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AN EIGENVALUE PROBLEM FOR A QUASILINEAR ELLIPTIC FIELD EQUATION ON $\mathbb{R}^n$

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Abstract. We study the field equation

$$-\Delta u + V(x)u + \varepsilon^r(-\Delta_p u + W'(u)) = \mu u$$

on $\mathbb{R}^n$, with ε positive parameter. The function W is singular in a point and so the configurations are characterized by a topological invariant: the topological charge. By a min-max method, for ε sufficiently small, there exists a finite number of solutions $(\mu(\varepsilon), u(\varepsilon))$ of the eigenvalue problem for any given charge $q \in \mathbb{Z} \setminus \{0\}$.

1. Introduction

In this paper we are concerned with the following nonlinear field equation:

$$(\mathrm{P}_\varepsilon) \qquad -\Delta u + V(x)u + \varepsilon^r\,(-\Delta_p u + W'(u)) = \mu u$$

where u is a function from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$ with $n \geq 3$, ε is a positive parameter and $p, r \in \mathbb{N}$ with $p > n$ and $r > p - n$. Here $\Delta u = (\Delta u_1, \ldots, \Delta u_{n+1})$, being $u = (u_1, \ldots, u_{n+1})$ and Δ the classical Laplacian operator. Moreover, $\Delta_p u$ denotes the $(n+1)$-vector, whose i-th component is given by

$$(\Delta_p u)_i = \nabla \cdot (|\nabla u_i|^{p-2} \nabla u_i).$$

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Finally, V is a real function $V : \mathbb{R}^n \to \mathbb{R}$ and W' is the gradient of a function $W : \mathbb{R}^{n+1} \setminus \{\xi_*\} \to \mathbb{R}$, where ξ_* is a point of $\mathbb{R}^{n+1}$ which for simplicity we choose on the $(n+1)$-th component:

$$\xi_* = (0, \overline{\xi}), \tag{1}$$

with $0 \in \mathbb{R}^n$ and $\overline{\xi} \in \mathbb{R}$, $\overline{\xi} > 0$.

The motivation for considering an eigenvalue problem relative to a nonlinear equation such as (P_ε) needs some explanations. Let us consider the nonlinear Schrödinger equation

$$i\psi_t = -\Delta\psi + V(x)\psi + \varepsilon^r N(\psi) \tag{2}$$

where $N(\psi)$ is a nonlinear differential operator. The standing waves

$$\psi(x,t) = u(x)e^{-i\mu t}$$

of equation (2) are determined by the solutions of the following nonlinear eigenvalue problem

$$-\Delta u + V(x)u + \varepsilon^r N(u) = \mu u \tag{3}$$

provided that

$$N(u(x)e^{-i\mu t}) = e^{-i\mu t} N(u(x)). \tag{4}$$

The nonlinear operator

$$N(u) = -\Delta_p u + W'(u) \tag{5}$$

can be extended to the complex functions in such a way to verify (4).

The choice of the operator (5) is due to the fact that in a paper of 1964 Derrick ([13]) pointed out by a simple rescaling argument that equation

$$-\Delta\varphi + \frac{1}{c^2}\varphi_{tt} + \frac{1}{2}f'(\varphi) = 0,$$

where f' is the gradient of a nonnegative C^1 real function f and the function φ has domain $\mathbb{R}^n$ with $n > 2$, has no nontrivial static solutions:

> "We are faced with the disconcerting fact that no equation of type
>
> $$\Delta\varphi - \frac{1}{c^2}\varphi_{tt} = \frac{1}{2}f'(\varphi)$$
>
> has any time-independent solutions which could reasonably be interpreted as elementary particles."

He presents some conjectures and the first one is to consider higher powers for the derivatives: in fact in [4] (see also [7]) the authors proved that equation

$$-\Delta\varphi - \Delta_p\varphi + W'(\varphi) = 0, \tag{6}$$

(where $\varphi : \mathbb{R}^3 \to \mathbb{R}^4$), has a family $\{\varphi_q\}_{q\in\mathbb{Z}\setminus\{0\}}$ of nontrivial solutions with the energy concentrated around the origin. These solutions are characterized by a topological invariant $\mathrm{ch}(\,\cdot\,)$, called topological charge, which takes integer values (see (9)). More precisely, for every $q \in \mathbb{Z}\setminus\{0\}$, there exists a solution φ_q with $\mathrm{ch}(\varphi_q) = q$. An interesting concentration problem has been studied in [2], where the authors consider some bound states of a field equation like (6) with the addition of a potential depending on a parameter.

Here we study the eigenvalue problem relative to equation (6), with the addition of a potential V; so we look for critical points of a suitable constrained functional and not only minima.

Throughout the paper we always assume these hypotheses on the function $V : \mathbb{R}^n \to \mathbb{R}$:

(V_1) $\lim\limits_{|x|\to\infty} V(x) = \infty$,

(V_2) $V(x)e^{-|x|} \in L^p(\mathbb{R}^n, \mathbb{R})$,

(V_3) $\operatorname*{ess\,inf}\limits_{x\in\mathbb{R}^n} V(x) > 0$.

We note that (V_2) is a technical hypothesis. We need it to prove the regularity of the eigenfunctions of the linear eigenvalue problem (see Lemma 2.8), but it may be weakened.

The assumptions on the function $W : \mathbb{R}^{n+1}\setminus\{\xi_*\} \to \mathbb{R}$ are the following:

(W_1) $W \in C^1(\mathbb{R}^{n+1}\setminus\{\xi_*\}, \mathbb{R})$,

(W_2) $W(\xi) \ge 0$ for all $\xi \in \mathbb{R}^{n+1}\setminus\{\xi_*\}$ and $W(0) = 0$,

(W_3) there exist two constants $c_1, c_2 > 0$ such that

$$\xi \in \mathbb{R}^{n+1},\ 0 < |\xi| < c_1 \Rightarrow W(\xi_* + \xi) \ge \frac{c_2}{|\xi|^{np/(p-n)}}$$

and $\overline{\xi} - c_1 > 0$,

(W_4) there exist two constants $c_3, c_4 > 0$ such that

$$\xi \in \mathbb{R}^{n+1},\ 0 \le |\xi| < c_3 \Rightarrow |W'(\xi)| \le c_4|\xi|.$$

The energy functional associated to the problem (P_ε) is:

$$J_\varepsilon(u) = \int_{\mathbb{R}^n} \left[\frac{1}{2}|\nabla u|^2 + \frac{1}{2}V(x)|u|^2 + \frac{\varepsilon^r}{p}|\nabla u|^p + \varepsilon^r W(u)\right] dx. \tag{7}$$

In [8] the authors proved the existence of solutions for the eigenvalue problem (P_ε) on a bounded domain Ω. In this paper we consider a more complex case, namely when the domain is $\mathbb{R}^n$ and the potential is coercive, i.e. $V(x) \to \infty$ for $|x| \to \infty$.

We state the following existence results (see Theorem 3.1 and Theorem 3.2): *Given $q \in \mathbb{Z}\setminus\{0\}$ and $k \in \mathbb{N}$, we consider $\xi_* = (0, \overline{\xi})$ with $0 \in \mathbb{R}^n$ and $\overline{\xi}$ large enough. Then for ε sufficiently small and for any $j \le k$ with $\widetilde{\lambda}_{j-1} < \widetilde{\lambda}_j$, there* *exist* $\mu_j(\varepsilon)$ *and* $u_j(\varepsilon)$ *respectively eigenvalue and eigenfunction of the problem* (P_ε)*, such that the topological charge of* $u_j(\varepsilon)$ *is* q.

Moreover, given $q \in \mathbb{Z}$*, for any* $\xi_* = (0, \overline{\xi})$ *(with* $0 \in \mathbb{R}^n$ *and* $\overline{\xi} > 0$*) and for any* $\varepsilon > 0$*, there exist* $\mu_1(\varepsilon)$ *and* $u_1(\varepsilon)$ *respectively eigenvalue and eigenfunction of the problem* (P_ε)*, such that the topological charge of* $u_1(\varepsilon)$ *is* q.

Here $\widetilde{\lambda}_j$ (see Subsection 2.4) are the eigenvalues of the linear problem $-\Delta u + V(x)u = \widetilde{\lambda} u$, since we have the discreteness of the spectrum of the Schrödinger operator $-\Delta + V$, with $\lim_{|x|\to\infty} V(x) = \infty$, by a compact embedding theorem (see e.g. [5] and Theorem 2.1).

Our aim is to find critical values of the energy functional J_ε in the intersection of any connected component, characterized by the topological charge, with the unitary sphere in $L^2(\mathbb{R}^n, \mathbb{R}^{n+1})$. The idea is to consider the functional J_ε as a perturbation of the symmetric functional

$$J_0(u) = \int_{\mathbb{R}^n} \frac{1}{2}[|\nabla u|^2 + V(x)|u|^2]\,dx.$$

Non-symmetric perturbations of a symmetric problem, in order to preserve critical values, have been studied by several authors. We omit for the sake of brevity a complete bibliography and we recall only [3], which seems to be the first work on the subject, and the recent papers [10] and [11]. In this paper we give a result of preservation for the functional J_ε of some critical values $\widetilde{\lambda}_j$ of the functional J_0 restricted on the unitary sphere of $L^2(\mathbb{R}^n, \mathbb{R}^{n+1})$ in the space $\Gamma(\mathbb{R}^n, \mathbb{R}^{n+1})$ (see Subsection 2.1).

