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EIGENVALUES AND BIFURCATION FOR ELLIPTIC EQUATIONS WITH MIXED DIRICHLET–NEUMANN BOUNDARY CONDITIONS RELATED TO CAFFARELLI–KOHN–NIRENBERG INEQUALITIES

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Abstract. This work deals with the analysis of eigenvalues, bifurcation and Hölder continuity of solutions to mixed problems like

$$\begin{cases} -\operatorname{div}(|x|^{-p\gamma}|\nabla u|^{p-2}\nabla u) = f_\lambda(x,u), & u > 0 \ \text{ in } \Omega, \\ u = 0 & \text{on } \Sigma_1, \\ |x|^{-p\gamma}|\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Sigma_2, \end{cases}$$

involving some potentials related with the Caffarelli–Kohn–Nirenberg inequalities, and with different kind of functions $f_\lambda(x,u)$.

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1. Introduction

In this paper we analyze the properties and behaviour of solutions to problem

$$(\mathrm{P}_\lambda) \qquad \begin{cases} -\Delta_{p,\gamma} u = f_\lambda(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ B(u) = 0 & \text{on } \partial\Omega, \end{cases}$$

where we assume $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain with $0 \in \Omega$, $\Delta_{p,\gamma} v = \operatorname{div}(|x|^{-p\gamma}|\nabla v|^{p-2}\nabla v)$, $1 < p < N$ and $-\infty < \gamma < (N-p)/p$, with different choices of f_λ. The boundary conditions are given by

$$B(u) = u\mathcal{X}_{\Sigma_1} + |x|^{-p\gamma}|\nabla u|^{p-2}\frac{\partial u}{\partial \nu}\mathcal{X}_{\Sigma_2},$$

where Σ_i, $i = 1, 2$, are smooth $(N-1)$-dimensional submanifolds of $\partial\Omega$ such that $\Sigma_1 \cap \Sigma_2 = \emptyset$, $\overline{\Sigma}_1 \cup \overline{\Sigma}_2 = \partial\Omega$ and $\overline{\Sigma}_1 \cap \overline{\Sigma}_2 = \Gamma$ is a smooth $(N-2)$-dimensional submanifold. We denote by ν the outward unitary normal to the boundary and by $\mathcal{X}_{\Sigma_i}$ the characteristic function of Σ_i.

Section 2 is devoted to study the functional framework needed to analyze these problems.

The main feature of this work is to consider these kind of mixed boundary problems with weights associated to Caffarelli–Kohn–Nirenberg inequalities [7]. Precisely, the function $f_\lambda(x,u)$ will be:

(1) $f_\lambda(x,u) = \lambda|u|^{p-2}u/|x|^{p\beta}$ with $\beta < (\gamma+1)$, $\lambda > 0$. In Section 3, we construct an eigenvalue sequence by minimax techniques in a similar way to García and Peral in [12], and we extend the classical properties to the first eigenvalue of the Dirichlet Problem with the Laplace operator to our framework, i.e. $\lambda_1(\beta)$ is positive, simple and isolated.

(2) $f_\lambda(x,u) = \lambda\omega_\beta(x)u^{s-1} + |x|^{-q\mu}u^{q-1}$, with different choices of the exponents s, q and $\beta \le (\gamma+1)$, $\mu < (\gamma+1)$, $\lambda > 0$, where

$$\omega_\beta(x) = \begin{cases} |x|^{-p\beta} & \text{if } x \in \Omega \cap \{|x| < 1\}, \\ |x|^{-p(\gamma+1)} & \text{if } x \in \Omega \cap \{|x| \ge 1\}. \end{cases} \tag{1.1}$$

