset \mathbb{C}_S = \{z \in \mathbb{C} \mid |z| < 1\}$, then matrix $\mathbf{M}$ is said to be Schur stable (abbreviated as SS).

(ix) If $\sigma(\mathbf{M}) \subset \mathbb{C}_H = \{z \in \mathbb{C} \mid \operatorname{Re} z < 0\}$, then matrix $\mathbf{M}$ is said to be Hurwitz stable (abbreviated as HS).

(x) If $\mathbf{M}$ is nonnegative (all entries are nonnegative), its spectral radius is a positive eigenvalue, denoted by $\lambda_{\max}(\mathbf{M})$, such that $|\lambda_i(\mathbf{M})| \le \lambda_{\max}(\mathbf{M})$, $i = 1, \ldots, n$.

(xi) If $\mathbf{M}$ is essentially nonnegative (all off-diagonal entries are nonnegative), then it has a real eigenvalue, denoted by $\lambda_{\max}(\mathbf{M})$, such that $\operatorname{Re}\{\lambda_i(\mathbf{M})\} \le \lambda_{\max}(\mathbf{M})$, $i = 1, \ldots, n$—for example, Lemma 1 in [28].

(xii) If $\mathbf{M}$ is symmetrical, then all its eigenvalues are real and there exists an eigenvalue denoted by $\lambda_{\max}(\mathbf{M})$, such that $\lambda_i(\mathbf{M}) \le \lambda_{\max}(\mathbf{M})$, $i = 1, \ldots, n$.

(xiii) "$\mathbf{M} \succ 0$", "$\mathbf{M} \prec 0$" mean that $\mathbf{M}$ is a positive-definite, negative-definite matrix.

(xiv) If the oriented graph of $\mathbf{M}$ is strongly connected, then $\mathbf{M}$ is called irreducible; otherwise $\mathbf{M}$ is called reducible.

(xv) For $p \in \{1, 2, \infty\}$, the matrix norms $\|\mathbf{M}\|_p$ and matrix measures $\mu_p(\mathbf{M})$ have the following expressions:

$$\|\mathbf{M}\|_p = \begin{cases} \max\limits_{1\le j\le n}\left\{\sum\limits_{i=1}^{n}\left|m_{ij}\right|\right\}, & p = 1, \\ \sqrt{\lambda_{\max}\left(\mathbf{M}^T\mathbf{M}\right)}, & p = 2, \\ \max\limits_{1\le i\le n}\left\{\sum\limits_{j=1}^{n}\left|m_{ij}\right|\right\}, & p = \infty, \end{cases} \tag{4-S}$$

$$\mu_p(\mathbf{M}) = \begin{cases} \max\limits_{1\le j\le n}\left\{m_{jj} + \sum\limits_{i=1, i\ne j}^{n}\left|m_{ij}\right|\right\}, & p = 1, \\ \dfrac{1}{2}\lambda_{\max}\left(\mathbf{M} + \mathbf{M}^T\right), & p = 2, \\ \max\limits_{1\le i\le n}\left\{m_{ii} + \sum\limits_{j=1, j\ne i}^{n}\left|m_{ij}\right|\right\}, & p = \infty. \end{cases} \tag{4-H}$$

(xvi) $|\mathbf{M}|$ denotes the matrix built with the absolute values of the entries of $\mathbf{M}$.

(xvii) $\mathbf{M}^S \in \mathbb{R}^{n\times n}$ (S superscript from Schur) denotes the nonnegative matrix defined by $\mathbf{M}^S = |\mathbf{M}|$.

(xviii) $\mathbf{M}^H \in \mathbb{R}^{n\times n}$ (H superscript from Hurwitz) denotes the essentially nonnegative matrix defined by $\mathbf{M}^H = \mathbf{M}^d + |\mathbf{M}^o|$, where $\mathbf{M}^d = \operatorname{diag}\{m_{11}, \ldots, m_{nn}\}$ and $\mathbf{M}^o = \mathbf{M} - \mathbf{M}^d$.

(xix) "$\mathbf{M} \le \mathbf{Q}$", "$\mathbf{M} < \mathbf{Q}$" mean componentwise inequalities, i.e. $m_{ij} \le q_{ij}$, $m_{ij} < q_{ij}$, $i, j = 1, \ldots, n$.

Throughout the text we shall write "X (resp., Y)" wherever "X" and "Y" are referred to in parallel.

2. Results on Matrix Polytopes

The current section explores the stability of matrix polytopes with a dominant vertex. For this class of polytopes, the standard Schur (Hurwitz) stability is proved to be equivalent to stronger stability properties, namely, diagonal stability relative to arbitrary Hölder p-norms $1 \le p \le \infty$.

2.1. Preliminaries

Definition 1 (S). Let us consider: $1 \le p \le \infty$; an arbitrary matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$; a matrix polytope $\mathscr{A}$ of form (1); a nonsingular matrix $\mathbf{Q} \in \mathbb{R}^{n\times n}$.

(a) The inequality

$$\left\|\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}\right\|_p < 1 \tag{5-S}$$

is called the Stein-type inequality relative to the p-norm associated with matrix $\mathbf{A}$; matrix $\mathbf{Q}$ is said to be a solution to this inequality.

(b) Matrix $\mathbf{Q}$ is said to be a solution to the Stein-type inequality relative to the p-norm associated with the polytope $\mathscr{A}$ if the following condition is fulfilled:

$$\forall \mathbf{A} \in \mathscr{A} : \left\|\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}\right\|_p < 1. \tag{6-S}$$

Definition 1 (H). Let us consider: $1 \le p \le \infty$; an arbitrary matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$; a matrix polytope $\mathscr{A}$ of form (1); a nonsingular matrix $\mathbf{Q} \in \mathbb{R}^{n\times n}$.

(a) The inequality

$$\mu_p\left(\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}\right) < 0 \tag{5-H}$$

is called the Lyapunov-type inequality relative to the p-norm associated with matrix $\mathbf{A}$; matrix $\mathbf{Q}$ is said to be a solution to this inequality.

(b) Matrix $\mathbf{Q}$ is said to be a solution to the Lyapunov-type inequality relative to the p-norm associated with the polytope $\mathscr{A}$ if the following condition is fulfilled:

$$\forall \mathbf{A} \in \mathscr{A} : \mu_p\left(\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}\right) < 0. \tag{6-H}$$

Remark 1. (i) The terminology introduced by Definition 1(S)(a) (resp., Definition 1(H)(a)) is motivated by the fact that inequality (5-S) (resp., (5-H)) with $p = 2$ is equivalent to the standard Stein inequality

$$\mathbf{A}^T\mathbf{P}\mathbf{A} - \mathbf{P} \prec 0, \qquad \mathbf{P} = \left(\mathbf{Q}^{-1}\right)^T\left(\mathbf{Q}^{-1}\right), \tag{7-S}$$

respectively the standard *Lyapunov inequality*

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} \prec 0, \qquad \mathbf{P} = \left(\mathbf{Q}^{-1}\right)^T\left(\mathbf{Q}^{-1}\right). \tag{7-H}$$

Indeed, for (5-S) with $p = 2$ we may write that $\|\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}\|_2 < 1$ is equivalent to

$$\begin{aligned} &\forall i = 1, \ldots, n : \lambda_i\left(\mathbf{Q}^T\mathbf{A}^T\left(\mathbf{Q}^{-1}\right)^T\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}\right) < 1 \\ &\Longleftrightarrow \lambda_i\left(\mathbf{Q}^T\mathbf{A}^T\left(\mathbf{Q}^{-1}\right)^T\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} - \mathbf{I}\right) < 0 \\ &\Longleftrightarrow \lambda_i\left(\mathbf{A}^T\left(\mathbf{Q}^{-1}\right)^T\left(\mathbf{Q}^{-1}\right)\mathbf{A} - \left(\mathbf{Q}^{-1}\right)^T\left(\mathbf{Q}^{-1}\right)\right) < 0, \end{aligned} \tag{8-S}$$

which means the fulfillment of (7-S).

Similarly, for (5-H) with $p = 2$ we may write that $\mu_2(\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}) < 0$ is equivalent to

$$\forall i = 1, \dots, n : \lambda_i \left(\mathbf{Q}^T\mathbf{A}^T\left(\mathbf{Q}^{-1}\right)^T + \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}\right) < 0$$
$$\Longleftrightarrow \lambda_i \left(\mathbf{A}^T\left(\mathbf{Q}^{-1}\right)^T\left(\mathbf{Q}^{-1}\right) + \left(\mathbf{Q}^{-1}\right)^T\left(\mathbf{Q}^{-1}\right)\mathbf{A}\right) < 0, \tag{8-H}$$

which means the fulfillment of (7-H).