The content of the paper is divided into the following sections. In Section 2 there is the description of the functional setting, the definition of a topological invariant, called topological charge, and some arguments of eigenvalues theory. The compactness, that we lose because of the unbounded domain $\mathbb{R}^n$, is recovered by the compact embedding of [5] (see Theorem 2.1). Then, by some technical devices, we obtain the Palais–Smale condition for the functional J_ε (defined in (7)). The addition of the potential V breaks the translation invariance, so that the technical lemmas require some care.

Section 3 is devoted to the proof of our main results. In Theorem 3.1 we state the existence of some critical values of the functional J_ε on every component of the unitary sphere, characterized by the value of the topological invariant "topological charge" (see (11), (8), (10)). These critical values $c^q_{\varepsilon,j}$ (see (28)) of the functional J_ε are of "min-max type". The construction of some suitable functions G^q_ε of topological charge q (see (26)) and some suitable manifolds $\mathcal{M}^q_{\varepsilon,j}$ (see (27)) is crucial in finding the critical values $c^q_{\varepsilon,j}$. In Theorem 3.2 we state the existence of the minimum of the functional J_ε on every component of the unitary sphere, characterized by the topological charge (see (10)).

Notations. We fix the following notations:

- $|x|$ is the Euclidean norm of $x \in \mathbb{R}^n$,
- if $\xi \in \mathbb{R}^{n+1}$ some times we will use the notation $\xi = (\widetilde{\xi}, \overline{\xi})$, where $\widetilde{\xi} \in \mathbb{R}^n$ and $\overline{\xi} \in \mathbb{R}$,
- if $x \in \mathbb{R}^n$ and $\rho > 0$, then $B(x, \rho)$ is the open ball with centre in x and radius ρ.

2. Functional setting

2.1. The space E. We shall consider the following functional spaces:

- $\Gamma(\mathbb{R}^n, \mathbb{R})$ the completion of $C_0^\infty(\mathbb{R}^n, \mathbb{R})$ with respect to the norm
$$\|z\|^2_{\Gamma(\mathbb{R}^n,\mathbb{R})} = \int_{\mathbb{R}^n} V(x)\,|z(x)|^2\,dx + \int_{\mathbb{R}^n} |\nabla z(x)|^2\,dx;$$
the space $\Gamma(\mathbb{R}^n, \mathbb{R})$ is then a Hilbert space, whose scalar product is denoted by $(z_1, z_2)_{\Gamma(\mathbb{R}^n,\mathbb{R})}$.
- $\Gamma(\mathbb{R}^n, \mathbb{R}^{n+1})$ the completion of $C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$ with respect to the norm
$$\|u\|^2_{\Gamma(\mathbb{R}^n,\mathbb{R}^{n+1})} = \int_{\mathbb{R}^n} V(x)\,|u(x)|^2\,dx + \int_{\mathbb{R}^n} |\nabla u(x)|^2\,dx;$$
where $|u|^2 = \sum_{i=1}^{n+1} |u_i|^2$ and $|\nabla u|^2 = \sum_{i=1}^{n} \sum_{j=1}^{n+1} |\partial u_j/\partial x_i|^2$; analogously the space $\Gamma(\mathbb{R}^n, \mathbb{R}^{n+1})$ is a Hilbert space, whose scalar product is denoted by $(u_1, u_2)_{\Gamma(\mathbb{R}^n,\mathbb{R}^{n+1})}$.

It is clear that the spaces $\Gamma(\mathbb{R}^n, \mathbb{R})$ and $\Gamma(\mathbb{R}^n, \mathbb{R}^{n+1})$ are continuously embedded respectively into the Sobolev spaces $H^1(\mathbb{R}^n, \mathbb{R})$ and $H^1(\mathbb{R}^n, \mathbb{R}^{n+1})$. At this point we recall a compact embedding theorem of Benci and Fortunato (see [5]), which will be important in the sequel:

Theorem 2.1. *The embedding of the space $\Gamma(\mathbb{R}^n, \mathbb{R})$ into the space $L^2(\mathbb{R}^n, \mathbb{R})$ is compact.*

We shall denote by:

- E the completion of $C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$ with respect to the norm

$$\|u\|^2_E = \int_{\mathbb{R}^n} V(x)\,|u(x)|^2\,dx + \int_{\mathbb{R}^n} |\nabla u(x)|^2\,dx + \left(\int_{\mathbb{R}^n} |\nabla u(x)|^p\,dx\right)^{2/p}.$$

The main properties of the Banach space E are summarized in the following lemma and corollary:

Lemma 2.1. *The Banach space E is continuously embedded into the space $L^s(\mathbb{R}^n, \mathbb{R}^{n+1})$ for $2 \le s \le \infty$.*

For the proof see [4].

Corollary 2.1.

(i) *The Banach space E is continuously embedded into the Sobolev space $W^{1,p}(\mathbb{R}^n, \mathbb{R}^{n+1})$.*

(ii) *There exist two constants $C_0, C_1 > 0$ such that, for every $u \in E$,*

$$\|u\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^{n+1})} \leq C_0 \|u\|_E,$$
$$|u(x) - u(y)| \leq C_1 |x - y|^{(p-n)/p} \|\nabla u\|_{L^p(\mathbb{R}^n,\mathbb{R}^{n+1})}.$$

(iii) *If $u \in E$ then $\lim_{|x|\to\infty} u(x) = 0$.*

2.2. Topological charge and connected components of Λ. In the space E we can consider the open subset

$$\Lambda = \{u \in E \mid \xi_* \notin u(\mathbb{R}^n)\}. \tag{8}$$

We recall now the definition of topological charge introduced by Benci, Fortunato and Pisani in [7] (we report here the definition given in [4]).

We write the $n + 1$ components of a function $u \in E$ in the following way:

$$u(x) = (\widetilde{u}(x), \overline{u}(x)),$$

where $\widetilde{u} : \mathbb{R}^n \to \mathbb{R}^n$ and $\overline{u} : \mathbb{R}^n \to \mathbb{R}$.

Definition 1. Let u be a function in $\Lambda \subset E$, then the support of u is the following set:

$$K_u = \{x \in \mathbb{R}^n \mid \overline{u}(x) > \overline{\xi}\},$$

where $\overline{\xi}$ is defined in (1). The topological charge of u is the following function:

$$\operatorname{ch}(u) = \begin{cases} \deg(\widetilde{u}, K_u, 0) & \text{if } K_u \neq \emptyset, \\ 0 & \text{if } K_u = \emptyset. \end{cases} \tag{9}$$

As a consequence of the fact that u is continuous and $\lim_{|x|\to\infty} u(x) = 0$ (see Corollary 2.1), K_u is an open bounded subset of $\mathbb{R}^n$. Since $u \in \Lambda$, if $x \in \partial K_u$, we have $\overline{u}(x) = \overline{\xi}$ and $\widetilde{u}(x) \neq 0$. Therefore the previous definition is well posed.

Moreover, the topological charge is continuous with respect to the uniform convergence (see [7]):

Lemma 2.2. *For every $u \in \Lambda$ there exists $r = r(u) > 0$ such that, for every $v \in \Lambda$,*

$$\|v - u\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^{n+1})} \leq r \Rightarrow \operatorname{ch}(u) = \operatorname{ch}(v).$$

The set $\Lambda \subset E$ is divided into connected components by the topological charge:

$$\Lambda = \bigcup_{q\in\mathbb{Z}} \Lambda_q,$$

where

$$\Lambda_q = \{u \in \Lambda \mid \mathrm{ch}\,(u) = q\}. \tag{10}$$

2.3. Palais–Smale condition for the energy functional. First of all we verify that the functional J_ε is well defined on the set Λ, that is:

$$J_\varepsilon(u) < \infty \quad \text{for all } u \in \Lambda.$$

It is enough to check that $\int_{\mathbb{R}^n} W(u(x))\,dx < \infty$. In fact by ($\mathrm{W}_2$) and ($\mathrm{W}_4$) we have that

$$\int_{\mathbb{R}^n} W(u(x))\,dx \leq \int_B c_4 |u(x)|^2\,dx + \int_{\mathbb{R}^n \setminus B} W(u(x))\,dx,$$

where $B = \{x \in \mathbb{R}^n \mid u(x) \in B(0, c_3)\}$. The first integral is bounded because $\int_B |u(x)|^2\,dx \leq \int_{\mathbb{R}^n} |u(x)|^2\,dx < \infty$. The second integral is bounded because by Corollary 2.1 the domain $\mathbb{R}^n \setminus B$ is bounded.

