Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 23, 2004, 339–356

GLOBAL EXISTENCE OF SOLUTIONS OF THE FREE BOUNDARY PROBLEM FOR THE EQUATIONS OF MAGNETOHYDRODYNAMIC INCOMPRESIBLE VISCOUS FLUID

Piotr Kacprzyk

Abstract. Global motion of magnetohydrodynamic fluid in a domain bounded by a free surface and under the external electrodynamic field is proved. The motion is such that velocity and magnetic field are small in H^3-space.

1. Introduction

In this paper we prove the existence of global solutions to equations describing the motion of magnetohydrodynamic incompresible viscous fluid in domain $\Omega_t \subset \mathbb{R}^3$ bounded by a free surface S_t. In the domain $D_t \subset \mathbb{R}^3$ which is exterior to Ω_t we have a gas under the constant pressure p_0. Moreover in the domain D_t we have an electromagnetic field which is generated by some currents which are located on a fixed boundary B on D_t.

2000 *Mathematics Subject Classification.* 35A05, 35R35, 76N10.

Key words and phrases. Free boundary, global existence, Sobolev spaces, magnetohydrodynamic incompresible fluid.

Supported by KBN grant No. 2/P03A/00/223.

In the domain Ω_t the motion is described by the following problem

$$
\begin{aligned}
&v_t + v\cdot\nabla v - \operatorname{div}\mathbb{T}(v,p) - \mu_1 \overset{1}{H}\cdot\nabla \overset{1}{H} + \mu_1\nabla \overset{1}{H}{}^2 = f && \text{in } \widetilde{\Omega}^T,\\
&\operatorname{div} v = 0, && \text{in } \widetilde{\Omega}^T\\
&\mu_1 \overset{1}{H}_t = -\operatorname{rot} \overset{1}{E} && \text{in } \widetilde{\Omega}^T,\\
&\operatorname{rot} \overset{1}{H} = \sigma_1(\overset{1}{E} + \mu_1 v\times \overset{1}{H}) && \text{in } \widetilde{\Omega}^T,\\
&\operatorname{div}(\mu_1 \overset{1}{H}) = 0 && \text{in } \widetilde{\Omega}^T,
\end{aligned}
\tag{1.1}
$$

where $\widetilde{\Omega}^T = \bigcup_{0\le t\le T}\Omega_t\times\{t\}$, $v = v(x,t)$ is the velocity of fluid, $p = p(x,t)$ is the pressure, $\overset{1}{H} = \overset{1}{H}(x,t)$ is the magnetic field, $f = f(x,t)$ is the external force field per unit mass, μ_1 is the constant magnetic permeability, σ_1 is the constant electric conductivity, $\overset{1}{E} = \overset{1}{E}(x,t)$ is the electric field,

$$
\mathbb{T}(v,p) = \{\nu(\partial_{x_i}v_j + \partial_{x_j}v_i) - p\delta_{ij}\}
\tag{1.2}
$$

is the stress tensor, where ν is the viscosity of the fluid. Moreover, by

$$
\mathbb{D}(v) = \{\nu(\partial_{x_i}v_j + \partial_{x_j}v_i)\}
\tag{1.3}
$$

we denote the dilatation tensor.

In the domain D_t in which there is a dielectric (gas) we assume that there is no fluid motion inside ($v = 0$). Therefore we have the electromagnetic field only described by the following system

$$
\begin{aligned}
&\mu_2 \overset{2}{H}_t = -\operatorname{rot} \overset{2}{E} && \text{in } \widetilde{D}^T,\\
&\operatorname{rot} \overset{2}{H} = \sigma_2 \overset{2}{E} && \text{in } \widetilde{D}^T,\\
&\operatorname{div}(\mu_2 \overset{2}{H}) = 0 && \text{in } \widetilde{D}^T,
\end{aligned}
\tag{1.4}
$$

where $\widetilde{D}^T = \bigcup_{0\le t\le T} D_t\times\{t\}$.

On $S_t = \partial\Omega_t\cap\partial D_t$ we assume the following transmition and boundary conditions

$$
\begin{aligned}
&n\cdot\mathbb{T}(v,p) = -p_0 n && \text{on } \widetilde{S}^T,\\
&\frac{1}{\sigma_1}\overset{1}{H} = \frac{1}{\sigma_2}\overset{2}{H} && \text{on } \widetilde{S}^T,\\
&\overset{1}{E}\cdot\tau_\alpha = \overset{2}{E}\cdot\tau_\alpha, \quad \alpha = 1,2, && \text{on } \widetilde{S}^T,\\
&vn = -\frac{\phi_t}{|\nabla\phi_t|} && \text{on } \widetilde{S}^T,
\end{aligned}
\tag{1.5}
$$

where $\widetilde{S}^T = \bigcup_{0\le t\le T} S_t\times\{t\}$, n is the unit outward vector to Ω_t and normal to S_t, τ_α, $\alpha = 1,2$ is the tangent vector to S_t, $\phi(x,t) = 0$ describes S_t at least locally.

Next we assume the boundary conditions on B

$$\overset{2}{H} = H_* \quad \text{on } B,$$
$$\overset{2}{E} = E_* \quad \text{on } B. \tag{1.6}$$

Finally we assume the initial conditions

$$\begin{aligned} \Omega_t|_{t=0} &= \Omega, \quad S_t|_{t=0} = S, \quad D_t|_{t=0} = D, \\ v|_{t=0} &= v_0, \quad \overset{1}{H}|_{t=0} = \overset{1}{H}_0, \quad \text{in } \Omega, \\ & \quad \overset{2}{H}|_{t=0} = \overset{2}{H}_0, \quad \text{in } D. \end{aligned} \tag{1.7}$$

To prove existence of solutions to the above problem we introduce the Lagrangian coordinates $\xi \in \Omega$. The Lagrangian coordinates are connected with the velocity v, are the initial data for the Cauchy problem

$$\frac{dx}{dt} = v(x,t), \quad x|_{t=0} = \xi \in \Omega. \tag{1.8}$$

Therefore $x_v(\xi,t) = \xi + \int_0^t \overline{v}(\xi,\tau)\,d\tau$, where

$$\overline{v}(\xi,t) = v(x_v(\xi,t),t).$$

To introduce the Lagrangian coordinates in D_t we extend v on D_t. Let us denote the extend function by v'. Then we define $\xi \in D$, by the Cauchy data to the problem

$$\frac{dx}{dt} = v'(x,t), \quad x|_{t=0} = \xi \in D. \tag{1.9}$$

Therefore $x_{v'}(\xi,t) = \xi + \int_0^t \overline{v}'(\xi,\tau)d\tau$, where $\overline{v}'(\xi,t) = v'(x_{v'}(\xi,t),t)$. Then by $(1.1)_5$

$$\Omega_t = \{x \in \mathbb{R}^3 : x = x_v(\xi,t),\ \xi \in \Omega\}, \quad S_t = \{x \in \mathbb{R}^3 : x = x_v(\xi,t),\ \xi \in S\}.$$

Since S_t is determined at least locally the by equation $\phi(x,t) = 0$, S is described by $\phi(x_v(\xi,t),t)|_{t=0} = 0$. Moreover, we have

$$\overline{n}_v = n(x_v(\xi,t),t) = \frac{\nabla_x \phi(x,t)}{|\nabla_x \phi(x,t)|}\bigg|_{x=x_v(\xi,t)}.$$

To simplify considerations we introduce the following notation

$$\begin{aligned} \|u\|_{l,Q} &= \|u\|_{H^l(Q)}, && Q \in \{\Omega, S, D, \Pi, B\},\ 0 \le l \in \mathbb{Z}, \\ \|u\|_{k,p,q,Q^T} &= \|u\|_{L_q(0,T,W_p^k(Q))}, && Q \in \{\Omega, S, D, \Pi, B\}, \\ p,q &\in [1,\infty], && 0 \le k \in \mathbb{Z}, \end{aligned}$$

where $Q^t = Q \times (0,t)$,

$$|u|_{p,Q} = \|u\|_{L_p(Q)}, \quad Q \in \{\Omega, S, D, \Pi, B\},\ p \in [1,\infty].$$

2. Weak solution

Weak solutions to problem (1.1)–(1.7) we formulate in the Lagrangian coordinates, where $(1.5)_1$ should be written in the form $\overline{n}\cdot\mathbb{T}(\overline{v},\overline{p}')=0$ where $\overline{p}=\overline{p}'+p_0$.