(ii) The existence of $\mathbf{P} \succ 0$ solving the standard Stein inequality (7-S) (resp., Lyapunov inequality (7-H)) is equivalent to Schur (resp., Hurwitz) stability of matrix $\mathbf{A}$.

(iii) Conditions (6-S) (resp., (6-H)) with $p = 2$ in Definition 1(S)(b) (resp., Definition 1(H)(b)) represent the definition of Schur (resp., Hurwitz) quadratic stability of the matrix polytope $\mathscr{A}$, for example, [37, page 213].

Definition 2 (S). Let $\mathscr{A}$ be a matrix polytope of form (1).

(a) $\mathscr{A}$ is called Schur stable (abbreviated as SS) if

$$\forall \mathbf{A} \in \mathscr{A} : \mathbf{A} \text{ is SS.} \tag{9-S}$$

(b) Let $1 \le p \le \infty$. $\mathscr{A}$ is called Schur diagonally stable relative to the p-norm (abbreviated as SDS_p) if there exists a diagonal positive-definite matrix $\mathbf{D} \succ 0$ that satisfies the Stein-type inequality relative to the p-norm associated with the polytope $\mathscr{A}$, i.e.

$$\forall \mathbf{A} \in \mathscr{A} : \left\|\mathbf{D}^{-1}\mathbf{A}\mathbf{D}\right\|_p < 1. \tag{10-S}$$

Definition 2 (H). Let $\mathscr{A}$ be a matrix polytope of form (1).

(a) $\mathscr{A}$ is called Hurwitz stable (abbreviated as HS) if

$$\forall \mathbf{A} \in \mathscr{A} : \mathbf{A} \text{ is HS.} \tag{9-H}$$

(b) Let $1 \le p \le \infty$. $\mathscr{A}$ is called Hurwitz diagonally stable relative to the p-norm (abbreviated as HDS_p) if there exists a diagonal positive-definite matrix $\mathbf{D} \succ 0$ that satisfies the Lyapunov-type inequality relative to the p-norm associated with the polytope $\mathscr{A}$, i.e.

$$\forall \mathbf{A} \in \mathscr{A} : \mu_p\left(\mathbf{D}^{-1}\mathbf{A}\mathbf{D}\right) < 0. \tag{10-H}$$

Remark 2. (i) If $\mathscr{A}$ is a trivial polytope defined by a single matrix $\mathbf{A}$ (i.e. $\mathbf{A}_1 = \mathbf{A}$, $K = 1$, in (1)), then Definition 2(S)(b) (resp., Definition 2(H)(b)) coincides with Definition 1 in [28].

(ii) Let $1 \le p \le \infty$. If $\mathscr{A}$ is a proper polytope (i.e. $\exists k_1 \ne k_2$, $k_1, k_2 \in \{1, \dots, K\}$: $\mathbf{A}_{k_1} \ne \mathbf{A}_{k_2}$ in (1)), then Definition 2(S)(b) (resp., Definition 2(H)(b)) proposes a meaningful extension of Definition 1 in [28]. Indeed, the simple use of Definition 1 in [28] does not necessarily imply the existence of a unique diagonal matrix $\mathbf{D} \succ 0$ that satisfies inequality (10-S) (resp., (10-H)) for all matrices $\mathbf{A} \in \mathscr{A}$.

(iii) Let $1 \le p \le \infty$. The SDS_p (resp., HDS_p) is a property of $\mathscr{A}$ stronger than SS (resp., HS). Indeed, each matrix $\mathbf{A} \in \mathscr{A}$ is SS (resp., HS) once it is SDS_p (resp., HDS_p) in accordance with Remark 2 in [28].

(iv) Let $p = 2$. Definition 2(S)(b) (resp., Definition 2(H)(b)) expresses a particular case of Schur (resp., Hurwitz) quadratic stability of the matrix polytope $\mathscr{A}$—see Remark 1(iii). Subsequently, the quadratic stability is a property of $\mathscr{A}$ stronger than SS (resp., HS) but, at the same time, weaker than SDS_2 (resp., HDS_2).

Definition 3 (S). Let $\mathscr{A}$ be a matrix polytope of form (1). If there exists a subscript $k^* \in \{1, \dots, K\}$, such that the vertex $\mathbf{A}_{k^*}$ fulfills one of the following two sets of componentwise inequalities:

$$\mathbf{A}_k^S \le \mathbf{A}_{k^*}, \quad k \ne k^*,\ k = 1, \dots, K, \tag{11-S-1}$$

$$\mathbf{A}_k^S \le -\mathbf{A}_{k^*}, \quad k \ne k^*,\ k = 1, \dots, K, \tag{11-S-2}$$

then $\mathscr{A}$ is called a matrix polytope with an S-dominant vertex, and $\mathbf{A}_{k^*}$ is called the S-dominant vertex of $\mathscr{A}$. (The notation $\mathbf{M}^S$ is introduced in Section 1.3.)

Definition 3 (H). If there exists a subscript $k^* \in \{1, \dots, K\}$, such that the vertex $\mathbf{A}_{k^*}$ fulfills the componentwise inequalities:

$$\mathbf{A}_k^H \le \mathbf{A}_{k^*}, \quad k \ne k^*,\ k = 1, \dots, K, \tag{11-H}$$

then $\mathscr{A}$ is called a matrix polytope with an H-dominant vertex, and $\mathbf{A}_{k^*}$ is called the H-dominant vertex of $\mathscr{A}$. (The notation $\mathbf{M}^H$ is introduced in Section 1.3.)

Remark 3. (i) If $\mathscr{A}$ is a matrix polytope with an S-dominant (resp., H-dominant) vertex $\mathbf{A}_{k^*}$, then Definition 3(S) (resp., Definition 3(H)) shows that matrix $\mathbf{A}_{k^*}$ is nonnegative—inequalities (11-S-1) or nonpositive—inequalities (11-S-2) (resp., essentially nonnegative—inequalities (11-H)).

(ii) In the remainder of the text, we mainly address the case of the S-dominant vertex defined by inequalities (11-S-1). The case based on inequalities (11-S-2) does not require a separate approach, since all the results we are going to use for $\mathbf{A}_{k^*}$ nonnegative remain valid for $-\mathbf{A}_{k^*}$ nonnegative.

(iii) A matrix polytope $\mathscr{A}$ may have two S-dominant vertices denoted as $\mathbf{A}_{k^*}$, $\mathbf{A}_{k^{**}}$ in the particular case when $\mathbf{A}_{k^*} = -\mathbf{A}_{k^{**}}$, $\mathbf{A}_{k^*} \ge 0$ satisfies inequalities (11-S-1), $\mathbf{A}_{k^{**}} \le 0$ satisfies inequalities (11-S-2). We still can refer to $\mathscr{A}$ as having "an S-dominant vertex," since the stability properties of $\mathscr{A}$ induced by $\mathbf{A}_{k^*}$ and by $\mathbf{A}_{k^{**}}$ are identical, as resulting from the further development of our paper.

2.2. Stability Analysis

Theorem 1 (S). *Let us consider:* $1 \le p \le \infty$*; a matrix polytope* $\mathscr{A}$ *with an S-dominant vertex* $\mathbf{A}_{k^*}$.

The following statements are equivalent.

(i) $\mathbf{A}_{k^*}$ *is SS.*

(ii) $\mathscr{A}$ *is SS.*

(iii) *There exists a* p, $1 \le p \le \infty$, *such that* $\mathscr{A}$ *is* SDS_p.

(iv) $\mathscr{A}$ *is* SDS_p *for all* p, $1 \le p \le \infty$.

(v) *There exists a diagonal matrix* $\mathbf{D} \succ 0$ *such that the union for all* $\mathbf{A} \in \mathscr{A}$ *of the generalized Gershgorin's* *disks written for columns is located inside the unit circle of the complex plane, i.e.* $\bigcup_{\mathbf{A}\in\mathscr{A}}\bigcup_{j=1}^{n} G_j^c(\mathbf{D}^{-1}\mathbf{A}\mathbf{D}) \subset \mathbb{C}_S$.

(vi) *There exists a diagonal matrix* $\mathbf{D} \succ 0$ *such that the union for all* $\mathbf{A} \in \mathscr{A}$ *of the generalized Gershgorin's disks written for rows is located inside the unit, circle of the complex plane, i.e.* $\bigcup_{\mathbf{A}\in\mathscr{A}}\bigcup_{i=1}^{n} G_i^r(\mathbf{D}^{-1}\mathbf{A}\mathbf{D}) \subset \mathbb{C}_S$.

Proof. See the Appendix. □

Theorem 1 (H). *Let us consider:* $1 \le p \le \infty$*; a matrix polytope* $\mathscr{A}$ *with an H-dominant vertex* $\mathbf{A}_{k^*}$.

The following statements are equivalent.

(i) $\mathbf{A}_{k^*}$ *is HS.*

(ii) $\mathscr{A}$ *is HS.*

(iii) *There exists a* p, $1 \le p \le \infty$, *such that* $\mathscr{A}$ *is* HDS_p.