Lemma 2.3. *The energy functional J_ε is of class C^1 on the open set Λ of E.*

Proof. The first part of the energy functional is clearly of class C^1. Then we consider $G(u) = \int_{\mathbb{R}^n} W(u(x))\,dx$. Now we want to prove the Gateaux differentiability, hence we show that

$$\lim_{t \to 0} \int_{\mathbb{R}^n} \left[\frac{W(u + tv) - W(u)}{t} - W'(u) \cdot v \right] dx = 0$$

for all $u \in \Lambda$ and for all $v \in E$. The integrand clearly tends to zero pointwise. By the Lagrange Theorem we have that

$$W(u(x) + tv(x)) - W(u(x)) = tW'(u(x) + \theta tv(x)) \cdot v(x)$$

for $t \in \mathbb{R}$ small enough, where $\theta = \theta(x, t) \in [0, 1]$. As $\lim_{|x| \to \infty} u(x) = 0$, there exists $R_1 > 0$ such that

$$x \in \mathbb{R}^n \setminus B(0, R_1) \Rightarrow \begin{cases} |u(x)| \leq c_3/2, \\ |u(x) + \theta tv(x)| \leq c_3, \end{cases}$$

for $|t| \leq \bar{t}$ with $\bar{t}$ suitably small. Then by (W_4), we have the following inequalities

$$|W'(u(x) + \theta tv(x)) \cdot v(x)| \\ \leq \begin{cases} c_4[|u(x)| + \bar{t}|v(x)|]|v(x)| & \text{for all } x \in \mathbb{R}^n \setminus B(0, R_1), \\ \mathrm{const}\,|v(x)| & \text{for all } x \in B(0, R_1). \end{cases}$$

There are analogous inequalities for $|W'(u(x)) \cdot v(x)|$. So we can apply the Lebesgue's dominated convergence theorem.

To have the Fréchet differentiability of the functional G it remains to show that the Gateaux derivative

$$v \to G'(u)(v) = \int_{\mathbb{R}^n} W'(u) \cdot v \, dx \quad u \in \Lambda, \ v \in E$$

is continuous with respect to u. Let $\{u_i\}_{i \in \mathbb{N}}$ be a sequence in Λ strongly converging to $u_0 \in \Lambda$, then we have

$$\begin{aligned}
\|G'(u_i) - G'(u_0)\|_{E^*} &= \sup_{\substack{v \in E \\ \|v\|_E = 1}} \left| \int_{\mathbb{R}^n} [W'(u_i) - W'(u_0)] \cdot v \right| \\
&\leq \sup_{\substack{v \in E \\ \|v\|_E = 1}} \left[\int_{\mathbb{R}^n} |W'(u_i) - W'(u_0)|^2 \right]^{1/2} \|v\|_{L^2(\mathbb{R}^n, \mathbb{R}^{n+1})} \\
&\leq C \left[\int_{\mathbb{R}^n} |W'(u_i) - W'(u_0)|^2 \right]^{1/2},
\end{aligned}$$

where C is a constant. Obviously we have that for all $x \in \mathbb{R}^n$ $|W'(u_i(x)) - W'(u_0(x))| \to 0$. Moreover, there exists $R_2 > 0$ such that

$$x \in \mathbb{R}^n \setminus B(0, R_2) \Rightarrow \begin{cases} |u_0(x)| \leq c_3/2, \\ |u_i(x)| \leq c_3, \end{cases}$$

for i large enough; hence, for i large enough, we have

$$|W'(u_i(x))| \leq \begin{cases} c_4 |u_i(x)| & \text{for all } x \in \mathbb{R}^n \setminus B(0, R_2), \\ \text{const} & \text{for all } x \in B(0, R_2), \end{cases}$$

$$|W'(u_0(x))| \leq \begin{cases} c_4 |u_0(x)| & \text{for all } x \in \mathbb{R}^n \setminus B(0, R_2), \\ \text{const} & \text{for all } x \in B(0, R_2), \end{cases}$$

and consequently

$$|W'(u_i(x)) - W'(u_0(x))|^2 \leq \begin{cases} c_4^2 (|u_i(x)| + |u_0(x)|)^2 & \text{for all } x \in \mathbb{R}^n \setminus B(0, R_2), \\ \text{const} & \text{for all } x \in B(0, R_2). \end{cases}$$

We can now apply the generalized version of the Lebesgue's dominated convergence theorem and conclude that $\|G'(u_i) - G'(u_0)\|_{E^*} \to 0$. $\square$

We put

$$S = \{u \in E \mid \|u\|_{L^2(\mathbb{R}^n, \mathbb{R}^{n+1})} = 1\}. \tag{11}$$

To get some critical points of the functional J_ε on the C^2 manifold $\Lambda \cap S$ we use the following version of Palais–Smale condition. For $J_\varepsilon \in C^1(\Lambda, \mathbb{R})$, the norm of the derivative at $u \in S$ of the restriction $\widehat{J}_\varepsilon = J_\varepsilon|_{\Lambda \cap S}$ is defined by

$$\|\widehat{J}'_\varepsilon(u)\|_* = \min_{t \in \mathbb{R}} \|J'_\varepsilon(u) - t g'(u)\|_{E^*},$$

where $g : E \to \mathbb{R}$ is the function defined by $g(u) = \int_{\mathbb{R}^n} |u(x)|^2 \, dx$.

Definition 2. The functional J_ε is said to satisfy the Palais–Smale condition in $c \in \mathbb{R}$ on $\Lambda \cap S$ (on $\Lambda_q \cap S$, for $q \in \mathbb{Z}$) if, for any sequence $\{u_i\}_{i\in\mathbb{N}} \subset \Lambda \cap S$ ($\{u_i\}_{i\in\mathbb{N}} \subset \Lambda_q \cap S$) such that $J_\varepsilon(u_i) \to c$ and $\|\widehat{J}'_\varepsilon(u_i)\|_* \to 0$, there exists a subsequence which converges to $u \in \Lambda \cap S$ ($u \in \Lambda_q \cap S$).

To obtain the Palais–Smale condition, we need a few technical lemmas.

Lemma 2.4. *Let $\{u_i\}_{i\in\mathbb{N}}$ be a sequence in Λ_q (with $q \in \mathbb{Z}$) such that the sequence $\{J_\varepsilon(u_i)\}_{i\in\mathbb{N}}$ is bounded. We consider the open bounded sets*

$$Z_i = \{x \in \mathbb{R}^n \mid |u_i(x)| > c_3\}. \tag{12}$$

Then the set $\bigcup_{i\in\mathbb{N}} Z_i \subset \mathbb{R}^n$ is bounded.

Proof. By contradiction we suppose that $\bigcup_{i\in\mathbb{N}} Z_i$ is unbounded; then there exist a sequence of indices $\nu_i \to \infty$ for $i \to \infty$ and a sequence of points $\{x_{\nu_i}\}_{i\in\mathbb{N}}$ such that $x_{\nu_i} \in Z_{\nu_i}$ and $|x_{\nu_i}| \to \infty$. By (12) we have:

$$|u_{\nu_i}(x_{\nu_i})| > c_3; \tag{13}$$

we consider the numbers $R_{\nu_i} = \sup\{R > 0 \mid \text{for all } x \in B(x_{\nu_i}, R)\ |u_{\nu_i}(x)| > c_3/2\}$. We claim that $R_{\nu_i} \to 0$ for $i \to \infty$. In fact, if $R_{\nu_i} \not\to 0$, there exists $M > 0$ such that $R_{\nu_i} > M$ for infinitely many indices. Then for such indices we have:

$$\begin{aligned}\int_{\mathbb{R}^n} V(x)\,|u_{\nu_i}(x)|^2\,dx &\geq \int_{B(x_{\nu_i},R_{\nu_i})} V(x)\,|u_{\nu_i}(x)|^2\,dx\\ &\geq \left(\frac{c_3}{2}\right)^2 \int_{B(x_{\nu_i},M)} V(x)\,dx,\end{aligned}$$

but $\int_{B(x_{\nu_i},M)} V(x)\,dx \to \infty$ and this is a contradiction.

We choose now for every $i \in \mathbb{N}$ a point $\widehat{x}_{\nu_i} \in \partial B(x_{\nu_i}, R_{\nu_i})$, i.e. such that

$$|u_{\nu_i}(\widehat{x}_{\nu_i})| = c_3/2; \tag{14}$$

it is clear that $|\widehat{x}_{\nu_i} - x_{\nu_i}| = R_{\nu_i} \to 0$. As the functions u_i are equiuniformly continuous, i.e. for all $x, y \in \mathbb{R}^n$ and for all $i \in \mathbb{N}$ (see (ii) of Corollary 2.1)

$$|u_i(x) - u_i(y)| \leq C_1|x-y|^{(p-n)/p}\,\|\nabla u_i\|_{L^p(\mathbb{R}^n,\mathbb{R}^{n+1})} \leq \text{const}\,|x-y|^{(p-n)/p},$$

then $|u_{\nu_i}(x_{\nu_i}) - u_{\nu_i}(\widehat{x}_{\nu_i})|$ tends to zero for $i \to \infty$. On the other hand, by (13) and (14), there holds:

$$|u_{\nu_i}(x_{\nu_i}) - u_{\nu_i}(\widehat{x}_{\nu_i})| \geq |u_{\nu_i}(x_{\nu_i})| - |u_{\nu_i}(\widehat{x}_{\nu_i})| > c_3/2. \qquad \square$$

The next two lemmas are the Propositions 3.8 and 3.9 of [7]. The addition of the potential V in our equation leads to the loss of translation invariance. Hence we give a proof of Lemma 2.6. (see Proposition 3.9 in [7]), because the arguments of [7] partially fall.