If $s = p$, $1 < q < p$ we have two more cases: for $\beta, \mu < (\gamma+1)$ we prove bifurcation from infinity, precisely, there exists a branch of solutions $(\lambda, u_\lambda) \in (0, \lambda_1(\omega_\beta)) \times L^\infty(\Omega)$ to problem (P_λ) bifurcating to the left from infinity at the associated first eigenvalue $\lambda_1(\omega_\beta)$. As a consequence, we prove that for $\beta = (\gamma+1)$, $\mu \le (\gamma+1)$, there exists a continuum of solutions $(\lambda, u_\lambda) \in (0, \lambda_{\gamma,N,p}) \times E^{p,\gamma}_{\Sigma_1}(\Omega)$ bifurcating to the left from infinity at $\lambda_{\gamma,N,p}$ given as the best constant in the associated Hardy–Sobolev inequality. The study of (P_λ) in these cases is done in Section 4.

When $\beta, \mu < (\gamma + 1)$ and $s = p < q < p^*_{\gamma,\mu}$ where:

$$p^*_{\gamma,\mu} = \begin{cases} \dfrac{pN}{N-p} & \text{if } \mu \leq \gamma, \\ \dfrac{pN}{N - p(\gamma + 1 - \mu)} & \text{if } \gamma < \mu < \gamma + 1, \end{cases} \tag{1.2}$$

we prove bifurcation from zero. Precisely, there exists an unbounded branch of solutions $\Gamma_\beta \subset (0, \lambda_1(\omega_\beta)) \times L^\infty(\Omega)$, bifurcating to the left from $(\lambda_1(\omega_\beta), 0)$.

If we consider $1 < s < p < q < p^*_{\gamma,\mu}$ and $\beta \leq (\gamma+1)$, $\mu < (\gamma+1)$, we prove that there exists a continuum of solutions bifurcating from $(0, 0)$; in $L^\infty(\Omega)$ if $\beta < (\gamma + 1)$ and in $E^{p,\gamma}_{\Sigma_1}(\Omega)$ if $\beta = (\gamma + 1)$, see Theorem 4.8 and Remark 4.13.

Finally, in Section 5, we extend the classical results about global Hölder continuity by using the De Giorgi and Stampacchia techniques ([11], [22]) adapted to our framework.

2. Functional setting, boundedness and compactness

We will note the weighted Lebesgue space

$$L^r_\alpha(\Omega) = \left\{ u\colon \Omega \to \mathbb{R} \text{ measurable}, \ \int_\Omega |u|^r |x|^{-r\alpha}\, dx < \infty \right\}$$

for some $\alpha \in \mathbb{R}$. Given $\varphi \in \mathcal{C}^\infty(\Omega)$ we set

$$\|\varphi\|_{p,\gamma} = \left(\int_\Omega (|\varphi|^p + |\nabla\varphi|^p)|x|^{-p\gamma}\, dx \right)^{1/p}.$$

The Sobolev space $\mathcal{D}^{1,p}_\gamma(\Omega)$ is defined as the completion of $\mathcal{C}^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{p,\gamma}$. In a natural way, we define $\mathcal{D}^{1,p}_{0,\gamma}(\Omega)$ as the completion of $\mathcal{C}^\infty_0(\Omega)$ under the norm $\|\cdot\|_{p,\gamma}$. In the space $\mathcal{D}^{1,p}_{0,\gamma}(\Omega)$, the norm $\|\cdot\|_{p,\gamma}$ is equivalent to the norm of the gradient in $L^p_\gamma(\Omega)$, because of the Poincarè inequality.