Definition 2.1. By *weak solutions problem* (1.1)–(1.7) we mean functions $\overline{v},\overline{H}$ which satisfy the integral identities

$$\begin{aligned}(2.1)\quad &\int_0^T\int_\Omega(-\overline{v}\cdot\overline{\varphi}_t+\mathbb{D}_v(\overline{v})\cdot\mathbb{D}_v(\overline{\varphi}))\,d\xi\,dt\\ &\quad-\int_0^T\int_\Omega(\mu_1\overset{1}{\overline{H}}\cdot\nabla_v\overset{1}{\overline{H}}\cdot\overline{\varphi}-\mu_1\nabla_v\overset{1}{\overline{H}}{}^2\cdot\overline{\varphi})\,d\xi\,dt\\ &\quad=\int_0^T\int_\Omega\overline{f}\cdot\overline{\varphi}\,d\xi\,dt-\int_0^T\int_S p_0\overline{n}_v\cdot\overline{\varphi}\,d\xi_S\,dt-\int_\Omega\overline{v}_0\cdot\overline{\varphi}(0)\,d\xi,\end{aligned}$$

$$\begin{aligned}(2.2)\quad &\int_0^T\int_\Pi\bigg(-\mu\overline{H}\cdot\overline{\psi}_t-\mu\overline{v}\cdot\nabla_v\overline{H}\cdot\overline{\psi}+\frac{1}{\sigma}\mathrm{rot}_v\overline{H}\cdot\mathrm{rot}_v\overline{\psi}\bigg)\,d\xi\,dt\\ &\quad-\int_0^T\int_\Omega\mu_1(\overline{v}\times\overset{1}{\overline{H}})\cdot\mathrm{rot}_v\overline{\psi}\,d\xi\,dt\\ &\quad=\frac{1}{\sigma_2}\int_0^T\int_B(\overline{n}_v\times\overline{E}_*)\cdot\overline{\psi}\,d\xi_B\,dt-\mu\int_\Pi\overline{H}_0\cdot\overline{\psi}(0)\,d\xi,\end{aligned}$$

where φ,ψ are sufficiently regular and $\varphi(x,T)=\psi(x,T)=0$, $\overline{n}_v$ is the unit outward vector normal to S or B.

In (2.1), (2.2) we use notation $\overline{A}(\xi,t)=A(x_v(\xi,t),t)$, $\overline{H}|_\Omega=\overset{1}{\overline{H}}$, $\overline{H}|_D=\overset{2}{\overline{H}}$, $\sigma|_\Omega=\sigma_1$, $\sigma|_D=\sigma_2$, $\Pi=\Omega\cup D$, $\mu|_\Omega=\mu_1$, $\mu|_D=\mu_2$, in (2.2) v is extention on Π,

$$\begin{gathered}\mathbb{D}_v(\overline{v})=\{\nu(\partial_{x_i}\xi_k\nabla_{\xi_k}\overline{v}_j+\partial_{x_j}\xi_k\nabla_{\xi_k}\overline{v}_i)\},\quad \mathrm{rot}_v\overline{v}=\nabla_v\times\overline{v},\\ \nabla_v=\partial_x\xi_i\nabla_{\xi_i},\quad \mathrm{div}_v\overline{v}=\nabla_v\cdot\overline{v}=\partial_{x_i}\xi_k\nabla_{\xi_k}\overline{v}_i,\quad \partial_{\xi_i}=\nabla_{\xi_i}.\end{gathered}$$

Let A be the Jakobi matrix of the transformation $x=x_v(\xi,t)$, then $\det A=\exp(\int_0^t\mathrm{div}_v\overline{v}\,d\tau)=1$. Moreover,

$$x^i_{\xi_j}=\delta_{ij}+\int_0^t\partial_{\xi_j}\overline{v}^i(\xi,\tau)\,d\tau\quad\text{and}\quad \xi_x=x_\xi^{-1}.$$

Then we get

$$\begin{aligned}\sup_{\xi\in\Omega}|x_\xi|&\le 1+\sup_{\xi\in\Omega}\int_0^t|\overline{v}(\xi,\tau)|\,d\tau\\ &\le 1+c\int_0^t\|\overline{v}\|_{3,\Omega}\,d\tau\le 1+c\sqrt{t}\sqrt{\int_0^t\|\overline{v}\|^2_{3,\Omega}\,d\tau}\le 1+c\sqrt{t}\|\overline{v}\|_{3,2,2,\Omega^t},\end{aligned}$$

then $\sup_{x\in\Omega_t}|\xi_x|\le\varphi(a)$, where $a=\sqrt{t}\|\overline{v}\|_{3,2,2,\Omega^t}$ and φ is an increasing positive function.

To prove the existence of the solution to the above problem we linarize (2.1), (2.2) to the form

$$
\begin{aligned}
(2.3)\quad &\int_0^T\int_\Omega(-\overline{v}\,\overline{\varphi}_t+\mathbb{D}_u(\overline{v})\mathbb{D}_u(\overline{\varphi}))\,d\xi\,dt\\
&\quad-\int_0^T\int_\Omega(\mu_1\overset{1}{\overline{H}}{}'\cdot\nabla_u\overset{1}{\overline{H}}{}'\cdot\overline{\varphi}-\mu_1\nabla_u\overset{1}{\overline{H}}{}'^{\,2}\cdot\overline{\varphi})\,d\xi\,dt\\
&\quad=\int_0^T\int_\Omega\overline{f}\overline{\varphi}\,d\xi\,dt-\int_0^T\int_S p_0\overline{n}_u\varphi\,d\xi_S\,dt-\int_\Omega\overline{v}_0\overline{\varphi}(0)\,d\xi,
\end{aligned}
$$

$$
\begin{aligned}
(2.4)\quad &\int_0^T\int_\Pi\Big(-\mu\overline{H}\,\overline{\psi}_t-\mu\overline{u}\nabla_u\overline{H}\,\overline{\psi}+\frac{1}{\sigma}\mathrm{rot}\,_u\overline{H}\mathrm{rot}\,_u\overline{\psi}\Big)\,d\xi\,dt\\
&\quad-\int_0^T\int_\Omega\mu_1(\overline{u}\times\overset{1}{\overline{H}})\mathrm{rot}\,_u\overline{\psi}\,d\xi\,dt\\
&\quad=\frac{1}{\sigma_2}\int_0^T\int_B(\overline{n}_u\times\overline{E}_*)\overline{\psi}\,d\xi_B\,dt-\mu\int_\Pi\overline{H}_0\overline{\psi}(0)\,d\xi,
\end{aligned}
$$

where $\overset{1}{\overline{H}}{}'$ and u, $\operatorname{div}u=0$ are given functions.

In [4] we proved

Theorem 2.2. *Assume that* $\overline{v}_0\in H^2(\Omega)$, $\overline{v}_t(0),\overline{v}_{tt}(0)\in L_2(\Omega)$, $\overline{f}_t,\overline{f}_{tt}\in L_2(0,T,L_2(\Omega))$, $\overline{f}\in L_2(0,T,H^2(\Omega))$, $\overline{H}_0\in H^2(\Pi)$, $\overline{H}_t(0)\in H^1(\Pi)$, $\overline{E}_*\in L_\infty(0,T,H^1(B))$, $\overline{E}_{*t},\overline{H}_{*tt}\in L_2(0,T,L_2(B))$, $\overline{H}_{*t}\in L_2(0,T,H^2(B))$, $\overline{H}_*\in L_2(0,T,H^3(B))$, $S,B\in H^{5/2}$. *Then there exists* $T^*>0$ *such that for* $T\le T^*$ *there exists a solution to problem* (1.1)–(1.7) *such that* $\overline{v}\in L_2(0,T,H^3(\Omega))\cap L_\infty(0,T,H^1(\Omega))$, $\overline{v}_t\in L_\infty(0,T,H^1(\Omega))\cap L_2(0,T,H^2(\Omega))$, $\overline{v}_{tt}\in L_\infty(0,T,L_2(\Omega))\cap L_2(0,T,H^1(\Omega))$, $\overline{p}\in L_2(0,T,H^2(\Omega))$, $\overline{p}_t\in L_2(0,T,H^1(\Omega))$, $\overline{H}\in L_2(0,T,H^3(\Pi))\cap L_\infty(0,T,H^1(\Pi))$, $\overline{H}_t\in L_\infty(0,T,H^1(\Pi))\cap L_2(0,T,H^2(\Pi))$, $\overline{H}_{tt}\in L_\infty(0,T,L_2(\Pi))\cap L_2(0,T,H^1(\Pi))$, *where* $(T^*)^\gamma(\varphi(0)+B)\le b$, $b>0$ *sufficiently small,* $\gamma>0$ *some constant and*

$$
\begin{aligned}
(2.5)\quad B=&\|\overline{E}_*\|^2_{0,2,2,B^t}+\|\overline{E}_{*t}\|^2_{0,2,2,B^t}+\|\overline{H}_*\|^2_{3,2,2,B^t}\\
&+\|\overline{H}_{*t}\|^2_{2,2,2,B^t}+\|\overline{H}_{*tt}\|^2_{0,2,2,B^t}+\|\overline{f}_t\|^2_{0,2,2,B^t}+\|\overline{f}\|^2_{1,2,2,B^t},
\end{aligned}
$$

$$
(2.6)\quad \varphi(0)=\sum_{i+k\le2}(\|\partial_t^i\overline{v}(0)\|^2_{k,\Omega}+\|\partial_t^i\overline{H}(0)\|^2_{k,\Pi}).
$$