Lemma 2.5. *Let $\{u_i\}_{i\in\mathbb{N}} \subset \Lambda$ be a sequence weakly converging to u and such that $\{J_\varepsilon(u_i)\}_{i\in\mathbb{N}} \subset \mathbb{R}$ is bounded, then $u \in \Lambda$.*

Lemma 2.6. *For any $a > 0$, there exists $d > 0$ such that for every $u \in \Lambda$*

$$J_\varepsilon(u) \le a \Rightarrow \inf_{x\in\mathbb{R}^n} |u(x) - \xi_*| \ge d.$$

Proof. By contradiction we suppose that there exist $a > 0$ and a sequence $\{u_i\}_{i\in\mathbb{N}} \subset \Lambda$ such that for any $i \in \mathbb{N}$ $J_\varepsilon(u_i) \le a$ and $\inf_{x\in\mathbb{R}^n} |u_i(x) - \xi_*| \le 1/i$. As we have $\|u_i\|_E \le \text{const}$, up to a subsequence u_i weakly converges to u in E. In particular u_i converges to u pointwise. Moreover, by Lemma 2.5, we know that $u \in \Lambda$. We denote by $\{x_i\}_{i\in\mathbb{N}}$ a sequence of points in $\mathbb{R}^n$ such that $u_i(x_i) \to \xi_*$. We claim that $\{x_i\}_{i\in\mathbb{N}}$ is bounded. By contradiction let $|x_i|$ tend to ∞. We consider now

$$R_i = \sup\{R \ge 0 \mid \text{for all } x \in B(x_i, R),\ u_i(x) \in B(\xi_*, c_1)\},$$

where c_1 is the constant defined in (W_3); proceeding in the same way as in the proof of Lemma 2.4, we obtain that $R_i \to 0$. For every $i \in \mathbb{N}$ we choose a point $\widehat{x}_i$ on the boundary of $B(x_i, R_i)$, i.e. $\widehat{x}_i$ is such that $|u_i(\widehat{x}_i) - \xi_*| = c_1$ and $|x_i - \widehat{x}_i| \to 0$. Now by the equiuniform continuity we have $|u_i(\widehat{x}_i) - u_i(x_i)| \to 0$, but this is absurd because

$$|u_i(\widehat{x}_i) - u_i(x_i)| = |u_i(\widehat{x}_i) - \xi_* + \xi_* - u_i(x_i)| \ge |c_1 - |u_i(x_i) - \xi_*||$$

and $|c_1 - |u_i(x_i) - \xi_*|| \to c_1 > 0$. Then $\{x_i\}_{i\in\mathbb{N}}$ is bounded and up to a subsequence $x_i \to x_0$. Since we have

$$|u_i(x_i) - u(x_0)| \le |u_i(x_i) - u_i(x_0)| + |u_i(x_0) - u(x_0)|,$$

by equiuniform continuity and by pointwise convergence we can conclude that $|u_i(x_i) - u(x_0)| \to 0$. This means that $u(x_0) = \xi_*$ and this is in contradiction with the fact that $u \in \Lambda$. $\square$

Proposition 2.1. *The functional J_ε satisfies the Palais–Smale condition on $\Lambda \cap S$ (on $\Lambda_q \cap S$ for $q \in \mathbb{Z}$) for any $c \in \mathbb{R}$ and $0 < \varepsilon \le 1$.*

Proof. It is immediate that every Palais–Smale sequence $\{u_i\}_{i\in\mathbb{N}}$ on $\Lambda \cap S$ is bounded in E. Hence we can choose a subsequence, which for simplicity we denote again $\{u_i\}_{i\in\mathbb{N}}$, converging to a function u weakly in E, strongly in $L^2(\mathbb{R}^n, \mathbb{R}^{n+1})$ (by Theorem 2.1) and uniformly on every compact subset of $\mathbb{R}^n$. As we have

$$\min_{t\in\mathbb{R}} \|J'_\varepsilon(u_i) - t g'(u_i)\|_{E^*} \to 0,$$

there is a sequence $\eta_i > 0$, with $\eta_i \to 0$ for $i \to \infty$ and a sequence $t_i \in \mathbb{R}$ such that for all $v \in E$

$$\tag{15} \bigg| \int_{\mathbb{R}^n} [\nabla u_i \cdot \nabla v + V(x) u_i \cdot v + \varepsilon^r |\nabla u_i|^{p-2} \nabla u_i \cdot \nabla v + \varepsilon^r W'(u_i) \cdot v] \, dx \\ - 2t_i \int_{\mathbb{R}^n} u_i \cdot v \, dx \bigg| \le \eta_i \|v\|_E.$$

From the substitution $v = u_i$ in (15), we obtain

$$\tag{16} \bigg| \int_{\mathbb{R}^n} [|\nabla u_i|^2 + V(x)|u_i|^2 + \varepsilon^r |\nabla u_i|^p + \varepsilon^r W'(u_i) \cdot u_i] \, dx - 2t_i \bigg| \le \eta_i \|u_i\|_E.$$

Since $\{u_i\}_{i\in\mathbb{N}}$ is bounded in E, the first three terms are bounded. Now, by Lemma 2.4 and by (W_4), we observe that

$$\bigg| \int_{\mathbb{R}^n} W'(u_i) \cdot u_i \, dx \bigg| \le \bigg| \int_{Z_i} W'(u_i) \cdot u_i \, dx \bigg| + \bigg| \int_{\mathbb{R}^n \setminus Z_i} W'(u_i) \cdot u_i \, dx \bigg| \\ \le \int_K |W'(u_i)||u_i| \, dx + c_4 \|u_i\|^2_{L^2(\mathbb{R}^n, \mathbb{R}^{n+1})}$$

where Z_i is defined in (12) and K is a compact subset of $\mathbb{R}^n$ such that $\bigcup_{i=1}^\infty Z_i \subset K$. Hence the fourth term of (16) is bounded too and so $\{t_i\}_{i\in\mathbb{N}}$ is bounded.

Substituting now $v = u_i - u$, we get:

$$\bigg| \int_{\mathbb{R}^n} [\nabla u_i \cdot \nabla (u_i - u) + V(x) u_i \cdot (u_i - u) + \varepsilon^r |\nabla u_i|^{p-2} \nabla u_i \cdot \nabla (u_i - u) \\ + \varepsilon^r W'(u_i) \cdot (u_i - u)] \, dx - 2t_i \int_{\mathbb{R}^n} u_i \cdot (u_i - u) \, dx \bigg| \le \eta_i \|u_i - u\|_E$$

and we write

$$\int_{\mathbb{R}^n} [\nabla u_i \cdot \nabla (u_i - u) + V(x)\, u_i \cdot (u_i - u) + \varepsilon^r |\nabla u_i|^{p-2} \nabla u_i \cdot \nabla (u_i - u)] \, dx \\ = -\varepsilon^r \int_{\mathbb{R}^n} [W'(u_i) \cdot (u_i - u) + 2t_i u_i \cdot (u_i - u)] \, dx + o(1).$$

As $\{t_i\}_{i\in\mathbb{N}}$ is bounded and $\{u_i\}_{i\in\mathbb{N}}$ converges to u in $L^2(\mathbb{R}^n, \mathbb{R}^{n+1})$ we get that $t_i \int_{\mathbb{R}^n} u_i \cdot (u_i - u) \, dx$ tends to zero. Moreover by Lemma 2.4 and (W_4)

$$\bigg| \int_{\mathbb{R}^n} W'(u_i) \cdot (u_i - u) \, dx \bigg| \le \bigg| \int_{Z_i} W'(u_i) \cdot (u_i - u) \, dx \bigg| \\ + c_4 \|u_i\|_{L^2(\mathbb{R}^n, \mathbb{R}^{n+1})} \|u_i - u\|_{L^2(\mathbb{R}^n, \mathbb{R}^{n+1})} \\ \le C \|u_i - u\|_{L^\infty(K, \mathbb{R}^{n+1})} \\ + c_4 \|u_i\|_{L^2(\mathbb{R}^n, \mathbb{R}^{n+1})} \|u_i - u\|_{L^2(\mathbb{R}^n, \mathbb{R}^{n+1})},$$

where Z_i is defined in (12), C is a constant and K is a compact subset of $\mathbb{R}^n$ such that $\bigcup_{i=1}^{\infty} Z_i \subset K$; hence this term tends to zero. Concluding

$$\int_{\mathbb{R}^n} [\nabla u_i \cdot \nabla(u_i - u) + V(x)\, u_i \cdot (u_i - u) + \varepsilon^r |\nabla u_i|^{p-2} \nabla u_i \cdot \nabla(u_i - u)]\, dx \to 0$$

for $i \to \infty$. At this point we recall that $-\Delta_p$ is a monotone operator (see [16] and [4]), and there exists $\nu > 0$ such that for all $u_1, u_2 \in E$

$$\int_{\mathbb{R}^n} [|\nabla(u_1 - u_2)|^2 + V(x)\, |u_1 - u_2|^2 + |\nabla u_1|^{p-2} \nabla u_1 \cdot \nabla(u_1 - u_2) \\ - |\nabla u_2|^{p-2} \nabla u_2 \cdot \nabla(u_1 - u_2)]\, dx \geq \|u_1 - u_2\|^2_{\Gamma(\mathbb{R}^n, \mathbb{R}^{n+1})} + \nu \|\nabla(u_1 - u_2)\|^p_{L^p}.$$