The natural space where we look for solutions to (P_λ) is

$$E^{p,\gamma}_{\Sigma_1}(\Omega) = \{ v \in \mathcal{D}^{1,p}_\gamma(\Omega) : v = 0 \text{ on } \Sigma_1 \}, \tag{2.1}$$

which can also be defined as the closure of $\mathcal{C}^1_c(\Omega \cup \Sigma_2)$ with the norm $\|\cdot\|_{p,\gamma}$. Taking into account that $\mathcal{H}_{N-1}(\Sigma_1) \neq 0$, we have that, as before, the norm $\|\cdot\|_{p,\gamma}$ in $E^{p,\gamma}_{\Sigma_1}(\Omega)$ is equivalent to the norm of the gradient in $L^p_\gamma(\Omega)$. As a consequence, from now on we shall use the notation

$$\|\varphi\|^p_{E^{p,\gamma}_{\Sigma_1}(\Omega)} = \|\varphi\|^p_{p,\gamma} = \int_\Omega |x|^{-p\gamma} |\nabla\varphi|^p\, dx \quad \text{for all } \varphi \in E^{p,\gamma}_{\Sigma_1}(\Omega).$$

We start formulating an extension of a Picone Identity in [19], that will be useful in the sequel.

Lemma 2.1 (Picone). *Let $v > 0$, $u \geq 0$ two differentiable functions, then*

$$|x|^{-p\gamma}|\nabla u|^p \geq |x|^{-p\gamma}\nabla\left(\frac{u^p}{v^{p-1}}\right)|\nabla v|^{p-2}\nabla v. \tag{2.2}$$

There is a more general form of the Picone Identity in [2] for entropy solutions. We will use the following particular case of the Caffarelli–Kohn–Nirenberg inequalities (see [7]).

Proposition 2.2 (Caffarelli–Kohn–Nirenberg). *Let r, γ and β be real constants such that*

$$p \geq 1 \quad \textit{and} \quad r,\ \frac{1}{p} - \frac{\gamma}{N},\ \frac{1}{r} - \frac{\beta}{N} > 0. \tag{2.3}$$

Then there exists a positive constant C such that for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ we have

$$\||x|^{-\beta}u\|_{L^r} \leq C\||x|^{-\gamma}|\nabla u|\|_{L^p}, \tag{2.4}$$

if and only if

$$\frac{1}{r} - \frac{\beta}{N} = \frac{1}{p} - \frac{\gamma+1}{N} \quad \textit{and} \quad 0 \leq \beta - \gamma \leq 1. \tag{2.5}$$

Furthermore, on any compact set in parameter space in which (2.3) *and* (2.5) *hold, the constant C is bounded.*

We observe that in the limit cases we have for $\beta = \gamma$, $r = p^*$ the corresponding Sobolev inequality. And the case $\beta = (\gamma + 1)$, $r = p$ which corresponds to the Hardy inequality.

Remarks 2.3. We define the constant

$$\lambda_{\gamma,N,p} = \inf_{\substack{u \in \mathcal{D}^{1,p}_{0,\gamma}(\Omega) \\ u \not\equiv 0}} \frac{\||x|^\gamma|\nabla u|\|_p^p}{\||x|^{\gamma+1}u\|_p^p},$$

taking into account that $-\infty < \gamma < (N-p)/p$, we have that $0 < \lambda_{\gamma,N,p} = ((N - p(\gamma+1))/p)^p < \infty$. In the case of Neumann problem, it is clear that $\lambda_{\gamma,N,p} = 0$, because the infimum is attained by constant functions. When we take the infimum over the space $E^{p,\gamma}_{\Sigma_1}(\Omega)$, we have that the new constant denoted by $\lambda_{\gamma,N,p}(\Sigma_1)$ verifies $\lambda_{\gamma,N,p}(\Sigma_1) \leq \lambda_{\gamma,N,p}$ because of the inclusion $\mathcal{D}^{1,p}_{0,\gamma}(\Omega) \subset E^{p,\gamma}_{\Sigma_1}(\Omega)$. Moreover, $\lambda_{\gamma,N,p}(\Sigma_1) > 0$, to prove it we use the Picone identity as follows: consider the quotient