Moreover, if $\varphi(0)$, B are sufficiently small then we get

$$
\begin{aligned}
(2.7)\quad & \|\overline{v}_t\|^2_{1,2,\infty,\Omega^T} + \|\overline{v}\|^2_{1,2,\infty,\Omega^T} + \|\overline{v}\|^2_{3,2,2,\Omega^T} + \|\overline{v}_t\|^2_{2,2,2,\Omega^T} + \|\overline{v}_{tt}\|^2_{1,2,2,\Omega^T} \\
& + \|\overline{p}\|^2_{2,2,2,\Omega^T} + \|\overline{p}_t\|^2_{1,2,2,\Omega^T} + \|\overline{H}_t\|^2_{1,2,\infty,\Pi^T} + \|\overline{H}\|^2_{1,2,\infty,\Pi^T} \\
& + \|\overline{H}\|^2_{3,2,2,\Pi^T} + \|\overline{H}_t\|^2_{2,2,2,\Pi^T} + \|\overline{H}_{tt}\|^2_{1,2,2,\Pi^T} \le c\varphi(0) + B.
\end{aligned}
$$

At first, in Section 3, we derive differential inequality (3.23) which makes possible an extension of the local solution of (1.1)–(1.7) step by step from interval $[0,T]$ to $[0,+\infty)$. In Section 4 we show Korn type inequalities which are necessary to prove inequality (3.23). In Section 5 we show the Main Theorem

THEOREM 2.3 (Main Theorem). *Assume that $f, f_t \in H^1(\Omega_t)$, $H_* \in H^3(B)$, $H_{*t} \in H^2(B)$, $H_{*tt} \in H^1(B)$, $(v(0), p(0), H(0)) \in \mathcal{N}(0)$, $b/2 \le \varphi(0)T^\gamma \le b$, $\varphi(0) \le \varepsilon_1$, where $b, \varepsilon_1 > 0$ are sufficiently small, $\gamma > 0$ same constant and $S, B \in H^{5/2}$. Then there exist a global solution of* (1.1)–(1.7) *such that*

$$(v(t), p(t), H(t)) \in \mathcal{M}(t), \quad t \in \mathbb{R}_+.$$

In Lemmas 3.1–3.12 we use

LEMMA 2.4. *For solution problem* (1.1)–(1.7) *we get*

$$
\begin{aligned}
\|H_t\|^2_{1,\Omega} &\le \alpha(a)(\|H_t\|^2_{1,\Omega_t} + \|H\|^2_{2,\Omega_t}\|v\|^2_{2,\Omega_t}), \\
\|\overline{H}\|^2_{i,\Omega} &\le \alpha(a)\|H\|^2_{i,\Omega_t}, \quad i = 1,2,3, \\
\|\overline{v}\|^2_{i,\Omega} &\le \alpha(a)\|v\|^2_{i,\Omega_t}, \quad i = 1,2,3, \\
\|\overline{H}_t\|^2_{2,\Omega} &\le \alpha(a)[\|H_t\|^2_{2,\Omega_t} + \|H\|^2_{3,\Omega_t}\|v\|^2_{2,\Omega_t} + \|H\|^2_{2,\Omega_t}\|v\|^2_{3,\Omega_t}], \\
\|\overline{v}_t\|^2_{2,\Omega_t} &\le \alpha(a)[\|v_t\|^2_{2,\Omega_t} + \|v\|^2_{3,\Omega_t}\|v\|^2_{2,\Omega_t}], \\
\|\overline{H}_{tt}\|^2_{0,\Omega} &\le \alpha(a)[\|H_{tt}\|^2_{0,\Omega_t} + \|v\|^2_{2,\Omega_t}(\|H_t\|^2_{1,\Omega_t} \\
&\quad + \|H\|^2_{2,\Omega_t}) + \|v_t\|^2_{1,\Omega_t}\|H\|^2_{2,\Omega_t} + \|H\|^2_{2,\Omega_t}\|v\|^4_{2,\Omega_t}], \\
\|\overline{v}_{tt}\|^2_{0,\Omega} &\le \alpha(a)[\|v_{tt}\|^2_{0,\Omega_t} + \|v\|^2_{2,\Omega_t}(\|v_t\|^2_{1,\Omega_t} + \|v\|^2_{2,\Omega_t}) + \|v\|^6_{2,\Omega_t}], \\
\|\overline{H}_{tt}\|^2_{1,\Omega} &\le \alpha(a)[\|H_{tt}\|^2_{1,\Omega_t} + \|v\|^2_{2,\Omega_t}(\|H_t\|^2_{2,\Omega_t} \\
&\quad + \|H\|^2_{3,\Omega_t}) + \|H\|^2_{3,\Omega_t}\|v_t\|^2_{2,\Omega_t} + \|H\|^2_{2,\Omega_t}\|v\|^4_{2,\Omega_t}], \\
\|\overline{v}_{tt}\|^2_{1,\Omega} &\le \alpha(a)[\|v_{tt}\|^2_{1,\Omega_t} + \|v\|^2_{2,\Omega_t}(\|v_t\|^2_{2,\Omega_t} + \|v\|^2_{3,\Omega_t}) + \|v\|^6_{2,\Omega_t}],
\end{aligned}
$$

where $a = \sqrt{t}\|\overline{v}\|_{3,2,2,\Omega^t}$ and α an increasing positive function.

PROOF. Differentiating $\overline{H}(\xi,t) = H(x(\xi,t),t)$ with respect to t and ξ we get $\overline{H}_t = H_x v + H_t$ and $\overline{H}_\xi = H_x x_\xi$. Then

$$
\begin{aligned}
\|\overline{H}_t\|^2_{1,\Omega} &\le \|H_x(x(\xi,t),t)v(x(\xi,t),t)\|^2_{1,\Omega} + \alpha(a)\|H_t\|^2_{1,\Omega_t} \\
&\le c\|H_x(x(\xi,t),t)\|^2_{1,\Omega}\|v(x(\xi,t),t)\|^2_{2,\Omega} + \alpha(a)\|H_t\|^2_{1,\Omega_t} \\
&\le \alpha(a)(\|H\|^2_{2,\Omega_t}\|v\|^2_{2\Omega_t} + \|H_t\|^2_{1,\Omega_t}).
\end{aligned}
$$

Hence the first inequality is proved. Similarly we can show another inequalities.$\square$

In Lemmas 3.7, 3.8, 3.10, 3.11 we are using

Lemma 2.5. *For solution problem* (1.1)–(1.7) *we get*

$$\|\overline{v}_t\|^2_{1,\Omega} \ge \inf_{\xi\in\Omega} x^2_\xi \|v_t\|^2_{1,\Omega_t} + 2\int_\Omega (v_{tx}x_\xi) b d\xi + \int_\Omega b^2 d\xi, \tag{2.8}$$

where $b = v_{xx}vx_\xi + v^2_x x_\xi$.

Proof. From $\overline{v}_{t\xi} = v_{tx}x_\xi + v_{xx}vx_\xi + v^2_x x_\xi$ we get

$$\int_\Omega \overline{v}^2_{t\xi}\, d\xi = \int_\Omega (v_{tx}x_\xi)^2\, d\xi + 2\int_\Omega (v_{tx}x_\xi) b\, d\xi + \int_\Omega b^2\, d\xi,$$

where $b = v_{xx}vx_\xi + v^2_x x_\xi$. Then we obtain (2.8). Similary we obtain inequality for

$$\|\overline{v}\|^2_{i,\Omega};\ \|\overline{H}\|^2_{i,\Omega}, \quad i = 1,2,3; \quad \|\overline{p}\|^2_{2,\Omega};\ \|\overline{v}_t\|^2_{2,\Omega};\ \|\overline{p}_t\|^2_{1,\Omega};\ \|\overline{H}_t\|^2_{2,\Omega}. \qquad \square$$

In Lemmas 3.10, 3.11 we are using inequalities (3.16), (3.19); (3.20) in local coordinates z, connected with $\{\xi\}$ (see [3]).