Hence we get our claim. □

2.4. Eigenvalues of the Schrödinger operator. By the compactness result cited in Theorem 2.1 we obtain the discreteness of the spectrum of the Schrödinger operator $-\Delta + V(x)$ on $L^2(\mathbb{R}^n, \mathbb{R})$ (which is the self-adjoint extension of the operator $-\Delta + V(x)$ on $C_0^\infty(\mathbb{R}^n, \mathbb{R})$). That is the spectrum of the operator $-\Delta + V(x)$ consists of a countable set of eigenvalues of finite multiplicity. The following sequence denotes the eigenvalues counted with their multiplicity:

$$\lambda_1 \leq \ldots \leq \lambda_k \leq \ldots$$

We denote by $\{e_i\}_{i \in \mathbb{N}}$ the sequence of the corresponding eigenfunctions, with $(e_i, e_j)_{L^2(\mathbb{R}^n, \mathbb{R})} = \delta_{ij}$.

We consider now the sequence

$$\widetilde{\lambda}_1 \leq \ldots \leq \widetilde{\lambda}_m \leq \ldots$$

of the eigenvalues of the problem

$$-\Delta u + V(x) u = \widetilde{\lambda} u \quad \text{with } u \in \Gamma(\mathbb{R}^n, \mathbb{R}^{n+1}). \tag{17}$$

If $u = (u_1, \ldots, u_{n+1})$, then (17) is equivalent to

$$-\Delta u_i + V(x) u_i = \widetilde{\lambda} u_i \quad \text{with } i = 1, \ldots, n+1.$$

It is trivial that $\lambda_1 = \widetilde{\lambda}_1 = \ldots = \widetilde{\lambda}_{n+1} \leq \widetilde{\lambda}_{n+2}$, in fact if λ is an eigenvalue of multiplicity ν of the problem

$$-\Delta z + V(x) z = \lambda z \quad \text{with } z \in \Gamma(\mathbb{R}^n, \mathbb{R}),$$

then it is an eigenvalue of (17) of multiplicity $(n+1)\nu$. Moreover, if $\lambda_k < \lambda_{k+1}$, then $\widetilde{\lambda}_{(n+1)k} < \widetilde{\lambda}_{(n+1)k+1}$.

If we set $\widetilde{e}_j = (e_j, 0, \ldots, 0)$, $\widetilde{e}_{j+1} = (0, e_j, \ldots, 0)$, $\ldots$, $\widetilde{e}_{j+n} = (0, 0, \ldots, e_j)$, it is clear what we mean by the sequence of the eigenvectors $\{\varphi_i\}_{i \in \mathbb{N}}$ corresponding to the sequence $\{\widetilde{\lambda}_i\}_{i \in \mathbb{N}}$, which is an orthonormal set in $L^2(\mathbb{R}^n, \mathbb{R}^{n+1})$.

The main properties of the eigenvalues $\{\lambda_i\}_{i\in\mathbb{N}}$ and $\{\widetilde{\lambda}_i\}_{i\in\mathbb{N}}$ are summarized in the following lemma:

LEMMA 2.7. *The following properties hold:*

$$\lambda_i = \min_{\substack{w\in\Gamma(\mathbb{R}^n,\mathbb{R}) \\ (w,e_j)_{L^2(\mathbb{R}^n,\mathbb{R})}=0 \\ \forall j=1,\dots,i-1}} \frac{\|w\|^2_{\Gamma(\mathbb{R}^n,\mathbb{R})}}{\|w\|^2_{L^2(\mathbb{R}^n,\mathbb{R})}},$$

$$\widetilde{\lambda}_i = \min_{\substack{u\in\Gamma(\mathbb{R}^n,\mathbb{R}^{n+1}) \\ (u,\varphi_j)_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})}=0 \\ \forall j=1,\dots,i-1}} \frac{\|u\|^2_{\Gamma(\mathbb{R}^n,\mathbb{R}^{n+1})}}{\|u\|^2_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})}},$$

and

$$(e_i,e_j)_{\Gamma(\mathbb{R}^n,\mathbb{R})} = \lambda_i\delta_{ij} \quad \text{for all } i,j\in\mathbb{N},$$
$$(\varphi_i,\varphi_j)_{\Gamma(\mathbb{R}^n,\mathbb{R}^{n+1})} = \widetilde{\lambda}_i\delta_{ij} \quad \text{for all } i,j\in\mathbb{N}.$$

If we set $E_m = \operatorname{span}[e_1,\dots,e_m]$ *and* $E_m^\perp = \{w\in\Gamma(\mathbb{R}^n,\mathbb{R}) \mid (w,e_i)_{L^2(\mathbb{R}^n,\mathbb{R})} = 0$ *for* $i=1,\dots,m\}$, *we get*

$$w\in E_m \Rightarrow \lambda_1 \le \frac{\|w\|^2_{\Gamma(\mathbb{R}^n,\mathbb{R})}}{\|w\|^2_{L^2(\mathbb{R}^n,\mathbb{R})}} \le \lambda_m,$$
$$w\in E_m^\perp \Rightarrow \frac{\|w\|^2_{\Gamma(\mathbb{R}^n,\mathbb{R})}}{\|w\|^2_{L^2(\mathbb{R}^n,\mathbb{R})}} \ge \lambda_{m+1}.$$

If we set, respectively, $F_m = \operatorname{span}[\varphi_1,\dots,\varphi_m]$ *and* $F_m^\perp = \{u\in\Gamma(\mathbb{R}^n,\mathbb{R}^{n+1}) \mid (u,\varphi_i)_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})} = 0$ *for* $i=1,\dots,m\}$, *we get*

$$u\in F_m \Rightarrow \widetilde{\lambda}_1 \le \frac{\|u\|^2_{\Gamma(\mathbb{R}^n,\mathbb{R}^{n+1})}}{\|u\|^2_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})}} \le \widetilde{\lambda}_m, \tag{18}$$

$$u\in F_m^\perp \Rightarrow \frac{\|u\|^2_{\Gamma(\mathbb{R}^n,\mathbb{R}^{n+1})}}{\|u\|^2_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})}} \ge \widetilde{\lambda}_{m+1}. \tag{19}$$

The proof is a direct consequence of classical argumentations of spectral theory.

Now we recall the following estimate about the eigenfunctions of the Schrödinger operator (see [9, p. 169]):

REMARK 1. If $z\in\Gamma(\mathbb{R}^n,\mathbb{R})$ is such that $-\Delta z + V(x)z = \lambda z$, then for any $a>0$ there exists a constant c_a such that

$$|z(x)| \le c_a e^{-a|x|}. \tag{20}$$

By this result and the regularity theorems we get the following lemma.

Lemma 2.8. *The eigenfunctions $\varphi_i \in \Gamma(\mathbb{R}^n, \mathbb{R}^{n+1})$ of the Schrödinger operator $-\Delta + V(x)$ belong to the Banach space E.*

Proof. By a regularity result, if $z \in \Gamma(\mathbb{R}^n, \mathbb{R})$ is such that $-\Delta z - \lambda z = -Vz$ and if $Vz \in L^2(\mathbb{R}^n, \mathbb{R}) \cap L^p(\mathbb{R}^n, \mathbb{R})$, then $z \in W^{2,p}(\mathbb{R}^n, \mathbb{R})$. By this fact the statement follows immediately.