$$Q(u) = \frac{\||x|^\gamma|\nabla u|\|_p^p}{\||x|^{\gamma+1}u\|_p^p},$$

and we take the infimum of Q in $E^{p,\gamma}_{\Sigma_1}(\Omega)$. Define $\omega(x)=|x|^\alpha$ for some constant $\alpha<0$ with $|\alpha|$ small enough such that $|\alpha|(p-1)<N-p(\gamma+1)$. Given a minimizing sequence $\{v_n\}$ to $\lambda_{\gamma,N,p}(\Sigma_1)$, by Picone identity (2.2),

$$\begin{aligned}\int_\Omega|\nabla v_n|^p|x|^{-p\gamma}\,dx&\geq\int_\Omega|x|^{-p\gamma}\bigg\langle\nabla\bigg(\frac{v_n^p}{\omega^{p-1}}\bigg),|\nabla\omega|^{p-2}\nabla\omega\bigg\rangle\,dx\\&=\int_\Omega(-\Delta_{p,\gamma}\omega)\frac{v_n^p|x|^{-p(\gamma+1)}}{\omega^{p-1}|x|^{-p(\gamma+1)}}\,dx-\int_{\Sigma_2}|x|^{-p\gamma}\frac{v_n^p}{\omega^{p-1}}\bigg|\frac{\partial\omega}{\partial\nu}\bigg|\,d\sigma(x)\\&\geq c(c_0,\alpha)\int_\Omega\frac{v_n^p}{|x|^{p(\gamma+1)}}\,dx-c(\Sigma_2,c_0,\alpha)\int_\Omega|\nabla v_n|^p|x|^{-p\gamma}\,dx,\end{aligned}$$

where we have used the Trace Theorem in the last inequality. Then we conclude that $Q(v_n)\geq C>0$ where C depends only on Σ_2, c_0 and α.

NOTATION. Along this work we will note $|E|_m=\int_E|x|^m\,dx$, for any measurable set $E\subset\mathbb{R}^N$.

Before proving the L^∞-regularity we enunciate an iteration lemma by Stampacchia (see [23]) that we will use.

LEMMA 2.4. *Let φ be a real function verifying:*

(a) *$\varphi(t)\geq 0$ for all $t\geq 0$,*
(b) *φ is non-increasing,*
(c) *if $h>k>k_0$, $\varphi(h)\leq C(\varphi(k))^\nu/(h-k)^\kappa$, with positive constants κ, ν, C.*

Then, there exist positive constants s_0, s_1, and s_2 such that:

(a) *if $\nu>1$, $\varphi(k_0+d)=0$, where $d^\kappa=s_0C(\varphi(k_0))^{\nu-1}$,*
(b) *if $\nu=1$, $\varphi(h)\leq\varphi(k_0)\exp[s_1-\rho(h-k_0)]$, where $\rho=(s_2C)^{-1/\kappa}$,*
(c) *if $\nu<1$ and $k_0>0$, $\varphi(h)\leq(s_1C^{\kappa/(1-\nu)}+s_2k_0^{\kappa/(1-\nu)}\varphi(k_0))h^{-\kappa/(1-\nu)}$.*

THEOREM 2.5. *Let $u\in E^{p,\gamma}_{\Sigma_1}(\Omega)$ be a solution to problem*

$$\begin{cases}-\Delta_{p,\gamma}u=f & \text{in }\Omega,\\ B(u)=0 & \text{on }\partial\Omega,\end{cases}$$

with $-\infty<\gamma<(N-p)/p$, $f\in L^r_\eta(\Omega)$ for some $r>N/p$, and $\eta=-p^*\gamma(r-1)/r$. Then $u\in L^\infty(\Omega)$.