3. Differential inequality

Assume that the existence of a sufficiently smooth local solution of problem (1.1)–(1.7) has been proved and

$$\overset{2}{H}|_B = 0 \quad \text{on } B; \quad n\cdot \mathbb{T}(v,p') = 0, \quad \text{on } \widetilde{S}^T, \quad \text{where } p' = p - p_0.$$

In this section we obtain a special differential inequality which enables us to prove a global solution.

Lemma 3.1. *For a sufficiently smooth solution* v, p', H *of* (1.1)–(1.7),

$$\frac{d}{dt}\|v\|^2_{0,\Omega_t} + \|v\|^2_{1,\Omega_t} \le c\,(\|\overset{1}{H}\|^4_{1,\Omega_t} + \|v\|^2_{2,\Omega_t} + \|f\|^2_{0,\Omega_t}). \tag{3.1}$$

Proof. Multipling $(1.1)_1$ by v and integrating over Ω_t we get

$$\begin{aligned}\frac{1}{2}\int_{\Omega_t} \partial_t v^2\, dx + \int_{\Omega_t} v\nabla v v\, dx + \int_{\Omega_t} \mathbb{D}^2(v)\, dx \\ - \mu_1 \int_{\Omega_t} \overset{1}{H}\,\nabla\,\overset{1}{H}\, v\, dx + \mu_1\int_{\Omega_t} \nabla\,\overset{1}{H}{}^2 v\, dx = \int_{\Omega_t} fv\, dx.\end{aligned} \tag{3.2}$$

Using

$$\int_{\Omega_t} vv_t\, dx = \frac{1}{2}\frac{d}{dt}\int_{\Omega_t} v^2\, dx - \int_{\Omega_t} v\nabla v v\, dx$$

we get (3.1). $\square$

Lemma 3.2. *For a sufficiently smooth solution v, p', H of* (1.1)–(1.7),

$$\frac{d}{dt}\|v_t\|^2_{0,\Omega_t}+\|v_t\|^2_{1,\Omega_t}\le c(\|v_t\|^2_{1,\Omega_t}\|v\|^2_{1,\Omega_t}+\|\overset{1}{H}_t\|^2_{1,\Omega_t}\|\overset{1}{H}\|^2_{1,\Omega_t} + (\|v\|^2_{1,\Omega_t}+\|H\|^2_{1,\Omega_t})^2+\|f_t\|^2_{0,\Omega_t})=X_1. \tag{3.3}$$

Proof. Differentiating $(1.1)_1$ with respect to t, multipling by v_t and integrating over Ω_t we get

$$\frac{1}{2}\int_{\Omega_t}(v_t)^2_t\,dx+\int_{\Omega_t}v_t\nabla vv_t+\int_{\Omega_t}v\nabla v_tv_t\,dx+\int_{\Omega_t}\mathbb{D}^2(v_t)\,dx - \mu_1\int_{\Omega_t}(\overset{1}{H}\nabla\overset{1}{H})_tv_t\,dx+\mu_1\int_{\Omega_t}(\nabla\overset{1}{H}{}^2)_tv_t\,dx=\int_{\Omega_t}f_tv_t\,dx. \tag{3.4}$$

Using

$$\int_{\Omega_t}v_tv_{tt}\,dx=\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}v_t^2\,dx-\int_{\Omega_t}v_t\nabla v_tv\,dx$$

and Lemma 3.1 we get (3.3). □

Lemma 3.3. *For a sufficiently smooth solution v, p', H of* (1.1)–(1.7)

$$\begin{aligned}\frac{d}{dt}\|v_{tt}\|^2_{0,\Omega_t}+\|v_{tt}\|^2_{1,\Omega_t}\le c[&\|v_{tt}\|^2_{1,\Omega_t}\|v\|^2_{2,\Omega_t}+\|v_t\|^4_{1,\Omega_t}\\&+\|\overset{1}{H}_{tt}\|^2_{0,\Omega_t}\|\overset{1}{H}\|^2_{1,\Omega_t}+\|\overset{1}{H}_t\|^4_{1,\Omega_t}\\&+(\|v_t\|^2_{1,\Omega_t}+\|v\|^2_{2,\Omega_t}(\|v\|^2_{1,\Omega_t}+1)\\&+\|\overset{1}{H}_t\|^2_{1,\Omega_t}+\|\overset{1}{H}\|^2_{1,\Omega_t})^2+\|f_{tt}\|^2_{0,\Omega_t}]=X_2.\end{aligned} \tag{3.5}$$

Proof. Differentiating $(1.1)_1$ two times with respect to t, multiplying by v_{tt}, integrating over Ω_t, using

$$\int_{\Omega_t}v_{tt}v_{ttt}\,dx=\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}(v_{tt})^2\,dx-\int_{\Omega_t}v_{tt}\nabla v_{tt}v\,dx,$$

and Lemma 3.2 we get (3.5). □

Lemma 3.4. *For a sufficiently smooth solution v, p', H of* (1.1)–(1.7)

$$\frac{d}{dt}\|H\|^2_{0,\Pi_t}+\|H\|^2_{1,\Pi_t}\le c\|H\|^2_{1,\Pi_t}\|v\|^2_{1,\Pi_t}. \tag{3.6}$$

Proof. Multiplying $(1.1)_{3,4}$ by H and integrating over Π_t, using

$$\int_{\Pi_t}HH_t\,dx=\frac{1}{2}\frac{d}{dt}\int_{\Pi_t}H^2\,dx-\int_{\Pi_t}H\nabla Hv\,dx,$$

we get (3.6). □

Lemma 3.5. *For a sufficiently smooth solution v, p', H of* (1.1)–(1.7)

$$\frac{d}{dt}\|H_t\|^2_{0,\Pi_t} + \|H_t\|^2_{1,\Pi_t} \le c(\|H_t\|^2_{1,\Pi_t}\|v\|^2_{1,\Pi_t} + \|v_t\|^2_{1,\Pi_t}\|H\|^2_{1,\Pi_t}). \tag{3.7}$$

Proof. Differentiating $(1.1)_{3,4}$ with respect to t multiplying by H_t, integrating over Π_t, using

$$\int_{\Pi_t} H_t H_{tt}\,dx = \frac{1}{2}\frac{d}{dt}\int_{\Pi_t}(H_t)^2\,dx - \int_{\Pi_t} H_t\nabla H_t v\,dx$$

we get (3.7). □

Lemma 3.6. *For a sufficiently smooth solution v, p', H of* (1.1)–(1.7)

$$\begin{aligned}\frac{d}{dt}\|H_{tt}\|^2_{0,\Pi_t} + \|H_{tt}\|^2_{1,\Pi_t} \le{}& c(\|H_{tt}\|^2_{1,\Pi_t}\|v\|^2_{1,\Pi_t} + \|v_{tt}\|^2_{1,\Pi_t}\|H\|^2_{1,\Pi_t})\\ &+ \|v_t\|^2_{1,\Pi_t}\|H_t\|^2_{1,\Pi_t}) = X_3.\end{aligned} \tag{3.8}$$

Proof. Differentiating $(1.1)_{3,4}$ two times with respect to t multiplying by H_{tt}, integrating over Π_t, using

$$\int_{\Pi_t} H_{tt}H_{ttt}\,dx = \frac{1}{2}\frac{d}{dt}\int_{\Pi_t}(H_{tt})^2\,dx - \int_{\Pi_t} H_{tt}\nabla H_{tt}v\,dx,$$

we get (3.8). □

Lemma 3.7. *For a sufficiently smooth solution v, p', H of* (1.1)–(1.7)

$$\begin{aligned}&\frac{d}{dt}\|v_t\|^2_{0,\Omega_t} + \frac{d}{dt}\|v\|^2_{2,\Omega_t} + \|v\|^2_{3,\Omega_t} + \|p'\|^2_{2,\Omega_t}\\ &\quad\le c[\varphi(a)\|\overset{1}{H}\|^2_{1,\Omega_t}\|\overset{1}{H}\|^2_{3,\Omega_t} + \varepsilon(\|v_t\|^2_{2,\Omega_t}\\ &\qquad + \|v\|^2_{3,\Omega_t}) + \|v\|^2_{3,\Omega_t}\|v\|^2_{1,\Omega_t} + X_1 + \|f\|^2_{1,\Omega_t}] = X_4.\end{aligned} \tag{3.9}$$