(iv) $\mathscr{A}$ *is* HDS_p *for all* p, $1 \le p \le \infty$.

(v) *There exists a diagonal matrix* $\mathbf{D} \succ 0$ *such that the union for all* $\mathbf{A} \in \mathscr{A}$ *of the generalized Gershgorin's disks written for columns is located in the left half plane of the complex plane, i.e.* $\bigcup_{\mathbf{A}\in\mathscr{A}}\bigcup_{j=1}^{n} G_j^c(\mathbf{D}^{-1}\mathbf{A}\mathbf{D}) \subset \mathbb{C}_H$.

(vi) *There exists a diagonal matrix* $\mathbf{D} \succ 0$ *such that the union for all* $\mathbf{A} \in \mathscr{A}$ *of the generalized Gershgorin's disks written for rows is located in the left half plane of the complex plane, i.e.* $\bigcup_{\mathbf{A}\in\mathscr{A}}\bigcup_{i=1}^{n} G_i^r(\mathbf{D}^{-1}\mathbf{A}\mathbf{D}) \subset \mathbb{C}_H$.

Proof. See the Appendix. □

2.3. Diagonal Solutions to Stein-Type and Lyapunov-Type Inequalities. The S-dominant (resp., H-dominant) vertex $\mathbf{A}_{k^*}$ of a matrix polytope $\mathscr{A}$ can be used not only for testing the properties SDS_p (resp., HDS_p) of $\mathscr{A}$ but also for finding concrete diagonal matrices $\mathbf{D} \succ 0$ that satisfy the inequality (10-S) in Definition 2(S) (resp., inequality (10-H) in Definition 2(H)).

Theorem 2 (S). *Let us consider:* $1 \le p \le \infty$*; a matrix polytope* $\mathscr{A}$ *with an S-dominant vertex* $\mathbf{A}_{k^*}$*; a diagonal positive-definite matrix* $\mathbf{D} \succ 0$.

Matrix $\mathbf{D}$ *is a solution to* (10-S) *(i.e.* $\mathbf{D}$ *satisfies the Stein-type inequality relative to the p-norm associated with the polytope* $\mathscr{A}$*) if and only if*

$$\left\|\mathbf{D}^{-1}\mathbf{A}_{k^*}\mathbf{D}\right\|_p < 1 \tag{12-S}$$

(i.e. $\mathbf{D}$ *satisfies the Stein-type inequality relative to the p-norm associated with the S-dominant vertex* $\mathbf{A}_{k^*}$*).*

Proof. See the Appendix. □

Theorem 2 (H). *Let us consider:* $1 \le p \le \infty$*; a matrix polytope* $\mathscr{A}$ *with an H-dominant vertex* $\mathbf{A}_{k^*}$*; a diagonal positive-definite matrix* $\mathbf{D} \succ 0$.

Matrix $\mathbf{D}$ *is a solution to* (10-H) *(i.e.* $\mathbf{D}$ *satisfies the Lyapunov-type inequality relative to the p-norm associated with the polytope* $\mathscr{A}$*) if and only if*

$$\mu_p\left(\mathbf{D}^{-1}\mathbf{A}_{k^*}\mathbf{D}\right) < 0 \tag{12-H}$$

(i.e. $\mathbf{D}$ *satisfies the Lyapunov-type inequality relative to the p-norm associated with the H-dominant vertex* $\mathbf{A}_{k^*}$*).*

Proof. See the Appendix. □

Remark 4. Let $1 \le p \le \infty$. Whenever $\mathbf{A}_{k^*}$ is SS (resp., HS) diagonal matrices $\mathbf{D} \succ 0$ that satisfy (12-S) (resp., (12-H) < ?cmd? >) can be built along the lines of Lemma 3 and Remark 3 in [28]. Further comments are available in the next section that discloses the role of $\mathbf{D} \succ 0$ in the dynamics of a polytopic system of form (3-S) (resp., (3-H)).

2.4. Stability Margins. The S-dominant (resp., H-dominant) vertex $\mathbf{A}_{k^*}$ of a matrix polytope $\mathscr{A}$ also allows one to develop a robustness analysis for SS and SDS_p of $\mathscr{A}$ defined by (1) and (11-S-1) or (11-S-2) (resp., HS and HDS_p of $\mathscr{A}$ defined by (1) and (11-H)).

Definition 4 (S). Let $\mathscr{A}$ be a matrix polytope with an S-dominant vertex $\mathbf{A}_{k^*}$.

(a) If $\mathbf{A}_{k^*}$ is SS, then

$$\rho_{\mathrm{SS}}\left(\mathbf{A}_{k^*}\right) = 1 - \max_{i=1,\dots,n}\left|\lambda_i\left(\mathbf{A}_{k^*}\right)\right| = 1 - \lambda_{\max}\left(\mathbf{A}_{k^*}\right) \tag{13-S}$$

is called the SS margin *of* $\mathbf{A}_{k^*}$.

(b) If $\mathscr{A}$ is SS, then

$$\rho_{\mathrm{SS}}(\mathscr{A}) = 1 - \max_{A\in\mathscr{A}}\max_{i=1,\dots,n}\left|\lambda_i(A)\right| \tag{14-S}$$

is called the SS margin *of* $\mathscr{A}$.

(c) Let $1 \le p \le \infty$. If $\mathscr{A}$ is SDS_p, then

$$\rho_{\mathrm{SDS}_p}(\mathscr{A}) = 1 - \inf_{\substack{\mathbf{D}\succ 0\\ \text{diagonal}}}\max_{\mathbf{A}\in\mathscr{A}}\left\|\mathbf{D}^{-1}\mathbf{A}\mathbf{D}\right\|_p \tag{15-S}$$

is called the SDS_p margin of $\mathscr{A}$.

Definition 4 (H). Let $\mathscr{A}$ be a matrix polytope with an H-dominant vertex $\mathbf{A}_{k^*}$.

(a) If $\mathbf{A}_{k^*}$ is HS, then

$$\rho_{\mathrm{HS}}\left(\mathbf{A}_{k^*}\right) = \left|\max_{i=1,\dots,n}\mathrm{Re}\left\{\lambda_i\left(\mathbf{A}_{k^*}\right)\right\}\right| = \left|\lambda_{\max}\left(\mathbf{A}_{k^*}\right)\right| \tag{13-H}$$

is called the HS margin *of* $\mathbf{A}_{k^*}$.

(b) If $\mathscr{A}$ is HS, then

$$\rho_{\mathrm{HS}}(\mathscr{A}) = \left|\max_{\mathbf{A}\in\mathscr{A}}\max_{i=1,\dots,n}\mathrm{Re}\left\{\lambda_i(\mathbf{A})\right\}\right| \tag{14-H}$$

is called the HS margin *of* $\mathscr{A}$.

(c) Let $1 \le p \le \infty$. If $\mathscr{A}$ is HDS_p, then

$$\rho_{\mathrm{HDS}_p}(\mathscr{A}) = \left|\inf_{\substack{\mathbf{D}\succ 0\\ \text{diagonal}}}\max_{\mathbf{A}\in\mathscr{A}}\mu_p\left(\mathbf{D}^{-1}\mathbf{A}\mathbf{D}\right)\right| \tag{15-H}$$

is called the HDS_p margin of $\mathscr{A}$.

Theorem 3 (S). *Let $\mathscr{A}$ be a matrix polytope with an S-dominant vertex $\mathbf{A}_{k^*}$. For any p, $1 \leq p \leq \infty$, the following equalities hold:*

$$\rho_{SDS_p}(\mathscr{A}) = \rho_{SS}(\mathscr{A}) = \rho_{SS}(\mathbf{A}_{k^*}). \tag{16-S}$$

Proof. See the Appendix. □

Theorem 3 (H). *Let $\mathscr{A}$ be a matrix polytope with an H-dominant vertex $\mathbf{A}_{k^*}$. For any p, $1 \leq p \leq \infty$, the following equalities hold:*

$$\rho_{HDS_p}(\mathscr{A}) = \rho_{HS}(\mathscr{A}) = \rho_{HS}(\mathbf{A}_{k^*}). \tag{16-H}$$

Proof. See the Appendix. □

Remark 5. (i) For each stability property of the polytope $\mathscr{A}$ discussed in Section 2.2, the corresponding margin (also called "degree" in the control-engineering literature) quantifies the distance between a matrix $\mathbf{A} \in \mathscr{A}$ representing the "worst case" relative to that property and the "limit situation" where that property is generically lost for an arbitrary matrix. Theorem 3(S) (resp., Theorem 3(H)) shows that the "worst case" of $\mathscr{A}$ relative to SS and SDS_p (resp., HS and HDS_p) is defined by the S-dominant (resp., H-dominant) vertex.