Now we verify that $Vz \in L^2(\mathbb{R}^n, \mathbb{R}) \cap L^p(\mathbb{R}^n, \mathbb{R})$. By Remark 1 and ($\mathrm{V}_2$) we get

$$\int_{\mathbb{R}^n} |V(x)z(x)|^p \, dx \leq \text{const} \, \|V(x)e^{-|x|}\|^p_{L^p(\mathbb{R}^n,\mathbb{R})} < \infty.$$

Moreover, if $R > 0$ is such that for $x \in \mathbb{R}^n \setminus B(0,R)$ $V(x) > 1$, we have

$$\int_{\mathbb{R}^n} |V(x)z(x)|^2 \, dx \\ \leq \text{const} \left(\int_{B(0,R)} |V(x)|^2 e^{-p|x|} \, dx + \int_{\mathbb{R}^n \setminus B(0,R)} |V(x)|^p e^{-p|x|} \, dx \right) < \infty. \quad \square$$

3. Critical values of the energy functional on every manifold $\Lambda^q \cap S$

3.1. The functions G^q_ε. Fixed an integer $k \in \mathbb{N}$, we define

$$M_k = \sup_{u \in S(k)} \|u\|_{L^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})} \tag{21}$$

where

$$S(k) = F_k \cap S \quad \text{for all } k \in \mathbb{N}. \tag{22}$$

At this point we choose the $(n+1)$-th coordinate $\overline{\xi}$ of the point ξ_* defined in (1) in such a way that

$$\overline{\xi} > 2M_k. \tag{23}$$

First of all we construct a function G_ρ depending on a parameter $\rho > 0$. We consider two functions φ_ρ, $\psi_\rho : \mathbb{R}^+ \to [0,1]$ of class C^∞ such that

$$\varphi_\rho(r) = \begin{cases} 1 & \text{for } 0 \leq r \leq \rho^2, \\ 0 & \text{for } r \geq 4\rho^2, \end{cases} \qquad \psi_\rho(r) = \begin{cases} 1 & \text{for } 0 \leq r \leq 9\rho^2, \\ 0 & \text{for } r \geq 16\rho^2. \end{cases} \tag{24}$$

Moreover, φ_ρ and ψ_ρ take values between 0 and 1 for $\rho^2 \leq r \leq 4\rho^2$ and $9\rho^2 \leq r \leq 16\rho^2$, respectively. We define:

$$\begin{aligned} G_\rho : B(0, 5\rho) \subset \mathbb{R}^n &\to (\mathbb{R}^n \times \mathbb{R}) \setminus \{\xi_*\}, \\ x &\mapsto \psi_\rho(|x|^2)\left(\frac{\overline{\xi}}{\rho}x, 2\overline{\xi}\varphi_\rho(|x|^2)\right). \end{aligned} \tag{25}$$

It is important to observe that the distance of the image of G_ρ from the point ξ_* is $\overline{\xi}$.

We can now introduce for any $q \in \mathbb{Z} \setminus \{0\}$ the functions G^q_ε.

DEFINITION 3. If $q \in \mathbb{Z} \setminus \{0\}$ and $0 < \varepsilon \leq 1$, we set

$$G^q_\varepsilon(x) = \begin{cases} G_{\rho_i}(\gamma_q(x - \widehat{x}_i)/\varepsilon) & \text{for } x \in B(\widehat{x}_i, 5\varepsilon\rho_i) \text{ and } i = 1, \dots, |q|, \\ 0 & \text{for } x \in \mathbb{R}^n \setminus \bigcup_{i=1}^{|q|} B(\widehat{x}_i, 5\varepsilon\rho_i), \end{cases} \tag{26}$$

where G_ρ is defined in (25), γ_q is the following function from $\mathbb{R}^n$ to $\mathbb{R}^n$

$$\gamma_q(x_1, x_2, \dots, x_n) = \begin{cases} (x_1, x_2, \dots, x_n) & \text{for } q > 0, \\ (-x_1, x_2, \dots, x_n) & \text{for } q < 0, \end{cases}$$

and the points $\widehat{x}_i$ and the radiuses ρ_i are such that

1. $B(\widehat{x}_i, \rho_i) \cap B(\widehat{x}_j, \rho_j) = \emptyset$ for all $i \neq j$, $i, j = 1, \dots, |q|$,
2. $\|G^q_1\|_{L^2(\mathbb{R}^n, \mathbb{R}^{n+1})} < 1$.

Finally, we define $G^q = G^q_1$.

REMARK 2. We note that by construction the image of G^q_ε does not intersect the point ξ_* and the distance of the image from the point is $\overline{\xi}$. Moreover, even if we expand the functions G^q_ε $(0 < \varepsilon \leq 1)$ of a factor $t \geq 1$, their image is such that they do not meet the point ξ_* and the distance is still $\overline{\xi}$. Hence $tG^q_\varepsilon \in \Lambda_q$ for all $t \geq 1$ and $\varepsilon \in (0, 1]$.

REMARK 3. By the definition of the functions G^q_ε and by Remark 2 we can conclude that for any $q \in \mathbb{Z}$ we have that $\Lambda_q \cap S \neq \emptyset$.

The following lemma presents some useful properties of the functions G^q_ε which will be crucial in the sequel:

LEMMA 3.1. *There exist $\widehat{\rho} > 0$ and $\overline{\varepsilon}$, with $0 < \overline{\varepsilon} \leq 1$, such that for all $0 < \varepsilon \leq \overline{\varepsilon}$ we have*

(i) $\|G^q_\varepsilon + \widehat{\rho}u\|_{L^2(\mathbb{R}^n, \mathbb{R}^{n+1})} \leq 1$ *for all* $u \in S(k)$,

(ii) $\displaystyle\inf_{\substack{\varepsilon \in (0, \overline{\varepsilon}] \\ u \in S(k)}} \|G^q_\varepsilon + \widehat{\rho}u\|_{L^2(\mathbb{R}^n, \mathbb{R}^{n+1})} > 0$,

(iii) $\displaystyle\inf_{\substack{x \in \mathbb{R}^n \\ \varepsilon \in (0, \overline{\varepsilon}] \\ u \in S(k)}} \left| \frac{G^q_\varepsilon(x) + \widehat{\rho}u(x)}{\|G^q_\varepsilon + \widehat{\rho}u\|_{L^2(\mathbb{R}^n, \mathbb{R}^{n+1})}} - \xi_* \right| > \frac{\overline{\xi}}{2}$,

(iv) $\displaystyle\frac{G^q_\varepsilon + \widehat{\rho}u}{\|G^q_\varepsilon + \widehat{\rho}u\|_{L^2(\mathbb{R}^n, \mathbb{R}^{n+1})}} \in \Lambda_q \cap S$ *for all* $u \in S(k)$.

For the proof see [8].

3.2. The critical values $c^q_{\varepsilon,j}$ of the energy functional on $\Lambda_q \cap S$. Now we can introduce some definitions which we will use to study multiplicity of solutions.

DEFINITION 4. Fixed $k \in \mathbb{N}$, $q \in \mathbb{Z} \setminus \{0\}$ and $0 < \varepsilon \leq \overline{\varepsilon}$, where $\overline{\varepsilon}$ is defined in Lemma 3.1, we set

$$\mathcal{M}^q_{\varepsilon,j} = \left\{ \frac{G^q_\varepsilon + \widehat{\rho}u}{\|G^q_\varepsilon + \widehat{\rho}u\|_{L^2(\mathbb{R}^n, \mathbb{R}^{n+1})}} \,\middle|\, u \in S(j) \right\} \tag{27}$$

with $j \le k$ and $\widehat{\rho}$ defined in Lemma 3.1.

Remark 4. It is trivial that for $j \le k$ we have $\mathcal{M}^q_{\varepsilon,j-1} \subset \mathcal{M}^q_{\varepsilon,j}$, where $\mathcal{M}^q_{\varepsilon,0} = \emptyset$. By Lemma 3.1 we can claim that $\mathcal{M}^q_{\varepsilon,j} \subset \Lambda_q \cap S$. Moreover, $\mathcal{M}^q_{\varepsilon,j}$ is a submanifold of $\Lambda_q \cap S$ for ε sufficiently small.

Definition 5. Fixed $k \in \mathbb{N}$, for all $q \in \mathbb{Z} \setminus \{0\}$, $j \le k$ and $0 < \varepsilon \le \overline{\varepsilon}$ ($\overline{\varepsilon}$ is defined in Lemma 3.1), we introduce the following values:

$$c^q_{\varepsilon,j} = \inf_{h \in \mathcal{H}^q_{\varepsilon,j}} \sup_{v \in \mathcal{M}^q_{\varepsilon,j}} J_\varepsilon(h(v)), \tag{28}$$

where $\mathcal{H}^q_{\varepsilon,j}$ are the following sets of continuous transformations:

$$\mathcal{H}^q_{\varepsilon,j} = \{h : \Lambda_q \cap S \to \Lambda_q \cap S \mid h \text{ continuous}, \ h|_{\mathcal{M}^q_{\varepsilon,j-1}} = \mathrm{id}_{\mathcal{M}^q_{\varepsilon,j-1}}\}.$$

We observe that $\mathcal{H}^q_{\varepsilon,j+1} \subset \mathcal{H}^q_{\varepsilon,j}$.

Lemma 3.2. *Fixed $k \in \mathbb{N}$, for all $q \in \mathbb{Z} \setminus \{0\}$, $j < k$ and $0 < \varepsilon \le \overline{\varepsilon}$, we have*

(i) $c^q_{\varepsilon,j} \le c^q_{\varepsilon,j+1}$,
(ii) $c^q_{\varepsilon,j} \in \mathbb{R}$.