Proof. From the inequalities (see [3])

$$\begin{aligned}&\|\overline{v}\|^2_{3,\Omega} + \|\overline{p}'\|^2_{2,\Omega} \le \varphi(a)\|\overline{H}\|^2_{1,\Omega}\|\overline{H}\|^2_{3,\Omega} + \|\overline{v}_t\|^2_{1,\Omega} + \|\overline{f}\|^2_{1,\Omega},\\ &\frac{d}{dt}\|v\|^2_{2,\Omega_t} \le c(\|v\|^2_{2,\Omega_t} + \varepsilon\|v_t\|^2_{2,\Omega_t} + \|v\|^2_{3,\Omega_t}\|v\|^2_{1,\Omega_t})\end{aligned} \tag{3.10}$$

and (3.3) we get (3.9). □

Lemma 3.8. *For a sufficiently smooth solution v, p', H of* (1.1)–(1.7)

$$\begin{aligned}&\frac{d}{dt}\|v_t\|^2_{1,\Omega_t} + \frac{d}{dt}\|v_{tt}\|^2_{0,\Omega_t} + \frac{d}{dt}\|v_t\|^2_{0,\Omega_t} + \|v_t\|^2_{2,\Omega_t} + \|p'_t\|^2_{1,\Omega_t}\\ &\le c[\varphi(a)(\|v\|^2_{2,\Omega_t}(\|v\|^2_{3,\Omega_t} + \|p'\|^2_{2,\Omega_t} + \|\overset{1}{H}\|^4_{2,\Omega_t})\\ &\quad + \|\overset{1}{H}_t\|^2_{1,\Omega_t}\|\overset{1}{H}\|^2_{2,\Omega_t} + \|\overset{1}{H}_t\|^2_{2,\Omega_t}\|\overset{1}{H}\|^2_{1,\Omega_t})+\\ &\quad + \varepsilon(\|v_{tt}\|^2_{1,\Omega_t} + \|v_t\|^2_{2,\Omega_t}) + \|v_t\|^2_{2,\Omega_t}\|v\|^2_{1,\Omega_t} + X_1 + X_2 + \|f_t\|^2_{0,\Omega_t}].\end{aligned} \tag{3.11}$$

Proof. From the inequality (see [3])

$$
\begin{aligned}
\|\overline{v}_t\|^2_{2,\Omega}+\|\overline{p}'_t\|^2_{1,\Omega}\le{}&\varphi(a)[\|\overline{v}\|^2_{2,\Omega}(\|\overline{v}\|^2_{3,\Omega}+\|\overline{p}'\|^2_{2,\Omega}+\|\overset{1}{\overline{H}}\|^4_{2,\Omega})\\
&+\|\overset{1}{\overline{H}}_t\|^2_{1,\Omega}\|\overset{1}{\overline{H}}\|^2_{2,\Omega}+\|\overset{1}{\overline{H}}_t\|^2_{2,\Omega}\|\overset{1}{\overline{H}}\|^2_{1,\Omega}]\\
&+\|\overline{v}_{tt}\|^2_{0,\Omega}+\|\overline{f}_t\|^2_{0,\Omega},\\
\frac{d}{dt}\|v_t\|^2_{1,\Omega_t}\le{}&\|v_t\|^2_{1,\Omega_t}+\varepsilon\|v_{tt}\|^2_{1,\Omega_t}+\|v_t\|^2_{2,\Omega_t}\|v\|^2_{1,\Omega_t}
\end{aligned}
\tag{3.12}
$$

and (3.3), (3.5) we get (3.11). □

Lemma 3.9. *For a sufficiently smooth solution v, p', H of* (1.1)–(1.7)

$$
\begin{aligned}
&\frac{d}{dt}\|v\|^2_{1,\Omega_t}+\frac{d}{dt}\|v_t\|^2_{0,\Omega_t}+\|v\|^2_{2,\Omega_t}\\
&\quad\le c[\varphi(a)(\|\overset{1}{H}\|^4_{1,\Omega_t}+\|\overset{1}{H}\|^2_{0,\Omega_t}\|\overset{1}{H}\|^2_{2,\Omega_t})\\
&\qquad+\varepsilon(\|v_t\|^2_{1,\Omega_t}+\|v\|^2_{2,\Omega_t})+\|v\|^2_{2,\Omega_t}\|v\|^2_{0,\Omega_t}+X_1+\|f\|^2_{0,\Omega_t}].
\end{aligned}
\tag{3.13}
$$

Proof. From the inequality (see [3])

$$
\begin{aligned}
\|\overline{v}\|^2_{2,\Omega_t}&\le\varphi(a)(\|\overset{1}{\overline{H}}\|^4_{1,\Omega}+\|\overset{1}{\overline{H}}\|^2_{0,\Omega}\|\overset{1}{\overline{H}}\|^2_{2,\Omega})+\|\overline{v}_t\|^2_{0,\Omega}+\|\overline{f}\|^2_{0,\Omega},\\
\frac{d}{dt}\|v\|^2_{1,\Omega_t}&\le\|v\|^2_{1,\Omega_t}+\varepsilon\|v_t\|^2_{1,\Omega_t}+\|v\|^2_{2,\Omega_t}\|v\|^2_{0,\Omega_t}
\end{aligned}
\tag{3.14}
$$

and (3.3), we get (3.13). □

Lemma 3.10. *For a sufficiently smooth solution v, p', H of* (1.1)–(1.7)

$$
\begin{aligned}
&\frac{d}{dt}\|H_t\|^2_{2,\Pi_t}+\frac{d}{dt}\|H_{tt}\|^2_{0,\Pi_t}+\frac{d}{dt}\|v_t\|^2_{0,\Omega_t}+\frac{d}{dt}\|v\|^2_{2,\Omega_t}+\|H_t\|^2_{2,\Pi_t}\\
&\quad\le c\varphi(a)[\|H_t\|^2_{2,\Pi_t}\|v\|^2_{1,\Pi_t}\\
&\qquad+\|H\|^2_{2,\Pi_t}(\|v\|^2_{3,\Pi_t}+\|v_t\|^2_{2,\Pi_t}+\|v\|^4_{2,\Pi_t})\\
&\qquad+\|v\|^2_{3,\Pi_t}(\|H_t\|^2_{1,\Pi_t}+\|H\|^2_{2,\Pi_t}\|v\|^2_{2,\Pi_t}+\|v\|^2_{2,\Pi_t}+\|v_t\|^2_{1,\Pi_t})\\
&\qquad+a^2\|H\|^2_{3,\Pi_t}+X_3+X_4]=X_5.
\end{aligned}
\tag{3.15}
$$

Proof. From inequality (see [3])

$$
\begin{aligned}
\frac{d}{dt}\|\widetilde{H}_{t\tau}\|^2_{0,\widehat{\Omega}}+\|\widetilde{H}_t\|^2_{2,\widehat{\Omega}}\le{}&\varphi(\widehat{a})[\|\widetilde{H}_{tt}\|^2_{0,\widehat{\Omega}}+\|\widehat{v}\|^2_{3,\widehat{\Omega}}(\|\widehat{H}\|^2_{1,\widehat{\Omega}}+\|\widehat{H}_t\|^2_{1,\widehat{\Omega}}\\
&+\|\widehat{v}_t\|^2_{1,\widehat{\Omega}}+\|\widehat{H}\|^2_{2,\widehat{\Omega}}\|\widehat{v}\|^2_{2,\widehat{\Omega}}\|\widehat{v}\|^2_{2,\widehat{\Omega}}+1)\\
&+\|\widehat{v}_t\|^2_{2,\widehat{\Omega}}+\|\widehat{H}\|^2_{2,\widehat{\Omega}}\|\widehat{v}_t\|^2_{2,\widehat{\Omega}}],\\
\frac{d}{dt}\|\widetilde{H}_{\tau t}\|^2_{0,\widehat{\Omega}}\le{}&c(\varepsilon\|\widehat{H}_t\|^2_{1,\widehat{\Omega}}+\|\widehat{H}_{tt}\|^2_{1,\widehat{\Omega}}),\\
\frac{d}{dt}\|H_t\|^2_{1,\Pi_t}\le{}&c(\varepsilon\|H_t\|^2_{1,\Pi_t}+\|H_{tt}\|^2_{1,\Pi_t}+\|H_t\|^2_{2,\Pi_t}\|v\|^2_{1,\Pi_t}),
\end{aligned}
\tag{3.16}
$$