(ii) For $p = 1$, Theorem 3(S) (resp., Theorem 3(H)) ensure the existence of a diagonal matrix $\mathbf{D} \succ 0$ such that the union for all $\mathbf{A} \in \mathscr{A}$ of the generalized Gershgorin's disks written for columns $\bigcup_{\mathbf{A}\in\mathscr{A}} \bigcup_{j=1}^{n} G_j^c(\mathbf{D}^{-1}\mathbf{A}\mathbf{D})$ is located in the region of the complex plane defined by $|z| \leq 1 - \rho_{\mathrm{SS}}(\mathbf{A}_{k^*})$ (resp., $\operatorname{Re} z \leq -\rho_{\mathrm{SS}}(\mathbf{A}_{k^*})$). The same location also corresponds to the union for all $\mathbf{A} \in \mathscr{A}$ of the generalized Gershgorin's disks written for rows $\bigcup_{\mathbf{A}\in\mathscr{A}} \bigcup_{i=1}^{n} G_i^r(\mathbf{D}^{-1}\mathbf{A}\mathbf{D})$, where the existence of the diagonal matrix $\mathbf{D} \succ 0$ is guaranteed by Theorem 3(S) (resp., Theorem 3(H)) with $p = \infty$. Obviously, the region $|z| \leq 1 - \rho_{\mathrm{SS}}(\mathbf{A}_{k^*})$ (resp., $\operatorname{Re} z \leq -\rho_{\mathrm{SS}}(\mathbf{A}_{k^*})$) refines the condition formulated by Theorem 1(S)(v)-(vi) (resp., Theorem 1(H)(v)-(vi)) for the location of the generalized Gershgorin's disks.

(iii) The equality (16-S) (resp., (16-H)) plays an important role in the characterization of the dynamic properties exhibited by the polytopic system (3-S) (resp., (3-H)). Further details on this role are available in Remark 6 of the next section.

(iv) For an arbitrary polytope $\mathscr{A}$ (without a dominant vertex), equality (16-S) (resp., (16-H)) does not hold true, in general. If, for a given p, $1 \leq p \leq \infty$, $\mathscr{A}$ is SDS_p (resp., HDS_p) then $\rho_{\mathrm{SDS}_p}(\mathscr{A}) \leq \rho_{\mathrm{SS}}(\mathscr{A})$ (resp., $\rho_{\mathrm{HDS}_p}(\mathscr{A}) \leq \rho_{\mathrm{HS}}(\mathscr{A})$), fact which was anticipated by Remark 2(iii) in general terms, without using this specific language of "stability margins." Moreover, if for given $p_1 \neq p_2$, $1 \leq p_1, p_2 \leq \infty$, $\mathscr{A}$ is diagonally stable relative to both p_1- and p_2-norm, then we may have $\rho_{\mathrm{SDS}_{p_1}}(\mathscr{A}) \neq \rho_{\mathrm{SDS}_{p_2}}(\mathscr{A})$ (resp., $\rho_{\mathrm{HDS}_{p_1}}(\mathscr{A}) \neq \rho_{\mathrm{HDS}_{p_2}}(\mathscr{A})$), as already suggested by our recent paper [30].

2.5. Particular Case of Interval Matrices with a Dominant Vertex. Theorems 1(S), 2(S), and 3(S) generalize the results reported in [27, Lemma 3.4.18], [29] for SS and SDS_p of interval matrices of form (2) with $\mathbf{A}_0$ or $-\mathbf{A}_0$ nonnegative, because these two types of interval matrices represent particular cases of matrix polytopes with an S-dominant vertex defined by inequalities (11-S-1) or (11-S-2).

Similarly, Theorems 1(H), 2(H), and 3(H) generalize results reported in [29] for HS and HDS_p of interval matrices of form (2) with $\mathbf{A}_0$ essentially nonnegative, since such interval matrices represent a particular case of matrix polytopes with an H-dominant vertex defined by inequalities (11-H).

3. Results on Polytopic Systems

The current section shows that a polytopic system defined by a matrix polytope with a dominant vertex may exhibit dynamical properties stronger than the standard concept of asymptotic stability; these dynamical properties are correlated, by equivalence, to the algebraic properties of the dominant vertex.

Theorem 4 (S). *Let us consider: $1 \leq p \leq \infty$; a discrete-time polytopic system of form* (3-S) *where polytope $\mathscr{A}$ has an S-dominant vertex $\mathbf{A}_{k^*}$; a positive-definite diagonal matrix $\mathbf{D} \succ 0$; a constant r satisfying $0 < r < 1$.*

The following statements are equivalent:

(i)

$$\left\| \mathbf{D}^{-1}\mathbf{A}_{k^*}\mathbf{D} \right\|_p \leq r. \tag{17-S}$$

(ii) *For the polytopic system* (3-S), *the functions*

$$\mathscr{V}_p^{\alpha} : \mathbb{R}^n \longrightarrow \mathbb{R}_+, \qquad \mathscr{V}_p^{\alpha}(\mathbf{x}) = \frac{1}{\alpha}\left\| \mathbf{D}^{-1}\mathbf{x} \right\|_p, \quad \alpha > 0 \tag{18-S}$$

are strong diagonal Lyapunov functions, with the decreasing rate r, i.e.

$$\begin{gathered} \forall \mathbf{x}(t) \text{ is a trajectory of system (3-S)}, \\ \forall t \in \mathbb{Z}_+,\ t \geq t_0 : \mathscr{V}_p^{\alpha}(\mathbf{x}(t+1)) \leq r\mathscr{V}_p^{\alpha}(\mathbf{x}(t)). \end{gathered} \tag{19-S}$$

(iii) *The contractive sets*

$$\begin{gathered} \mathscr{X}_p^{\alpha}(t; t_0) = \left\{ \mathbf{x} \in \mathbb{R}^n \mid \left\| \mathbf{D}^{-1}\mathbf{x} \right\|_p \leq \alpha r^{(t-t_0)} \right\}, \\ t, t_0 \in \mathbb{Z}_+,\ t \geq t_0,\ \alpha > 0 \end{gathered} \tag{20-S}$$

are invariant with respect to the state-space trajectories (solutions) of the polytopic system (3-S), *i.e.*

$$\begin{gathered} \forall \alpha > 0,\ \forall t, t_0 \in \mathbb{Z}_+,\ t \geq t_0,\ \forall \mathbf{x}_0 = \mathbf{x}(t_0) \in \mathbb{R}^n : \\ \left\| \mathbf{D}^{-1}\mathbf{x}_0 \right\|_p \leq \alpha \Longrightarrow \left\| \mathbf{D}^{-1}\mathbf{x}(t; t_0, \mathbf{x}_0) \right\|_p \leq \alpha r^{(t-t_0)}. \end{gathered} \tag{21-S}$$

Proof. See the Appendix. □

Theorem 4 (H). *Let us consider: $1 \leq p \leq \infty$; a continuous-time polytopic system of form* (3-H) *where polytope $\mathscr{A}$ has an H-dominant vertex $\mathbf{A}_{k^*}$; a positive-definite diagonal matrix $\mathbf{D} \succ 0$; a constant r satisfying $r < 0$.*

The following statements are equivalent:

(i)

$$\mu_p\left(\mathbf{D}^{-1}\mathbf{A}_{k^*}\mathbf{D}\right) \le r. \quad (17\text{-H})$$

(ii) *For the polytopic system* (3-H), *the functions*

$$\mathscr{V}_p^{\alpha} : \mathbb{R}^n \longrightarrow \mathbb{R}_+, \qquad \mathscr{V}_p^{\alpha}(\mathbf{x}) = \frac{1}{\alpha}\left\|\mathbf{D}^{-1}\mathbf{x}\right\|_p, \quad \alpha > 0 \quad (18\text{-H})$$

are strong diagonal Lyapunov functions, with the decreasing rate r, i.e.

$$\forall \mathbf{x}(t) \text{ is a trajectory of system (3-H)}, \ \forall t \in \mathbb{R}_+, \ t \ge t_0 :$$

$$D^+\mathscr{V}_p^{\alpha}(\mathbf{x}(t)) = \lim_{h\downarrow 0}\frac{1}{h}\left[\mathscr{V}_p^{\alpha}(\mathbf{x}(t+h)) - \mathscr{V}_p^{\alpha}(\mathbf{x}(t))\right] \le r\mathscr{V}_p^{\alpha}(\mathbf{x}(t)). \quad (19\text{-H})$$