In the following we will use the version of the deformation lemma on a C^2 manifold which we now recall (see for example [14], [18] and [19]).

Lemma 3.3 (Deformation Lemma). *Let J be a C^1-functional defined on a C^2-Finsler manifold M. Let c be a regular value for J. We assume that:*

(i) *J satisfies the Palais–Smale condition in c on M,*
(ii) *there exists $k > 0$ such that the sublevel J^{c+k} is complete.*

Then there exist $\delta > 0$ and a deformation $\eta : [0,1] \times M \longrightarrow M$ such that:

$$\begin{aligned} \eta(0,u) &= u \quad \text{for all } u \in M, \\ \eta(t,u) &= u \quad \text{for all } t \in [0,1] \text{ and all } u \in J^{c-2\delta}, \\ \eta(1, J^{c+\delta}) &\subset J^{c-\delta}. \end{aligned}$$

Lemma 3.4. *For any $q \in \mathbb{Z}$, $\varepsilon \in (0,1]$ and $a \in \mathbb{R}$, the subset $\Lambda_q \cap S \cap J^a_\varepsilon$ of the Banach space E is complete.*

We give some notations: if $u \in E$ we set

$$P_{F_j} u = \sum_{i=1}^{j} (u, \varphi_i)_{\Gamma(\mathbb{R}^n, \mathbb{R}^{n+1})} \varphi_i \quad \text{and} \quad Q_{F_j} u = u - P_{F_j} u. \tag{29}$$

It is immediate that

$$(Q_{F_j} u, \varphi_i)_{\Gamma(\mathbb{R}^n, \mathbb{R}^{n+1})} = \widetilde{\lambda}_i (Q_{F_j} u, \varphi_i)_{L^2(\mathbb{R}^n, \mathbb{R}^{n+1})} = 0 \quad \text{for all } i = 1, \ldots, j. \tag{30}$$

We can now prove the main result:

Theorem 3.1. *Given $q \in \mathbb{Z}\setminus\{0\}$ and $k \in \mathbb{N}$, we consider $\xi_* = (0, \overline{\xi}) \in \mathbb{R}^{n+1}$ with $\overline{\xi} > 2M_k$, where $M_k = \sup_{u\in S(k)} \|u\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^{n+1})}$.*

Then there exists $\widehat{\varepsilon} \in (0,1]$ such that for any $\varepsilon \in (0,\widehat{\varepsilon}]$ and for any $j \le k$ with $\widetilde{\lambda}_{j-1} < \widetilde{\lambda}_j$, we get that $c^q_{\varepsilon,j}$ is a critical value for the functional J_ε restricted to the manifold $\Lambda_q \cap S$. Moreover, $c^q_{\varepsilon,j-1} < c^q_{\varepsilon,j}$ and $c^q_{\varepsilon,j} \to \widetilde{\lambda}_j$ for $\varepsilon \to 0$.

Proof. In the following proof we will denote by $\|\cdot\|_{L^q}$ and $\|\cdot\|_\Gamma$ the norms respectively in $L^q(\mathbb{R}^n,\mathbb{R}^{n+1})$ and in $\Gamma(\mathbb{R}^n,\mathbb{R}^{n+1})$.
We divide the argument into three steps.

Step 1. We prove that

$$\sup_{v\in\mathcal{M}^q_{\varepsilon,j}} J_\varepsilon(v) \le \widetilde{\lambda}_j + \sigma(\varepsilon), \tag{31}$$

$$c^q_{\varepsilon,j} \le \widetilde{\lambda}_j + \sigma(\varepsilon), \tag{32}$$

where $\lim_{\varepsilon\to 0}\sigma(\varepsilon) = 0$.

First of all we verify that

$$\sup_{v\in\mathcal{M}^q_{\varepsilon,j}} J_0(v) \le \widetilde{\lambda}_j + \sup_{u\in S(j)} \frac{\|Q_{F_j}G^q_\varepsilon\|^2_\Gamma}{\|P_{F_j}G^q_\varepsilon + \widehat{\rho}u\|^2_{L^2} + \|Q_{F_j}G^q_\varepsilon\|^2_{L^2}}. \tag{33}$$

In fact by Definition 4, (29) and (30) we have:

$$\begin{aligned}
\sup_{v\in\mathcal{M}^q_{\varepsilon,j}} J_0(v) &= \sup_{u\in S(j)} \left\| \frac{G^q_\varepsilon + \widehat{\rho}u}{\|G^q_\varepsilon + \widehat{\rho}u\|_{L^2}} \right\|^2_\Gamma = \sup_{u\in S(j)} \frac{\|P_{F_j}G^q_\varepsilon + \widehat{\rho}u\|^2_\Gamma + \|Q_{F_j}G^q_\varepsilon\|^2_\Gamma}{\|P_{F_j}G^q_\varepsilon + \widehat{\rho}u\|^2_{L^2} + \|Q_{F_j}G^q_\varepsilon\|^2_{L^2}} \\
&\le \sup_{u\in S(j)} \left(\frac{\|P_{F_j}G^q_\varepsilon + \widehat{\rho}u\|^2_\Gamma}{\|P_{F_j}G^q_\varepsilon + \widehat{\rho}u\|^2_{L^2}} + \frac{\|Q_{F_j}G^q_\varepsilon\|^2_\Gamma}{\|P_{F_j}G^q_\varepsilon + \widehat{\rho}u\|^2_{L^2} + \|Q_{F_j}G^q_\varepsilon\|^2_{L^2}} \right) \\
&\le \widetilde{\lambda}_j + \sup_{u\in S(j)} \frac{\|Q_{F_j}G^q_\varepsilon\|^2_\Gamma}{\|P_{F_j}G^q_\varepsilon + \widehat{\rho}u\|^2_{L^2} + \|Q_{F_j}G^q_\varepsilon\|^2_{L^2}}.
\end{aligned}$$

Now, by definition of J_ε and (33), we prove the following inequalities:

$$\begin{aligned}
c^q_{\varepsilon,j} &= \inf_{h\in\mathcal{H}^q_{\varepsilon,j}} \sup_{v\in\mathcal{M}^q_{\varepsilon,j}} J_\varepsilon(h(v)) \le \sup_{v\in\mathcal{M}^q_{\varepsilon,j}} J_\varepsilon(v) \\
&\le \sup_{v\in\mathcal{M}^q_{\varepsilon,j}} J_0(v) + \varepsilon^r \sup_{v\in\mathcal{M}^q_{\varepsilon,j}} \int_{\mathbb{R}^n} \left(\frac{1}{p}|\nabla v|^p + W(v)\right) dx \\
&\le \widetilde{\lambda}_j + \sup_{u\in S(j)} \frac{\|Q_{F_j}G^q_\varepsilon\|^2_\Gamma}{\|P_{F_j}G^q_\varepsilon + \widehat{\rho}u\|^2_{L^2} + \|Q_{F_j}G^q_\varepsilon\|^2_{L^2}} \\
&\quad + \frac{\varepsilon^r}{p} \sup_{u\in S(j)} \frac{\int_{\mathbb{R}^n} |\nabla(G^q_\varepsilon + \widehat{\rho}u)|^p\,dx}{\|G^q_\varepsilon + \widehat{\rho}u\|^p_{L^2}} \\
&\quad + \varepsilon^r \sup_{u\in S(j)} \int_{\mathbb{R}^n} W\left(\frac{G^q_\varepsilon + \widehat{\rho}u}{\|G^q_\varepsilon + \widehat{\rho}u\|_{L^2}}\right) dx.
\end{aligned} \tag{34}$$

At this point we note that $\lim_{\varepsilon\to 0}\|Q_{F_j}G^q_\varepsilon\|^2_\Gamma = 0$; in fact by (29) and (30), by the fact that the support of G^q_ε is contained in the support of G^q for all $\varepsilon < 1$ and by the fact that $V \in L^2_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R})$, we have

$$\begin{aligned}\|Q_{F_j}G^q_\varepsilon\|^2_\Gamma \le \|G^q_\varepsilon\|^2_\Gamma &\le \int_{\mathbb{R}^n} |\nabla G^q_\varepsilon|^2\,dx + \|V\|_{L^2(\Omega,\mathbb{R})}\|G^q_\varepsilon\|^2_{L^4} \\ &= \varepsilon^{n-2}\int_{\mathbb{R}^n}|\nabla G^q|^2\,dx + \varepsilon^{\frac{n}{2}}\|V\|_{L^2(\Omega,\mathbb{R})}\|G^q\|^2_{L^4}\end{aligned}$$

where $\Omega \subset \mathbb{R}^n$ is the support of G^q.