PROOF. Consider $v_k=\operatorname{sign}(u)(|u|-k)^+$, then $v_k\in E^{p,\gamma}_{\Sigma_1}(\Omega)$ and $u_{x_i}=(v_k)_{x_i}$ in $A(k)=\{x\in\Omega:|u(x)|>k\}$. Using v_k as test function in $-\Delta_{p,\gamma}u=f$ we obtain

$$\begin{aligned}&\int_{A(k)}|\nabla v_k|^p|x|^{-p\gamma}\,dx=\lambda\int_\Omega fv_k\,dx\\&\quad\leq\bigg(\int_{A(k)}v_k^{p^*}|x|^{-p^*\gamma}\,dx\bigg)^{1/p^*}\bigg(\int_\Omega|f|^r|x|^{(\gamma+\varepsilon)r}\,dx\bigg)^{1/r}\bigg(\int_{A(k)}\frac{dx}{|x|^{\varepsilon s}}\bigg)^{1/s},\end{aligned}\tag{2.6}$$

with $1/p^* + 1/r + 1/s = 1$, $\varepsilon > 0$. Then we conclude that

$$\bigg(\int_{A(k)} |v_k|^{p^*} |x|^{-p^*\gamma}\, dx\bigg)^{(p-1)/p^*} \leq C\bigg(\int_{A(k)} \frac{dx}{|x|^{\varepsilon s}}\bigg)^{1/s}.$$

Assume that $0 < k < h$, then $A(h) \subset A(k)$ and as a consequence,

$$(h-k)^{p-1}\bigg(\int_{A(h)} |x|^{-p^*\gamma}\, dx\bigg)^{(p-1)/p^*} \leq \bigg(\int_{A(k)} |v_k|^{p^*} |x|^{-p^*\gamma}\, dx\bigg)^{(p-1)/p^*}.$$

By the last two inequalities and taking $\varepsilon s = p^*\gamma$, we get

$$(h-k)^{p-1}|A(h)|_{-p^*\gamma}^{(p-1)/p^*} \leq C\mu |A(k)|_{-p^*\gamma}^{1/s},$$

defining $\phi(t) = |A(t)|_{-p^*\gamma}$ we have

$$\phi(h) \leq \frac{C}{(h-k)^{p^*}}[\phi(k)]^{p^*/(s(p-1))}.$$

Now we conclude by Lemma 2.4(a), because $\nu = p^*/(s(p-1)) > 1$ if and only if $r > N/p$, where $\varepsilon = p^*\gamma/s$ and $\gamma + \varepsilon = \gamma p^*(r-1)/r$. □

When the second member, f, is a power of u, we prove the next result.

Lemma 2.6. *Let $u \in E^{p,\gamma}_{\Sigma_1}(\Omega)$ be a solution to problem*

$$\begin{cases} -\Delta_{p,\gamma} u = |x|^{-q\mu}|u|^{q-2}u & \text{in } \Omega, \\ B(u) = 0 & \text{on } \partial\Omega, \end{cases}$$

with $q < p^*_{\gamma,\mu}$, $\mu < (\gamma+1)$. Then $u \in L^\infty(\Omega)$.

Proof. Following the notation of the proof to Theorem 2.5, taking v_k as a test function and the measure $d\varrho = dx/|x|^{p^*_{\gamma,\mu}\mu}$,

$$\begin{aligned}
(2.7)\quad \int_{A(k)} |\nabla v_k|^p |x|^{-p\gamma}\, dx &= \int_{A(k)} |u|^{q-2}uv_k|x|^{-q\mu}\, dx \\
&= \int_{A(k)} |u|^{q-2}uv_k|x|^{-q\mu}|x|^{p^*_{\gamma,\mu}\mu}\frac{dx}{|x|^{p^*_{\gamma,\mu}\mu}} \\
&\leq \bigg(\int_{A(k)} |u|^{p^*_{\gamma,\mu}}\, d\varrho\bigg)^{(q-1)/p^*_{\gamma,\mu}} \bigg(\int_{A(k)} v_k^{p^*_{\gamma,\mu}}\, d\varrho\bigg)^{1/p^*_{\gamma,\mu}} \\
&\quad\cdot \bigg(\int_{\Omega} |x|^{(p^*_{\gamma,\mu}\mu - q\mu)r}\, d\varrho\bigg)^{1/r} \bigg(\int_{A(k)} d\varrho\bigg)^{1/s}
\end{aligned}$$