(iii) *The contractive sets*

$$\mathscr{X}_p^{\alpha}(t;t_0) = \left\{\mathbf{x} \in \mathbb{R}^n \mid \left\|\mathbf{D}^{-1}\mathbf{x}\right\|_p \le \alpha e^{r(t-t_0)}\right\}, \quad t, t_0 \in \mathbb{R}_+, \ t \ge t_0, \ \alpha > 0 \quad (20\text{-H})$$

are invariant with respect to the state-space trajectories (solutions) of the polytopic system (3-H), *i.e.*

$$\forall \alpha > 0, \ \forall t, t_0 \in \mathbb{R}_+, \ t \ge t_0, \ \forall \mathbf{x}_0 = \mathbf{x}(t_0) \in \mathbb{R}^n : \quad \left\|\mathbf{D}^{-1}\mathbf{x}_0\right\|_p \le \alpha \Longrightarrow \left\|\mathbf{D}^{-1}\mathbf{x}(t;t_0,\mathbf{x}_0)\right\| \le \alpha e^{r(t-t_0)}. \quad (21\text{-H})$$

Proof. See the Appendix. □

Remark 6. (i) Let $1 \le p \le \infty$. The exploration of the dynamical properties of a polytopic system via Theorem 4(S) (resp., Theorem 4(H)) outlines the importance of the concrete value $0 < r < 1$ (resp., $r < 0$) in the right hand side of inequality (17-S) (resp., (17-H)). This concrete value r does not appear explicitly in the Stein-type inequality (12-S) (resp., Lyapunov-type inequality (12-H)); the existence of a diagonal matrix $\mathbf{D} \succ 0$ that solves inequality (12-S) (resp., (12-H)) represents a necessary and a sufficient condition for the SDS_p (resp., HDS_p) of a matrix polytope $\mathscr{A}$. For a polytopic system (3-S) (resp., (3-H)) a complete description of the dynamics implies the knowledge of pairs formed by r and $\mathbf{D}$ that satisfy Theorem 4(S) (resp., Theorem 4(H)).

(ii) The constant $0 < r < 1$ (resp., $r < 0$) in Theorem 4(S) (resp., Theorem 4(H)) represents a decreasing rate for the diagonal Lyapunov functions and for the contractive invariant sets. We are going to prove that for any p, $1 \le p \le \infty$, the value of the fastest decreasing rate is given by the $\lambda_{\max}(\mathbf{A}_{k^*})$, regardless of the discrete-time or continuous-time nature of the dynamics. Indeed, for $r < \lambda_{\max}(\mathbf{A}_{k^*})$, there exists no diagonal matrix $\mathbf{D} \succ 0$ satisfying inequality (17-S) (resp., (17-H)). For $r \ge \lambda_{\max}(\mathbf{A}_{k^*})$, we use Lemma 3 and Remark 3 in [28] that yield the following discussion on the irreducibility/reducibility of $\mathbf{A}_{k^*}$.

Case 1 (matrix $\mathbf{A}_{k^*}$ is irreducible). (The definition of an irreducible matrix is introduced in Section 1.3.) Denote by $\mathbf{v} = [v_1 \cdots v_n]^T > 0$ and $\mathbf{w} = [w_1 \cdots w_n]^T > 0$ its right and left Perron eigenvectors, respectively. Given p, $1 \le p \le \infty$, we construct the diagonal matrix $\mathbf{D}_p = \mathrm{diag}\{v_1^{1/q}/w_1^{1/p}, \ldots, v_n^{1/q}/w_n^{1/p}\} \succ 0$, where (i) $(1/p)+(1/q) = 1$ if $1 < p < \infty$; (ii) $1/p = 1$, $1/q = 0$ if $p = 1$; (iii) $1/p = 0$, $1/q = 1$ if $p = \infty$. Matrix $\mathbf{D}_p \succ 0$ fulfills the following equality:

(i) $\|\mathbf{D}_p^{-1}\mathbf{A}_{k^*}\mathbf{D}_p\|_p = \lambda_{\max}(\mathbf{A}_{k^*})$ if $\mathbf{A}_{k^*}$ is nonnegative, i.e. $\mathbf{A}_{k^*}$ is defined by (11-S-1);

(ii) $\|\mathbf{D}_p^{-1}\mathbf{A}_{k^*}\mathbf{D}_p\|_p = \lambda_{\max}(-\mathbf{A}_{k^*})$ if $\mathbf{A}_{k^*}$ is nonpositive, i.e. $\mathbf{A}_{k^*}$ is defined by (11-S-2);

(iii) $\mu_p(\mathbf{D}_p^{-1}\mathbf{A}_{k^*}\mathbf{D}_p) = \lambda_{\max}(\mathbf{A}_{k^*})$ if $\mathbf{A}_{k^*}$ is essentially nonnegative, i.e. $\mathbf{A}_{k^*}$ is defined by (11-H).

Case 2 (matrix $\mathbf{A}_{k^*}$ is reducible). For any $r > \lambda_{\max}(\mathbf{A}_{k^*})$, there exists $\varepsilon > 0$ such that $\lambda_{\max}(\mathbf{A}_{k^*}) < \lambda_{\max}(\mathbf{A}_{k^*} + \varepsilon\mathbf{J}) < r$, where $\mathbf{J}$ is the n by n matrix with all its entries 1. We apply the procedure presented by Case 1 to the irreducible matrix $\mathbf{A}_{k^*} + \varepsilon\mathbf{J}$, and the resulting diagonal matrix $\mathbf{D}_p \succ 0$ fulfills $\|\mathbf{D}_p^{-1}(\mathbf{A}_{k^*}+\varepsilon\mathbf{J})\mathbf{D}_p\|_p = \lambda_{\max}(\mathbf{A}_{k^*}+\varepsilon\mathbf{J}) < r$, $\mu_p(\mathbf{D}_p^{-1}(\mathbf{A}_{k^*}+\varepsilon\mathbf{J})\mathbf{D}_p) = \lambda_{\max}(\mathbf{A}_{k^*} + \varepsilon\mathbf{J}) < r$, respectively. On the other hand, we have the norm inequality $\|\mathbf{D}_p^{-1}\mathbf{A}_{k^*}\mathbf{D}_p\|_p \le \|\mathbf{D}_p^{-1}(\mathbf{A}_{k^*} + \varepsilon\mathbf{J})\mathbf{D}_p\|_p$, the measure inequality $\mu_p(\mathbf{D}_p^{-1}\mathbf{A}_{k^*}\mathbf{D}_p) \le \mu_p(\mathbf{D}_p^{-1}(\mathbf{A}_{k^*} + \varepsilon\mathbf{J})\mathbf{D}_p)$, respectively. Subsequently, we obtain $\lambda_{\max}(\mathbf{A}_{k^*}) < \|\mathbf{D}_p^{-1}\mathbf{A}_{k^*}\mathbf{D}_p\|_p < r$, $\lambda_{\max}(\mathbf{A}_{k^*}) < \mu_p(\mathbf{D}_p^{-1}\mathbf{A}_{k^*}\mathbf{D}_p) < r$, respectively.

Thus, for any p, $1 \le p \le \infty$, we can construct a diagonal matrix $\mathbf{D}_p \succ 0$ such that inequality (17-S) (resp., (17-H)) is fulfilled with $r = \lambda_{\max}(\mathbf{A}_{k^*})$—for $\mathbf{A}_{k^*}$ irreducible and with $r > \lambda_{\max}(\mathbf{A}_{k^*})$ but as close to $\lambda_{\max}(\mathbf{A}_{k^*})$ as we want—for $\mathbf{A}_{k^*}$ reducible.

(iii) The fastest decreasing rate of the diagonal Lyapunov functions and of the contractive invariant sets can be expressed in terms of the stability margins (discussed in Section 2.4), as $1 - \rho_{\mathrm{SS}}(\mathbf{A}_{k^*})$ for the discrete-time case and $-\rho_{\mathrm{SS}}(\mathbf{A}_{k^*})$ for the continuous-time case. This point of view shows that the stability margins provide an algebraic characterization for the polytope $\mathscr{A}$ and, concomitantly, allow the evaluation of the dynamical properties of the polytopic system (3-S) (resp., (3-H)).

(iv) For an arbitrary polytope $\mathscr{A}$ (without a dominant vertex), the fastest decreasing rate may depend on the p-norm that defines the Lyapunov function and the invariant sets (if exist). If, for a given p, $1 \le p \le \infty$, $\mathscr{A}$ is SDS_p (resp., HDS_p) then the fastest decreasing rate corresponding to the p-norm of the polytopic system can be expressed in terms of the stability margins as $1 - \rho_{\mathrm{SDS}_p}(\mathscr{A})$ (resp., $-\rho_{\mathrm{HDS}_p}(\mathscr{A})$). The fastest decreasing rate corresponding to the p-norm can be fairly estimated by a computational procedure based on a bisection method presented in [30].

4. Illustrative Examples

This section presents two examples that illustrate the usefulness of the theoretical results developed by our work. Example 1 explores (i) the stability of a matrix polytope with an S-dominant vertex of form (1) and (11-S-1); (ii) the dynamical properties of the discrete-time polytopic system of form (3-S) defined by the considered polytope. Example 2 explores (i) the stability of a matrix polytope with an H-dominant vertex of form (1) and (11-H); (ii) the dynamical properties of the continuous-time polytopic system of form (3-H) defined by the considered polytope.