Moreover, by (ii) of Lemma 3.1 we obtain

$$\sup_{0<\varepsilon\le\overline{\varepsilon}}\ \sup_{u\in S(j)} \frac{1}{\|P_{F_j}G^q_\varepsilon + \widehat{\rho}u\|^2_{L^2} + \|Q_{F_j}G^q_\varepsilon\|^2_{L^2}} < \infty,$$

in fact $\|P_{F_j}G^q_\varepsilon\|^2_{L^2} \le \varepsilon^n\|G^q\|^2_{L^2}$ and $\|Q_{F_j}G^q_\varepsilon\|^2_{L^2} \le \varepsilon^n\|G^q\|^2_{L^2}$. Therefore the second term of the last inequality of (34) goes to zero when ε goes to zero. Now we observe that the following inequality holds:

$$\int_{\mathbb{R}^n}|\nabla(G^q_\varepsilon + \widehat{\rho}u)|^p\,dx \le \mathrm{const}\left(\varepsilon^{n-p}\int_{\mathbb{R}^n}|\nabla G^q|^p\,dx + \widehat{\rho}^p\int_{\mathbb{R}^n}|\nabla u|^p\,dx\right).$$

Then by this inequality and (ii) of Lemma 3.1 (we recall that $r > p-n$), we have that the third term of the last inequality of (34) tends to zero when ε tends to zero.

As regards the last term, we verify that $\int_{\mathbb{R}^n} W((G^q_\varepsilon + \widehat{\rho}u)/\|G^q_\varepsilon + \widehat{\rho}u\|_{L^2})\,dx$ is bounded. In fact by definition of G^q_ε and by the exponential decay of the eigenfunctions (see Remark 1) there exists a ball $B(0,R)$ such that, if we write $u = \sum_{i=1}^j a_i\varphi_i$ with $\sum_{i=1}^j a_i^2 = 1$, for all $x \in \mathbb{R}^n\setminus B(0,R)$ the following inequalities hold

$$\left|\frac{G^q_\varepsilon(x) + \widehat{\rho}u(x)}{\|G^q_\varepsilon + \widehat{\rho}u\|_{L^2}}\right| = \frac{\widehat{\rho}|u(x)|}{\|G^q_\varepsilon + \widehat{\rho}u\|_{L^2}} \le \frac{\mathrm{const}\,\widehat{\rho}(\sum_{i=1}^j|a_i|)e^{-|x|}}{\|G^q_\varepsilon + \widehat{\rho}u\|_{L^2}} \le Me^{-|x|} < c_3$$

where the constant M does not depend on $u \in S(j)$ nor on ε for ε small enough (see the point (ii) of Lemma 3.1). By (W_4) we get

$$\left|W\left(\frac{G^q_\varepsilon(x) + \widehat{\rho}u(x)}{\|G^q_\varepsilon + \widehat{\rho}u\|_{L^2}}\right)\right| \le c_4\frac{|G^q_\varepsilon(x) + \widehat{\rho}u(x)|^2}{\|G^q_\varepsilon + \widehat{\rho}u\|^2_{L^2}}$$

for any $x \in \mathbb{R}^n\setminus B(0,R)$. Concluding we have

$$\left|\int_{\mathbb{R}^n} W\left(\frac{G^q_\varepsilon + \widehat{\rho}u}{\|G^q_\varepsilon + \widehat{\rho}u\|_{L^2}}\right)dx\right| \le c_4 + \int_{B(0,R)}\left|W\left(\frac{G^q_\varepsilon + \widehat{\rho}u}{\|G^q_\varepsilon + \widehat{\rho}u\|_{L^2}}\right)\right|dx$$

where the integral on the right hand side is bounded by (iii) of Lemma 3.1. So we have the claim.

Step 2. We prove that $c^q_{\varepsilon,j} \ge \widetilde{\lambda}_j$.

By positivity of W the following inequalities hold

$$c^q_{\varepsilon,j} \geq \inf_{h\in\mathcal{H}^q_{\varepsilon,j}} \sup_{v\in\mathcal{M}^q_{\varepsilon,j}} \|h(v)\|^2_\Gamma \geq \inf_{h\in\mathcal{H}^q_{\varepsilon,j}} \sup_{\substack{v\in\mathcal{M}^q_{\varepsilon,j} \\ P_{F_{j-1}}h(v)=0}} \|h(v)\|^2_\Gamma.$$

By an argument of degree theory we get that for any $h \in \mathcal{H}^q_{\varepsilon,j}$ the intersection of the set $h(\mathcal{M}^q_{\varepsilon,j})$ with the set $\{u \in E \mid (u,\varphi_i)_\Gamma = 0 \text{ for all } i = 1,\dots,j-1\}$ is not empty, that is there exists $v \in \mathcal{M}^q_{\varepsilon,j}$ such that $P_{F_{j-1}}h(v) = 0$ (for the proof see [8]). Now by (19) in Lemma 2.7 we obtain $c^q_{\varepsilon,j} \geq \widetilde{\lambda}_j$.

Step 3. If $\widetilde{\lambda}_{j-1} < \widetilde{\lambda}_j$, then $c^q_{\varepsilon,j}$ is a critical value for the functional J_ε on the manifold $\Lambda_q \cap S$ and $c^q_{\varepsilon,j-1} < c^q_{\varepsilon,j}$ for ε small enough.

We begin by noting that

$$c^q_{\varepsilon,j-1} < c^q_{\varepsilon,j}, \tag{35}$$

$$\sup_{v\in\mathcal{M}^q_{\varepsilon,j-1}} J_\varepsilon(v) < c^q_{\varepsilon,j}; \tag{36}$$

in fact, by Steps 1 and 2, we obtain for ε sufficiently small,

$$\begin{aligned} c^q_{\varepsilon,j-1} &\leq \widetilde{\lambda}_{j-1} + \sigma(\varepsilon) < \widetilde{\lambda}_j \leq c^q_{\varepsilon,j}, \\ \sup_{v\in\mathcal{M}^q_{\varepsilon,j-1}} J_\varepsilon(v) &\leq \widetilde{\lambda}_{j-1} + \sigma(\varepsilon) < \widetilde{\lambda}_j \leq c^q_{\varepsilon,j}. \end{aligned}$$

Now we suppose by contradiction that $c^q_{\varepsilon,j}$ is a regular value for J_ε on $\Lambda_q \cap S$. By Proposition 2.1 and Lemmas 3.3, 3.4 there exist $\delta > 0$ and a deformation $\eta : [0,1] \times \Lambda_q \cap S \to \Lambda_q \cap S$ such that

$$\begin{aligned} &\eta(0,u) = u && \text{for all } u \in \Lambda_q \cap S, \\ &\eta(t,u) = u && \text{for all } t \in [0,1] \text{ and all } u \in J_\varepsilon^{c^q_{\varepsilon,j}-2\delta}, \\ &\eta(1, J_\varepsilon^{c^q_{\varepsilon,j}+\delta}) \subset J_\varepsilon^{c^q_{\varepsilon,j}-\delta}. \end{aligned}$$

By (36) we can suppose

$$\sup_{v\in\mathcal{M}^q_{\varepsilon,j-1}} J_\varepsilon(v) < c^q_{\varepsilon,j} - 2\delta. \tag{37}$$

Moreover, by definition of $c^q_{\varepsilon,j}$ there exists a transformation $\widehat{h} \in \mathcal{H}^q_{\varepsilon,j}$ such that $\sup_{v\in\mathcal{M}^q_{\varepsilon,j}} J_\varepsilon(\widehat{h}(v)) < c^q_{\varepsilon,j} + \delta$. Now by the properties of the deformation η and by (37) we get $\eta(1,\widehat{h}(\cdot)) \in \mathcal{H}^q_{\varepsilon,j}$ and $\sup_{v\in\mathcal{M}^q_{\varepsilon,j}} J_\varepsilon(\eta(1,\widehat{h}(v))) < c^q_{\varepsilon,j} - \delta$ and this is a contradiction. □

3.3. Minima of the energy functional on $\Lambda_q \cap S$. Finally we can get the minimum values of the functional J_ε on each manifold $\Lambda_q \cap S$, with $q \in \mathbb{Z}$, for any $\varepsilon > 0$ and for any $\xi_* = (0,\overline{\xi})$.

Theorem 3.2. *Given $q \in \mathbb{Z}$, for any $\xi_* = (0, \overline{\xi})$ with $0 \in \mathbb{R}^n$ and $\overline{\xi} > 0$ and for any $\varepsilon > 0$, there exists a minimum for the functional J_ε on the submanifold $\Lambda_q \cap S$ of $\Lambda \cap S$.*

Proof. The claim follows by the fact that $\Lambda_q \cap S$ is not empty (see Remark 3) and the functional J_ε is bounded from below and satisfies the Palais–Smale condition on $\Lambda_q \cap S$ (see Proposition 2.1). □

Remark 5. The minimum critical value of J_ε on $\Lambda_q \cap S$ is not obtained by Theorem 3.1 and coincides by definition with $c^q_{\varepsilon,1}$ (Definition 5). Moreover, the minimum critical value $c^q_{\varepsilon,1}$ tends to $\widetilde{\lambda}_1$ for ε that tends to 0.

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