we get

$$
(3.17)\quad \begin{aligned}
&\frac{d}{dt}\|H_t\|^2_{1,\Pi_t} + \|H_t\|^2_{2,\Pi_t}\\
&\quad\le \varphi(a)[\|H_{tt}\|^2_{1,\Pi_t} + \|H_t\|^2_{2,\Pi_t}\|v\|^2_{1,\Pi_t}\\
&\qquad + \|H\|^2_{2,\Pi_t}(\|v\|^2_{3,\Pi_t} + \|v_t\|^2_{2,\Pi_t} + \|v\|^4_{2,\Pi_t})\\
&\qquad + \|v\|^2_{3,\Pi_t}(\|H_t\|^2_{1,\Pi_t} + \|H\|^2_{2,\Pi_t}\|v\|^2_{2,\Pi_t} + \|v\|^2_{2,\Pi_t} + \|v_t\|^2_{2,\Pi_t})\\
&\qquad + \|v\|^2_{2,\Pi_t} + a^2\|H\|^2_{3,\Pi_t}],
\end{aligned}
$$

using (3.8), (3.9) we get (3.15). □

LEMMA 3.11. *For a sufficiently smooth solution v, p', H of* (1.1)–(1.7)

$$
(3.18)\quad \begin{aligned}
&\frac{d}{dt}\|H\|^2_{2,\Pi_t} + \frac{d}{dt}\|H_{tt}\|^2_{0,\Pi_t} + \frac{d}{dt}\|v_t\|^2_{0,\Omega_t} + \frac{d}{dt}\|v\|^2_{2,\Omega_t}\\
&\quad + \|H_t\|^2_{2,\Pi_t} + \|H\|^2_{3,\Pi_t} \le c\varphi(a)[\|H\|^2_{2,\Pi_t}\|v\|^2_{2,\Pi_t}\\
&\qquad + \|H\|^2_{2,\Pi_t}\|v\|^2_{3,\Pi_t} + \|H\|^2_{3,\Pi_t}\|v\|^2_{2,\Pi_t} + X_4 + X_5] = X_6.
\end{aligned}
$$

PROOF. From inequalities (see [3])

$$
(3.19)\quad \begin{aligned}
&\frac{d}{dt}\|\widetilde{H}_\tau\|^2_{0,\widehat{\Omega}} + \|\widetilde{H}\|^2_{2,\widehat{\Omega}}\\
&\quad\le \varphi_1(\widehat{a})[\|\widehat{H}\|^2_{1,\widehat{\Omega}}\|\widehat{v}\|^2_{2,\widehat{\Omega}} + \|\widehat{v}\|^2_{2,\widehat{\Omega}} + \|\widehat{H}\|^2_{1,\widehat{\Omega}} + \|\widehat{H}_t\|^2_{0,\widehat{\Omega}} + \widehat{a}^2\|\widehat{H}\|^2_{3,\widehat{\Omega}}],
\end{aligned}
$$

$$
(3.20)\quad \begin{aligned}
\frac{d}{dt}\|\widetilde{H}_{\tau\tau}\|^2_{0,\widehat{\Omega}} + \|\widetilde{H}\|^2_{3,\widehat{\Omega}} &\le \varphi_2(\widehat{a})[\|\widehat{H}\|^2_{2,\widehat{\Omega}}\|\widehat{v}\|^2_{2,\widehat{\Omega}} + \|\widehat{v}\|^2_{2,\widehat{\Omega}} + \|\widehat{H}\|^2_{2,\widehat{\Omega}}\\
&\quad + \|\widehat{H}\|^2_{1,\widehat{\Omega}}\|\widehat{v}\|^2_{3,\widehat{\Omega}} + \|\widehat{H}_t\|^2_{1,\widehat{\Omega}}],\\
\frac{d}{dt}\|\widetilde{H}_\tau\|^2_{0,\widehat{\Omega}} &\le c(\varepsilon\|\widehat{H}\|^2_{1,\widehat{\Omega}} + \|\widehat{H}_t\|^2_{1,\widehat{\Omega}}),\\
\frac{d}{dt}\|\widetilde{H}_{\tau\tau}\|^2_{0,\widehat{\Omega}} &\le c(\varepsilon\|\widehat{H}\|^2_{2,\widehat{\Omega}} + \|\widehat{H}_t\|^2_{2,\widehat{\Omega}}),
\end{aligned}
$$

we get

$$
(3.21)\quad \begin{aligned}
\|H\|^2_{3,\Pi_t} &\le c\varphi_3(a)[\|H\|^2_{2,\Pi_t}\|v\|^2_{2,\Pi_t} + \|v\|^2_{2,\Pi_t}\\
&\quad + \|H\|^2_{2,\Pi_t}\|v\|^2_{3,\Pi_t} + \|H_t\|^2_{2,\Pi_t} + \|H\|^2_{3,\Pi_t}\|v\|^2_{2,\Pi_t}].
\end{aligned}
$$

Then from the inequality

$$
\frac{d}{dt}\|H\|^2_{2,\Pi_t} \le \varepsilon\|H\|^2_{2,\Pi_t} + \|H_t\|^2_{2,\Pi_t} + \|H\|^2_{3,\Pi_t}\|v\|^2_{1,\Pi_t}
$$

and (3.9), (3.15) we get (3.18). □

Lemma 3.12. *For a sufficiently smooth solution v, p', H of* (1.1)–(1.7)

$$\frac{d}{dt}\|H\|^2_{1,\Pi_t}+\frac{d}{dt}\|H\|^2_{2,\Pi_t}+\frac{d}{dt}\|H_{tt}\|^2_{0,\Pi_t} + \frac{d}{dt}\|v_t\|^2_{0,\Omega_t}+\frac{d}{dt}\|v\|^2_{2,\Omega_t}+\|H_t\|^2_{2,\Pi_t}+\|H\|^2_{3,\Pi_t}\le X_6. \tag{3.22}$$

Proof. From the inequality

$$\frac{d}{dt}\|H\|^2_{1,\Pi_t}\le\|H\|^2_{1,\Pi_t}+\varepsilon\|H_t\|^2_{1,\Pi_t}+\|H\|^2_{2,\Pi_t}\|v\|^2_{0,\Pi_t}$$

and from (3.18) we get (3.22). □

Now let $\overset{2}{H}=H_*$, on B, then from Lemmas 3.1–3.12 we get

Lemma 3.13. *For a sufficiently smooth solution v, p, H of* (1.1)–(1.7)

$$\frac{d}{dt}\varphi+\phi\le c[\gamma(a)\phi\varphi(1+\varphi)+\|f\|^2_{1,\Omega_t}+\|f_t\|^2_{1,\Omega_t}+\|H_*\|^2_{3,B} \\ \|H_*\|^4_{1,B}+\|H_{*t}\|^2_{2,B}(1+\|H_{*t}\|^2_{2,B})+\|H_{*tt}\|^2_{1,B}], \tag{3.23}$$

where γ is an increasing positive function,

$$\begin{aligned}\varphi(t)&=\sum_{i+k\le 2}(\|\partial_t^i v\|^2_{k,\Omega_t}+\|\partial_t^i H\|^2_{k,\Pi_t}),\\ \phi(t)&=\sum_{\substack{i+k\le 3\\ i\le 2}}(\|\partial_t^i v\|^2_{k,\Omega_t}+\|\partial_t^i H\|^2_{k,\Omega_t})+\|p'\|^2_{2,\Omega_t}+\|p'_t\|^2_{1,\Omega_t}.\end{aligned} \tag{3.24}$$

4. Korn inequality

Lemma 4.1. *Let $\Omega_t\subset\mathbb{R}^3$ be a bounded domain, v, p', H solution of* (1.1)–(1.7) *and*

$$\mathbb{E}_{\Omega_t}(v_t)=\int_{\Omega_t}(\partial_{x_i}v_t^j+\partial x_j v_t^i)^2\,dx<\infty. \tag{4.1}$$

Then there exists constant c such that

$$\|v_t\|^2_{1,\Omega_t}\le c(\mathbb{E}_{\Omega_t}(v_t)+\|v\|^4_{1,\Omega_t}+\|\overset{1}{H}\|^4_{1,\Omega_t}). \tag{4.2}$$

Proof. Introduce a function u by

$$u=\sum_{i=1}^3 b_i\varphi_i(x)+v_t, \tag{4.3}$$

where

$$\begin{aligned}\varphi_i&=(x-\overline{x})\times e_i,\\ \overline{x}&=\frac{1}{|\Omega_t|}\bigg(\int_{\Omega_t}x^1\,dx,\int_{\Omega_t}x^2\,dx,\int_{\Omega_t}x^3\,dx\bigg),\\ e_i&=(\delta_{i1},\delta_{i2},\delta_{i3})\quad i=1,2,3.\end{aligned} \tag{4.4}$$

Define $b = (b_1, b_2, b_3)$ by

$$b = \frac{1}{2|\Omega_t|} \int_{\Omega_t} \operatorname{rot} v_t \, dx. \tag{4.5}$$