Example 1. It is adapted from [27]. Let

$$\begin{aligned}\mathcal{S}^{[-1,1]} &= \left\{\mathbf{S} \in \mathbb{R}^{n\times n} \mid \mathbf{S} = \operatorname{diag}\left\{s_1,\dots,s_n\right\}, \left|s_i\right| \le 1, i=1,\dots,n\right\} \\ &= \left\{\mathbf{S} \in \mathbb{R}^{n\times n} \mid |\mathbf{S}| \le \mathbf{I}_n\right\}\end{aligned} \tag{22}$$

be the set of diagonal matrices whose elements are subunitary. $\mathcal{S}^{[-1,1]}$ is the convex hull generated by the set of 2^n vertices $\mathcal{S}_{\text{sgn}} = \{\mathbf{S} \in \mathbb{R}^{n\times n} \mid |\mathbf{S}| = \mathbf{I}_n\}$, also called the class of signature matrices.

Given a nonnegative matrix $\mathbf{A}^* \in \mathbb{R}^{n\times n}$, the set

$$\mathcal{A} = \mathcal{R}\left(\mathbf{A}^*\right) = \left\{\mathbf{A} = \mathbf{A}^*\mathbf{S} \mid \mathbf{S} \in \mathcal{S}^{[-1,1]}\right\} \tag{23}$$

is a matrix polytope of form (1) with the vertices $\mathbf{A}_i = \mathbf{A}^*\mathbf{S}_i$, where $\mathbf{S}_i \in \mathcal{S}_{\text{sgn}}$, $i = 1,\dots,2^n$. The set $\mathcal{A} = \mathcal{R}(\mathbf{A}^*)$ defined by (23) is a matrix polytope with an S-dominant vertex because matrix $\mathbf{A}^*$ is a vertex that satisfies condition (11-S-1).

Theorem 1(S) shows that the Schur stability of $\mathbf{A}^*$ is a necessary and a sufficient condition for the Schur diagonal stability of the polytope $\mathcal{A} = \mathcal{R}(\mathbf{A}^*)$ relative to any p-norm, $1 \le p \le \infty$. According to Theorem 2(S), a diagonal positive-definite matrix $\mathbf{D}_p \succ 0$ satisfies the Stein-type inequality relative to the p-norm associated with $\mathcal{A} = \mathcal{R}(\mathbf{A}^*)$ if and only if $\mathbf{D}_p \succ 0$ satisfies the Stein-type inequality relative to the p-norm associated with $\mathbf{A}^*$. This property of $\mathcal{A} = \mathcal{R}(\mathbf{A}^*)$ is guaranteed for any p-norm by Theorem 2(S), whereas Proposition 2.5.8 in [27] can guarantee only the particular case corresponding to $p = 2$.

If $\mathbf{A}^*$ is Schur stable, then the eigenvalue $\lambda_{\max}(\mathbf{A}^*)$ allows one to investigate the following properties: (i) for the polytope $\mathcal{A} = \mathcal{R}(\mathbf{A}^*)$, the SDS_p margins are given by relation (16-S) in Theorem 3(S) i.e. $\rho_{\text{SDS}_p}(\mathcal{R}(\mathbf{A}^*)) = 1 - \lambda_{\max}(\mathbf{A}^*)$, for any p, $1 \le p \le \infty$; (ii) for the discrete-time polytopic system of form (3-S) defined by $\mathcal{A} = \mathcal{R}(\mathbf{A}^*)$, regardless of the p-norm considered in Theorem 4(S) for the Lyapunov function (18-S), and for the contractive invariant sets (20-S), the fastest decreasing rate is $\lambda_{\max}(\mathbf{A}^*)$ if $\mathbf{A}^*$ is irreducible and arbitrarily close to $\lambda_{\max}(\mathbf{A}^*)$ if $\mathbf{A}^*$ is reducible—as per Remark 6(ii).

Finally, we notice that the nonpositive matrix $\mathbf{A}^{**} = -\mathbf{A}^*$ is also a vertex of the polytope $\mathcal{A} = \mathcal{R}(\mathbf{A}^*)$ and it satisfies the dominance condition (11-S-2). This means that the polytope $\mathcal{A} = \mathcal{R}(\mathbf{A}^*)$ fits in the particular context commented by Remark 3(iii). It is obvious that the analysis presented by the current example is complete, in the sense that the vertex $\mathbf{A}^{**} \le 0$ brings no supplementary information (since the matrix $-\mathbf{A}^{**}$ is nonnegative, there exists $\lambda_{\max}(-\mathbf{A}^{**}) = \lambda_{\max}(\mathbf{A}^*)$, and for all p, $1 \le p \le \infty$, the equality $\|\mathbf{D}^{-1}\mathbf{A}^{**}\mathbf{D}\|_p = \|\mathbf{D}^{-1}\mathbf{A}^*\mathbf{D}\|_p$ holds true).

The above approach applies *mutatis-mutandis* to the investigation of the matrix polytope $\mathcal{L}(\mathbf{A}^*) = \{\mathbf{A} = \mathbf{S}\mathbf{A}^* \mid \mathbf{S} \in \mathcal{S}^{[-1,1]}\}$. In this case, Theorem 2(S) generalizes for $1 \le p \le \infty$ the result that can be obtained when $p = 2$ by using Proposition 2.5.9 in [27] for matrix $\mathbf{A}^*$ and polytope $\mathcal{L}(\mathbf{A}^*)$.

Example 2. Let us consider the interval matrix [14]:

$$\begin{gathered}\mathcal{A} = \left\{\mathbf{A} \in \mathbb{R}^{2\times 2} \mid \mathbf{A} = \mathbf{A}_0 + \Delta,\ |\Delta| \le \mathbf{R}\right\}, \\ \mathbf{A}_0 = \begin{bmatrix} -3.8 & 1.6 \\ 0.6 & -4.2 \end{bmatrix}, \qquad \mathbf{R} = 0.17\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},\end{gathered} \tag{24}$$

that is a matrix polytope with $K = 2^4$ vertices:

$$\mathbf{A}_k = \mathbf{A}_0 + \Delta_k, \quad k = 1,\dots,16, \text{ where } \Delta_k = 0.17\begin{bmatrix} \pm 1 & \pm 1 \\ \pm 1 & \pm 1 \end{bmatrix}. \tag{25}$$

$\mathcal{A}$ has an H-dominant vertex

$$\mathbf{A}^* = \mathbf{A}_0 + \mathbf{R} = \begin{bmatrix} -3.63 & 1.77 \\ 0.77 & -4.03 \end{bmatrix}, \tag{26}$$

which satisfies inequalities (11-H). Note that $\mathbf{A}^*$ is essentially positive and has the Perron-Frobenius eigenvalue $\lambda_{\max}(\mathbf{A}^*) = -2.6456$. Theorem 1(H) shows that the Hurwitz stability of $\mathbf{A}^*$ ensures the Hurwitz diagonal stability of the polytope $\mathcal{A}$ relative to any p-norm, $1 \le p \le \infty$.

The stability margins of the polytope $\mathcal{A}$ are given by relation (16-H) in Theorem 2, i.e. $\rho_{\text{HDS}_p}(\mathcal{A}) = \rho_{\text{HS}}(\mathcal{A}) = \rho_{\text{HS}}(\mathbf{A}^*) = |\lambda_{\max}(\mathbf{A}^*)| = 2.6456$, for any p, $1 \le p \le \infty$.

For the qualitative analysis of the continuous-time polytopic system defined by (3-H) and (24), we can apply Theorem 4(H). Remark 6(ii) shows that for any p, $1 \le p \le \infty$, the fastest decreasing rate for the diagonal Lyapunov functions and for the contractive invariant sets is exactly $\lambda_{\max}(\mathbf{A}^*) = -2.6456$, since $\mathbf{A}^*$ is irreducible. We apply Case 1 of the procedure presented in Remark 6(ii) and relying on the right and left Perron eigenvectors of $\mathbf{A}^*$ ($\mathbf{v} = [1 \quad 0.5562]^T$ and $\mathbf{w} = [0.7822 \quad 1]^T$), we construct the diagonal matrices $\mathbf{D}_p \succ 0$ corresponding to the fastest decreasing rate. For $p \in \{1, 2, \infty\}$, these diagonal matrices are $\mathbf{D}_1 = \operatorname{diag}\{1.2785, 1\}$, $\mathbf{D}_2 = \operatorname{diag}\{1.1307, 0.7458\}$, and $\mathbf{D}_\infty = \operatorname{diag}\{1, 0.5562\}$, and they satisfy Theorem 4(H) with $r = \lambda_{\max}(\mathbf{A}^*) = -2.6456$.