Since $\operatorname{rot} \varphi_i = 2e_i$ $i = 1, 2, 3$, quations (4.3) and (4.4) imply

$$\int_{\Omega_t} \operatorname{rot} u \, dx = 0. \tag{4.6}$$

From (4.4) we have $\int_{\Omega_t} \varphi_i dx = 0$ for $i = 1, 2, 3$ so

$$\int_{\Omega_t} u dx = \int_{\Omega_t} v_t dx, \quad \text{and also} \quad \mathbb{E}_{\Omega_t}(\varphi_i) = 0, \quad i = 1, 2, 3, \tag{4.7}$$

so

$$\mathbb{E}_{\Omega_t}(u) = \mathbb{E}_{\Omega_t}(v_t). \tag{4.8}$$

By Theorem 1 of [6] we have

$$\partial x^j w_i = \varepsilon_{ikl} \partial x^k S_{jl}, \quad i = 1, 2, 3, \quad w = \operatorname{rot} u, \quad S_{ij} = \partial_{x^i} u^j + \partial_{x^j} u^i, \tag{4.9}$$

so by (4.6) and Lemma 2.4 of [5] it follows that

$$\|\operatorname{rot} u\|^2_{0,\Omega_t} \leq c \sum_{i,j=1}^{3} \|S_{ij}\|^2_{0,\Omega_t} = c \, \mathbb{E}_{\Omega_t}(u) = c\mathbb{E}_{\Omega_t}(v_t). \tag{4.10}$$

Employing the identity

$$\partial_{x^j} u_i = \frac{1}{2}(\partial_{x^j} u^i + \partial_{x^i} u^j) + \frac{1}{2}(\partial_{x^j} u^i - \partial_{x^i} u^j)$$

and (3.10) we have

$$\|\nabla u\|^2_{0,\Omega_t} \leq c(\mathbb{E}_{\Omega_t}(u) + \|\operatorname{rot} u\|^2_{0,\Omega_t}) \leq c\mathbb{E}_{\Omega_t}(u) = c\mathbb{E}_{\Omega_t}(v_t). \tag{4.11}$$

Using (4.3) we obtain

$$\|\nabla v_t\|^2_{0,\Omega_t} \leq c(\mathbb{E}_{\Omega_t}(v_t) + |b|). \tag{4.12}$$

Integrating $(1.1)_1$ over Ω_t we get

$$\int_{\Omega_t} v_t \, dx = -\int_{\Omega_t} v \nabla v \, dx + \int_{\Omega_t} \mu_1 (\overset{1}{H} \nabla \overset{1}{H} - \nabla \overset{1}{H}{}^2) \, dx \tag{4.13}$$

and multiplying $(1.1)_1$ by φ_i, $i = 1, 2, 3$ and integrating over Ω_t from (4.3) we get systems of equations

$$\begin{aligned} \sum_{i=1}^{3} b_i \int_{\Omega_t} \varphi_i \cdot \varphi_j \, dx \\ = \int_{\Omega_t} u \varphi_j \, dx + \int_{\Omega_t} v \nabla v \, dx + \int_{\Omega_t} \mu_1 (\overset{1}{H} \nabla \overset{1}{H} - \nabla \overset{1}{H}{}^2) \, dx, \end{aligned} \tag{4.14}$$

for $j = 1,2,3$. Since $\det\Gamma \neq 0$, where $\Gamma = \{\Gamma_{ij}\}$, $\Gamma_{ij} = \int_{\Omega_t} \varphi_i\varphi_j\,dx$, we can calculate b from (4.14), so

$$|b|^2 \le c(\|u\|^2_{0,\Omega_t} + \|v\|^4_{1,\Omega_t} + \|\overset{1}{H}\|^4_{1,\Omega_t}). \tag{4.15}$$

Now by Poincare inequality and (4.7), (4.8) we obtain

$$\begin{aligned}\|u\|^2_{0,\Omega_t} &\le 2\Big\|u - \frac{1}{|\Omega_t|}\int_{\Omega_t} u\,dx\Big\|^2_{0,\Omega_t} + 2\Big\|\frac{1}{|\Omega_t|}\int_{\Omega_t} u\,dx\Big\|^2_{0,\Omega_t} \\ &\le c\bigg(\|\nabla u\|^2_{0,\Omega_t} + \bigg\|\frac{1}{|\Omega_t|}\int_{\Omega_t} v_t\,dx\bigg\|^2_{0,\Omega_t}\bigg) \\ &\le c(\mathbb{E}_{\Omega_t}(u) + \|v\|^4_{1,\Omega_t} + \|\overset{1}{H}\|^4_{1,\Omega_t}) \\ &= c(\mathbb{E}_{\Omega_t}(v_t) + \|v\|^4_{1,\Omega_t} + \|\overset{1}{H}\|^4_{1,\Omega_t}).\end{aligned} \tag{4.16}$$

Using (4.15) and (4.16) in (4.12) we obtain (4.2). □

Lemma 4.2. *Let $\Omega_t \subset \mathbb{R}^3$ be a bounded domain, v, p', H solution of* (1.1)–(1.7) *and*

$$\mathbb{E}_{\Omega_t}(v_{tt}) = \int_{\Omega_t} (\partial_{x^i} v^j_{tt} + \partial_{x^j} v^i_{tt})^2\,dx < \infty. \tag{4.17}$$

Then there exists constant c such that

$$\begin{aligned}\|v_{tt}\|^2_{1,\Omega_t} \le c[&\mathbb{E}_{\Omega_t}(v_t) + \|v_t\|^2_{1,\Omega_t}\|v\|^2_{1,\Omega_t} + \|v\|^4_{2,\Omega_t}(\|v\|^2_{1,\Omega_t} + 1) \\ &+ \|\overset{1}{H}\|^2_{2,\Omega_t}(\|\overset{1}{H}_t\|^2_{1,\Omega_t} + \|v\|^2_{1,\Omega_t} + \|\overset{1}{H}\|^2_{1,\Omega_t})].\end{aligned} \tag{4.18}$$

Proof. Let $u = \sum_{i=1}^3 b_i\varphi_i(x) + v_{tt}$, where φ_i are described by(4.4). The rest of the argument is as in Lemma 4.1. □

5. Global existence

To prove global existence we introduce the spaces

$$\begin{aligned}\mathcal{N}(t) &= \{(v,p',H) : \varphi(t) < \infty\}, \\ \mathcal{M}(t) &= \bigg\{(v,p',H) : \varphi(t) + \int_0^t \phi(t)\,dt < \infty\bigg\},\end{aligned}$$

where $\varphi(t), \phi(t)$ be defined by (3.24). From Theorem 2.2 we get

Lemma 5.1. *Let $(v(0),p'(0),H(0)) \in \mathcal{N}(0)$ and $\varphi(0) < \varepsilon_1$. Then $(v(t),p'(t),H(t)) \in \mathcal{M}(t)$, $t < T$ where T is the time of local existence and*

$$\begin{aligned}\varphi(t) + \int_0^T \phi(\tau)\,d\tau \le c\varepsilon_1 &+ \|E_*\|^2_{0,2,2,B_T} + \|E_{*t}\|^2_{0,2,2,B_T} \\ &+ \|H_*\|^2_{3,2,2,B_T} + \|H_{*t}\|^2_{2,2,2,B_T} + \|H_{*tt}\|^2_{0,2,2,B_T} \\ &+ \|f_t\|^2_{0,2,2,\Omega_T} + \|f\|^2_{1,2,2,\Omega_T} \equiv \beta + c\varepsilon_1.\end{aligned} \tag{5.1}$$

PROOF. From inequalities

$$\begin{aligned}\|v\|^2_{2,\Omega} &\le \varepsilon\|\overline{v}_t\|^2_{2,2,2,\Omega^t} + c(\varepsilon)\|\overline{v}\|^2_{2,2,2,\Omega^t} + \|\overline{v}(0)\|^2_{2,\Omega},\\ \|\overline{H}\|^2_{2,\Pi} &\le \varepsilon\|\overline{H}_t\|^2_{2,2,2,\Pi^t} + c(\varepsilon)\|\overline{H}\|^2_{2,2,2,\Pi^t} + \|\overline{H}(0)\|^2_{2,\Pi},\end{aligned}$$

and Theorem 1.1 we get (5.1). □

LEMMA 5.2. *Assume that there exist a local solution of* (1.1)–(1.7) *in* $\mathcal{M}(t)$, $0 \le t \le T$ *with initial data in* $\mathcal{N}(0)$ *sufficiently small and*

$$\begin{aligned}\alpha(t) = \|f\|^2_{1,\Omega_t} + \|f_t\|^2_{1,\Omega_t} + \|H_*\|^2_{3,B} + \|H_*\|^4_{1,B}&\\ + \|H_{*t}\|^2_{2,B}(1 + \|H_{*t}\|^2_{2,B}) + \|H_{*tt}\|^2_{1,B} \le ce^{-\mu t},&\end{aligned} \tag{5.2}$$