Note that all the above results remain valid if instead of $\mathcal{A}$ defined by (24), we consider the matrix polytope

$$\begin{gathered}\mathcal{A}' = \left\{\mathbf{A} \in \mathbb{R}^{2\times 2} \mid \mathbf{A}^- \le \mathbf{A} \le \mathbf{A}^*\right\}, \\ \mathbf{A}^* = \begin{bmatrix} -3.63 & 1.77 \\ 0.77 & -4.03 \end{bmatrix}, \qquad \mathbf{A}^- = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \\ a \le -3.63,\ |b| \le 1.77,\ |c| \le 0.77,\ d \le -4.03,\end{gathered} \tag{27}$$

which has the same dominant vertex $\mathbf{A}^*$ (26).

5. Conclusions

The paper provides analysis instruments for the stability of matrix polytopes with a dominant vertex, as well as for the dynamics of discrete- and continuous-time uncertain systems defined by such polytopes. These analysis instruments are formulated as necessary and sufficient conditions exclusively based on the characteristics of the dominant vertex. Thus, the dominant vertex represents the only test matrix used for studying the following properties of a matrix polytope and its associated dynamical system: (i) Schur (resp., Hurwitz) stability (including the computation of the corresponding margin); (ii) Schur (resp., Hurwitz) diagonal stability relative to a p-norm (including the computation of the corresponding margin); (iii) existence of diagonal positive-definite matrices solving the Stein-type (resp., Lyapunov-type) inequalities relative to a p-norm; (iv) existence of diagonal-type Lyapunov functions and contractive invariant sets defined by a p-norm and decreasing with a given rate. A global result of our work is the proof that stability and diagonal stability relative to an arbitrary p-norm are equivalent for the considered class of matrix polytopes (fact which is not true for general matrix polytopes).

Appendix

Proof of Theorem 1(S). (i) ⇒(ii).

It results from the following:

$$\begin{aligned}\forall \mathbf{A} \in \mathscr{A} : \mathbf{A} = \sum_{k=1}^{K} \gamma_k \mathbf{A}_k &\Longrightarrow \mathbf{A} \le \mathbf{A}^S \le \sum_{k=1}^{K} \gamma_k \mathbf{A}_k^S \\ &\le \sum_{k=1}^{K} \gamma_k \mathbf{A}_{k^*} = \left(\sum_{k=1}^{K} \gamma_k\right) \mathbf{A}_{k^*} = \mathbf{A}_{k^*} \\ &\Longrightarrow \max_{i=1,\ldots,n} \left|\lambda_i\left(\mathbf{A}\right)\right| \le \lambda_{\max}\left(\mathbf{A}_{k^*}\right) < 1,\end{aligned} \tag{A.1}$$

since $\mathbf{A}^S$, $\mathbf{A}_{k^*}$ are nonnegative and we can apply Theorem 8.1.18 and Corollary 8.1.19 in [38].

(ii) ⇒(i) It is obvious, because $\mathbf{A}_{k^*} \in \mathscr{A}$.

(i) ⇒(iv) Let $1 \le p \le \infty$. Lemma 3 in [28] ensures the existence of a diagonal matrix $\mathbf{D} \succ 0$ such that $\lambda_{\max}(\mathbf{A}_{k^*}) = \lambda_{\max}(\mathbf{D}^{-1}\mathbf{A}_{k^*}\mathbf{D}) \le \|\mathbf{D}^{-1}\mathbf{A}_{k^*}\mathbf{D}\|_p < 1$. On the other hand, we have the implication

$$\begin{aligned}\forall \mathbf{A} \in \mathscr{A} : \mathbf{A} = \sum_{k=1}^{K} \gamma_k \mathbf{A}_k &\Longrightarrow \left\|\mathbf{D}^{-1}\mathbf{A}\mathbf{D}\right\|_p \le \left\|\mathbf{D}^{-1}\mathbf{A}^S\mathbf{D}\right\|_p \\ &\le \sum_{k=1}^{K} \gamma_k \left\|\mathbf{D}^{-1}\mathbf{A}_k^S\mathbf{D}\right\|_p \le \sum_{k=1}^{K} \gamma_k \left\|\mathbf{D}^{-1}\mathbf{A}_{k^*}\mathbf{D}\right\|_p \\ &= \left(\sum_{k=1}^{K} \gamma_k\right) \left\|\mathbf{D}^{-1}\mathbf{A}_{k^*}\mathbf{D}\right\|_p = \left\|\mathbf{D}^{-1}\mathbf{A}_{k^*}\mathbf{D}\right\|_p,\end{aligned} \tag{A.2}$$

as per Lemma 4 in [28]. We conclude that for all $\mathbf{A} \in \mathscr{A}$: $\|\mathbf{D}^{-1}\mathbf{A}\mathbf{D}\|_p < 1$.

(iv) ⇒(iii) It is obvious.

(iii) ⇒(ii) It follows from Remark 2(iii).

(v) ⇔(iii) with $p = 1$. It results from the equivalence

$$\begin{aligned}&\forall \mathbf{A} \in \mathscr{A} : G_j^c\left(\mathbf{D}^{-1}\mathbf{A}\mathbf{D}\right) \subset \mathbf{C}_S, \quad j = 1, \ldots, n \\ &\Longleftrightarrow -1 < a_{jj} - \sum_{i=1, i \ne j}^{n} \frac{d_j}{d_i} \left|a_{ij}\right|, \\ &a_{jj} + \sum_{i=1, i \ne j}^{n} \frac{d_j}{d_i} \left|a_{ij}\right| < 1, \quad j = 1, \ldots, n \\ &\Longleftrightarrow \left\|\mathbf{D}^{-1}\mathbf{A}\mathbf{D}\right\|_p < 1.\end{aligned} \tag{A.3}$$

(vi) ⇔(iii) with $p = \infty$. It is similar to (v) ⇔(iii) with $p = 1$. □

Proof of Theorem 1(H). (i) ⇒(ii).

It results from the following:

$$\begin{aligned}\forall \mathbf{A} \in \mathscr{A} : \mathbf{A} = \sum_{k=1}^{K} \gamma_k \mathbf{A}_k &\Longrightarrow \mathbf{A} \le \mathbf{A}^H \le \sum_{k=1}^{K} \gamma_k \mathbf{A}_k^H \\ &\le \sum_{k=1}^{K} \gamma_k \mathbf{A}_{k^*} = \left(\sum_{k=1}^{K} \gamma_k\right) \mathbf{A}_{k^*} = \mathbf{A}_{k^*}.\end{aligned} \tag{A.4}$$

Take $s > 0$ so that $s\mathbf{I} + \mathbf{A}^H \ge 0$. Hence, $|s\mathbf{I} + \mathbf{A}| = s\mathbf{I} + \mathbf{A}^H \le s\mathbf{I} + \mathbf{A}_{k^*}$. By applying Theorem 8.1.18 and Corollary 8.1.19 in [38], we get $s + \max_{i=1,\ldots,n} \mathrm{Re}\{\lambda_i(\mathbf{A})\} \le s + \lambda_{\max}(\mathbf{A}_{k^*})$, which implies that $\max_{i=1,\ldots,n} \mathrm{Re}\{\lambda_i(\mathrm{A})\} \le \lambda_{\max}(\mathbf{A}_{k^*}) < 0$.

(ii) ⇒(i) It is obvious, because $\mathbf{A}_{k^*} \in \mathscr{A}$.

(i) ⇒(iv) Let $1 \le p \le \infty$. Lemma 3 in [28] ensures the existence of a diagonal matrix $\mathbf{D} \succ 0$ such that $\lambda_{\max}(\mathbf{A}_{k^*}) \le \mu_p(\mathbf{D}^{-1}\mathbf{A}_{k^*}\mathbf{D}) < 0$. On the other hand, we have the implication

$$\begin{aligned}\forall \mathbf{A} \in \mathscr{A} : \mathbf{A} = \sum_{k=1}^{K} \gamma_k \mathbf{A}_k &\Longrightarrow \mu_p\left(\mathbf{D}^{-1}\mathbf{A}\mathbf{D}\right) \le \mu_p\left(\mathbf{D}^{-1}\mathbf{A}^H\mathbf{D}\right) \\ &\le \sum_{k=1}^{K} \gamma_k\, \mu_p\left(\mathbf{D}^{-1}\mathbf{A}_k^H\mathbf{D}\right) \le \sum_{k=1}^{K} \gamma_k \mu_p\left(\mathbf{D}^{-1}\mathbf{A}_{k^*}\mathbf{D}\right) \\ &= \left(\sum_{k=1}^{K} \gamma_k\right) \mu_p\left(\mathbf{D}^{-1}\mathbf{A}_{k^*}\mathbf{D}\right) = \mu_p\left(\mathbf{D}^{-1}\mathbf{A}_{k^*}\mathbf{D}\right),\end{aligned} \tag{A.5}$$

as per Lemma 4 in [28]. We conclude that for all $\mathbf{A} \in \mathscr{A}$: $\mu_p(\mathbf{D}^{-1}\mathbf{A}\mathbf{D}) < 0$.