$0 \le t \le T$, *where* $\mu > 1/2$. *Then*

$$\varphi(t) \le ce^{-t/2}\left(\varphi(0) + \frac{1}{\mu - 1/2}\right). \tag{5.3}$$

PROOF. From (3.23) and (5.2) we get

$$\frac{d}{dt}\varphi + \phi \le c(\gamma(a)\phi\varphi(1+\varphi) + e^{-\mu t}). \tag{5.4}$$

From Lemma 5.1 we have $c\gamma(a)\phi\varphi(1+\varphi) \le \phi/2$ if β is sufficiently small. Then from (5.4) we get

$$\frac{d}{dt}\varphi + \frac{1}{2}\phi \le ce^{-\mu t}. \tag{5.5}$$

We have $\varphi \le \phi$ then from (5.5)

$$\frac{d}{dt}\varphi + \frac{1}{2}\varphi \le ce^{-\mu t}. \tag{5.6}$$

From (5.6) we get (5.3). □

LEMMA 5.3. *Let assumptions of Lemma* 5.2 *be satisfied and* $\varphi(0) < \varepsilon_1$. *Then* $\varphi(t) \le \varepsilon_1$ *for* $t \in [0, T]$, $T > 0$ *is the time of the local existence.*

PROOF. Let $T > 0$ be so large that

$$ce^{-T/2} \le \frac{1}{2}, \quad c\frac{1}{\mu - 1/2}e^{-T/2} \le \frac{1}{4}b\frac{1}{T^\gamma}$$

where $\gamma > 0$ and, from Theorem 2.2, $b/2 \le \varphi(0)T^\gamma < b$. Then we get

$$\varphi(T) \le ce^{-T/2}\varphi(0) + c\frac{1}{\mu - 1/2}e^{-T/2} \le \varphi(0).$$ □

Theorem 5.1. *Assume that* $f, f_t \in H^1(\Omega_t)$, $H_* \in H^3(B)$, $H_{*t} \in H^2(B)$, $H_{*tt} \in H^1(B)$, $(v(0), p'(0), H(0)) \in \mathcal{N}(0)$ *and*

$$\varphi(0) \leq \varepsilon_1, \tag{5.7}$$

where ε_1 *is sufficiently small. Assume also that* $S, B \in H^{5/2}$. *Then there exists a global solution of* (1.1)–(1.7) *such that*

$$(v(t), p'(t), H(t)) \in \mathcal{M}(t), \quad t \in \mathbb{R}_+.$$

Proof. The theorem is proved step by step using local existence in a fixed time interval. Under the assumptions that

$$(v(0), p'(0), H(0)) \in \mathcal{N}(0), \tag{5.8}$$

Theorem 5.1 and Lemma 5.1 yield local existence of solutions of (1.1)–(1.7).

By (5.8) and Lemma 5.1 implies that the local solution belongs to $\mathcal{M}(t)$, $t \leq T$. For small ε_1 the existence time T is correspondingly large, so we can assume it is a fixed positive number. To prove the last result we needed the Korn inequalities (see Section 4) and imbeedding theorems. The constants in those theorems depend on Ω_t and shape of S_t, so generally they are functions of t.

But in view of (5.1) with sufficiently small ε_1, β we obtain

$$\left| \int_0^t v \, d\tau \right| \leq c\beta, \quad t \in [0, T]. \tag{5.9}$$

Hence from the relation

$$x = \xi + \int_0^t v(x(\xi, \tau), \tau) \, d\tau, \quad \xi \in S, \ t \leq T, \tag{5.10}$$

for sufficiently small ε_1, β and fixed T, the shape of Ω_t, $t \leq T$ does not change too much, so the constants from the immbeding theorems can be chosen independent of time. Now we wish to extend the solution to the interval [T, 2T]. Using Lemma 5.3 we can prove the existence of local solution in $\mathcal{M}(t)$, $T \leq t \leq 2T$. To prove

$$\varphi(2T) \leq \varepsilon_1 \tag{5.11}$$

we need inequality (3.23) where the constants depend on the constants from the imbedding theorems and Korn inequalities for $t \in [T, 2T]$. Therefore we have to show that the shape of S_t, $t \leq 2T$, does not change more than for $t \leq T$. Assume that there exists a local solution in the interval $[0, kT]$. Then in view of

Lemma 5.2, we have for $t \in [0, kT]$

$$
\begin{aligned}
(5.12)\quad \left|\int_0^t v\,d\tau\right| + \left|\int_0^t H\,d\tau\right| &\le \int_0^t (\|v\|_{2,\Omega_\tau} + \|H\|_{2,\Pi_\tau})\,d\tau \\
&\le c_1 \sum_{i=0}^{k-1} \int_{iT}^{(i+1)T} (\|v\|_{2,\Omega_t} + \|H\|_{2,\Pi_t})\,dt \\
&\le c_1 T^{1/2} \sum_{i=0}^{k-1} \left[\left(\int_{iT}^{(i+1)T} \|v\|^2_{2,\Omega_t}\,dt\right)^{1/2} \left(\int_{iT}^{(i+1)T} \|H\|^2_{2,\Pi_t}\,dt\right)^{1/2}\right] \\
&\le c_1 T^{\frac{1}{2}} \sum_{i=0}^{k-1} \left(\int_{iT}^{(i+1)T} \varphi(t)\,dt\right)^{1/2} \\
&\le c_1 T^{1/2} \sum_{i=0}^{k-1} \sqrt{2}\left[\frac{1}{\mu - 1/2} e^{-iT\mu} + \varphi(iT)\right]^{1/2} \\
&\le \sqrt{2} c_1 T^{1/2} \left[\frac{e^{-T/2}}{1 - e^{-T/2}}\left(\frac{2}{\mu - 1/2} + \varphi(0)\right)\right]^{1/2}.
\end{aligned}
$$

Taking $k = 2$, ε_1, sufficiently small and T, μ sufficiently large we see that $\int_0^t v(x(\xi, t), t)\,dt$ is small for any $t \in [0, 2T]$, so (5.11) implies that the shape of S_t changes no more than in $[0, T]$, and then the differential inequality (3.23) can olso be shown for this interval with the same constans. Hence in view of Lemma 5.1 the solutions of (1.1)–(1.7) belongs to $\mathcal{M}(t)$, $t \in [T, 2T]$. Next Lemmas 5.1–5.3 imply (5.12).

Repeating the above considerations for the intervals $[kT, (k+1)T]$, $k \ge 2$, we prove the existence for all $t \in \mathbb{R}_+$ $\square$

Acknowledgments. The author thanks to Prof. W. Zajączkowski for very fruitfull discussions during preparation of this paper.

References

[1] O. V. Besov, V. P. Ilying and S. M. Nikol'skiĭ, *Integral Representations of Functions and Imbedding Theorems*, Nauka, Moscow, 1975.

[2] L. Landau and E. Lifschits, *Electrodynamics of Continuum Media*, Nauka, Moscov, 1957.

[3] P. Kacprzyk, *Local existence of solutions of the free boundary problem for the equations of magnetohydrodynamic incompresible fluid*, Appl. Math. (1677).

[4] ———, *Almost global existence of solutions of the free boundary problem for the equations of magnetohydrodynamic incompresible fluid*, Appl. Math. (1697).

[5] O. Ladyzhenskaya and V. A. Solonnikov, *On some problems of vector analysis and generalized formulations of boundary problems for Navier–Stokes equations*, Zap. Nauchn. Sem., LOMI **59** (1976).

[6] V. A. Solonnikov, *On an unsteady motion of an isolated volume of a viscous incompresible fluid*, Izv. Akad. Nauk SSSR Ser. Mat. **51** (1987).

[7] G. Ströhmer and W. M. Zajaczkowski, *Local existence of solutions of the free boundary problem for the equations of compresible barotropic viscous self-gravitationg fluids*, Appl. Math. **26** (1999), 1–31.

[8] W. M. Zajączkowski, *On nonstationary motion of a compresible barotropic viscous fluid bounded by a free surface*, Dissertationes Math. **324** (1993).

Manuscript received October 6, 2003

Piotr Kacprzyk
Institute of Mathematics
and Cryptology Cybernetics Faculty
Military University of Technology
S. Kaliskiego 2
00-908 Warsaw, POLAND

E-mail address: p.kacprzyk@imbo.wat.edu.pl