(iv) ⇒(iii) It is obvious.

(iii) ⇒(ii) It follows from Remark 2(iii).

(v) ⇔(iii) with $p = 1$. It results from the equivalence

$$\begin{aligned}&\forall \mathbf{A} \in \mathscr{A} : G_j^c\left(\mathbf{D}^{-1}\mathbf{A}\mathbf{D}\right) \subset \mathbf{C}_H, \quad j = 1, \ldots, n \\ &\Longleftrightarrow a_{jj} + \sum_{i=1, i \ne j}^{n} \frac{d_j}{d_i} \left|a_{ij}\right| < 0, j = 1, \ldots, n \\ &\Longleftrightarrow \mu_1\left(\mathbf{D}^{-1}\mathbf{A}\mathbf{D}\right) < 0.\end{aligned} \tag{A.6}$$

(vi) ⇔(iii) With $p = \infty$. It is similar to (v) ⇔(iii) with $p = 1$. □

Proof of Theorem 2(S). From the proof (i) ⇒(iv) of Theorem 1(S), we can write $\|\mathbf{D}^{-1}\mathbf{A}\mathbf{D}\|_p \leq \|\mathbf{D}^{-1}\mathbf{A}_{k^*}\mathbf{D}\|_p$ for all $\mathbf{A} \in \mathscr{A}$. Thus, if $\mathbf{D} \succ 0$ satisfies inequality (12-S) (i.e. the Stein-type inequality relative to the p-norm associated with the matrix $\mathbf{A}_{k^*}$), then $\mathbf{D} \succ 0$ satisfies inequality (10-S) (i.e. the Stein-type inequality relative to the p-norm associated with the polytope $\mathscr{A}$). The converse part is obvious, since $\mathbf{A}_{k^*} \in \mathscr{A}$. □

Proof of Theorem 2(H). From the proof (i) ⇒(iv) of Theorem 1(H), we can write $\mu_p(\mathbf{D}^{-1}\mathbf{A}\mathbf{D}) \leq \mu_p(\mathbf{D}^{-1}\mathbf{A}_{k^*}\mathbf{D})$ for all $\mathbf{A} \in \mathscr{A}$. Thus, if $\mathbf{D} \succ 0$ satisfies inequality (12-H) (i.e. the Lyapunov-type inequality relative to the p-norm associated with the matrix $\mathbf{A}_{k^*}$), then $\mathbf{D} \succ 0$ satisfies inequality (10-H) (i.e. the Lyapunov-type inequality relative to the p-norm associated with the polytope $\mathscr{A}$). The converse part is obvious, since $\mathbf{A}_{k^*} \in \mathscr{A}$. □

Proof of Theorem 3(S). From the proof (i) ⇒(ii) of Theorem 1(S), we have $\rho_{\mathrm{SS}}(\mathscr{A}) \geq \rho_{\mathrm{SS}}(\mathbf{A}_{k^*})$, and from $\mathbf{A}_{k^*} \in \mathscr{A}$, we get $\rho_{\mathrm{SS}}(\mathscr{A}) \leq \rho_{\mathrm{SS}}(\mathbf{A}_{k^*})$, such that we can conclude that $\rho_{\mathrm{SS}}(\mathscr{A}) = \rho_{\mathrm{SS}}(\mathbf{A}_{k^*})$. Let $1 \leq p \leq \infty$ and $\varepsilon > 0$. Lemma 3 in [28] ensures the existence of a diagonal matrix $\mathbf{D} \succ 0$ such that $\lambda_{\max}(\mathbf{A}_{k^*}) \leq \|\mathbf{D}^{-1}\mathbf{A}_{k^*}\mathbf{D}\|_p < \lambda_{\max}(\mathbf{A}_{k^*}) + \varepsilon$. At the same time, from the proof (i) ⇒(iv) of Theorem 1(S) we have $\|\mathbf{D}^{-1}\mathbf{A}\mathbf{D}\|_p \leq \|\mathbf{D}^{-1}\mathbf{A}_{k^*}\mathbf{D}\|_p$ for all $\mathbf{A} \in \mathscr{A}$. As $\mathbf{A}_{k^*} \in \mathscr{A}$, we may write $\inf_{\substack{\mathbf{D}\succ 0\\ \text{diagonal}}} \max_{\mathbf{A}\in\mathscr{A}}\|\mathbf{D}^{-1}\mathbf{A}\mathbf{D}\|_p = \inf_{\substack{\mathbf{D}\succ 0\\ \text{diagonal}}} \|\mathbf{D}^{-1}\mathbf{A}_{k^*}\mathbf{D}\|_p = \lambda_{\max}(\mathbf{A}_{k^*})$ and, subsequently, $\rho_{\mathrm{SDS}_p}(\mathscr{A}) = \rho_{\mathrm{SS}}(\mathbf{A}_{k^*})$. □

Proof of Theorem 3(H). From the proof (i) ⇒(ii) of Theorem 1(H), we have $\rho_{\mathrm{HS}}(\mathscr{A}) \geq \rho_{\mathrm{HS}}(\mathbf{A}_{k^*})$, and from $\mathbf{A}_{k^*} \in \mathscr{A}$, we get $\rho_{\mathrm{HS}}(\mathscr{A}) \leq \rho_{\mathrm{HS}}(\mathbf{A}_{k^*})$, such that we can conclude that $\rho_{\mathrm{HS}}(\mathscr{A}) = \rho_{\mathrm{HS}}(\mathbf{A}_{k^*})$. Let $1 \leq p \leq \infty$ and $\varepsilon > 0$. Lemma 3 in [28] ensures the existence of a diagonal matrix $\mathbf{D} \succ 0$ such that $\lambda_{\max}(\mathbf{A}_{k^*}) \leq \mu_p(\mathbf{D}^{-1}\mathbf{A}_{k^*}\mathbf{D}) < \lambda_{\max}(\mathbf{A}_{k^*}) + \varepsilon$. At the same time, from the proof (i)⇒(iv) of Theorem 1(H), we have $\mu_p(\mathbf{D}^{-1}\mathbf{A}\mathbf{D}) \leq \mu_p(\mathbf{D}^{-1}\mathbf{A}_{k^*}\mathbf{D})$ for all $\mathbf{A} \in \mathscr{A}$. As $\mathbf{A}_{k^*} \in \mathscr{A}$, we may write $|\inf_{\substack{\mathbf{D}\succ 0\\ \text{diagonal}}} \max_{\mathbf{A}\in\mathscr{A}}\mu_p(\mathbf{D}^{-1}\mathbf{A}\mathbf{D})| = |\inf_{\substack{\mathbf{D}\succ 0\\ \text{diagonal}}} \mu_p(\mathbf{D}^{-1}\mathbf{A}_{k^*}\mathbf{D})| = |\lambda_{\max}(\mathbf{A}_{k^*})|$ and, subsequently, $\rho_{\mathrm{HDS}_p}(\mathscr{A}) = \rho_{\mathrm{HS}}(\mathrm{A}_{k^*})$. □

Proof of Theorem 4(S). Inequality (17-S) is equivalent to the statement $\|\mathbf{D}^{-1}\mathbf{A}\mathbf{D}\|_p \leq r$ for all $\mathbf{A} \in \mathscr{A}$. This results from the inequality $\|\mathbf{D}^{-1}\mathbf{A}\mathbf{D}\|_p \leq \|\mathbf{D}^{-1}\mathbf{A}_{k^*}\mathbf{D}\|_p$, for all $\mathbf{A} \in \mathscr{A}$ that was obtained in the proof for (i) ⇒(iv) of Theorem 1(S). Then we apply Theorem 2 in [28] to all matrices $\mathbf{A}$ in $\mathscr{A}$, and we get the equivalence (17-S) ⇔(19-S) ⇔(21-S). □

Proof of Theorem 4(H). Inequality (17-H) is equivalent to the statement $\mu_p(\mathbf{D}^{-1}\mathbf{A}\mathbf{D}) \leq r$ for all $\mathbf{A} \in \mathscr{A}$. This results from the inequality $\mu_p(\mathbf{D}^{-1}\mathbf{A}\mathbf{D}) \leq \mu_p(\mathbf{D}^{-1}\mathbf{A}_{k^*}\mathbf{D})$, for all $\mathbf{A} \in \mathscr{A}$ that was obtained in the proof for (i) ⇒(iv) of Theorem 1(H). Then, we apply Theorem 2 in [28] to all matrices $\mathbf{A}$ in $\mathscr{A}$, and we get the equivalence (17-H) ⇔(19-H) ⇔(21-H). □

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