where $1/r + q/p^*_{\gamma,\mu} + 1/s = 1$. To do that we need to prove

$$1 - \bigg(\frac{1}{r} + \frac{q}{p^*_{\gamma,\mu}}\bigg) > 0 \quad \text{and} \quad r > 1,$$

but this is clear because the first inequality is equivalent to $r > p^*_{\gamma,\mu}/(p^*_{\gamma,\mu} - q)$, and then $r > 1$. Also, we point out that the first term on the right hand side of (2.7) is bounded because $u \in E^{p,\gamma}_{\Sigma_1}(\Omega)$, and taking into account that

$(p^*_{\gamma,\mu}-q)\mu r-p^*_{\gamma,\mu}\gamma>-N$ is equivalent to $0>-N$ by taking $r=p^*_{\gamma,\mu}/(p^*_{\gamma,\mu}-q)$, then the third term on the right hand side of (2.7) is bounded, hence we conclude that

$$\left(\int_{A(k)} v_k^{p^*_{\gamma,\mu}}\,d\varrho\right)^{(p-1)/p^*_{\gamma,\mu}} \le C\left(\int_{A(k)} d\varrho\right)^{1/s}. \tag{2.8}$$

On the other hand,

$$(h-k)^{p-1}|A(h)|_{-p^*_{\gamma,\mu}\mu}^{(p-1)/p^*_{\gamma,\mu}} \le \left(\int_{A(k)} v_k^{p^*_{\gamma,\mu}}\,d\varrho\right)^{(p-1)/p^*_{\gamma,\mu}} \tag{2.9}$$

for all $h>k>0$, hence by (2.8), (2.9) we obtain

$$|A(h)|_{-p^*_{\gamma,\mu}\mu} \le \frac{C}{(h-k)^{p^*_{\gamma,\mu}}}|A(k)|_{-p^*_{\gamma,\mu}\mu}^{(p^*_{\gamma,\mu}/(p-1))(1-1/r-q/p^*_{\gamma,\mu})}. \tag{2.10}$$

In order to apply Lemma 2.4 we take

$$\varphi(t)=|A(t)|_{-p^*_{\gamma,\mu}\mu},\quad \kappa=p^*_{\gamma,\mu}\quad\text{and}\quad \nu=\frac{p^*_{\gamma,\mu}}{p-1}\left(1-\frac{1}{r}-\frac{q}{p^*_{\gamma,\mu}}\right)>0.$$

Now if $\nu>1$, we conclude by Lemma 2.4(a). If $\nu\le 1$, Lemma 2.4 permit us to improve the integrability of u by an exponent greater than p^*. Then we can iterate the process of estimate $|A(h)|_{-p^*_{\gamma,\mu}\mu}$ and the exponent increases in each step. Therefore, in the case $\nu=1$ we finish in the first iteration, and if $\nu<1$, defining

$$\nu_0=\frac{p^*_{\gamma,\mu}}{p-1}\left(1-\frac{1}{r}-\frac{q}{p^*_{\gamma,\mu}}\right),$$

by Lemma 2.4(c), the new exponent of integrability q_0 of u is in the interval $q_0\in(p^*_{\gamma,\mu},p^*_{\gamma,\mu}/(1-\nu_0))$. Iterating the argument, in a finite number of steps, we finish. □

COROLLARY 2.7. *Let $u\in E^{p,\gamma}_{\Sigma_1}(\Omega)$ be a solution to problem*

$$\begin{cases} -\Delta_{p,\gamma}u=|x|^{-p\beta}|u|^{p-2}u & \text{in } \Omega,\\ B(u)=0 & \text{on } \partial\Omega,\end{cases}$$

with $\beta<(\gamma+1)$. Then $u\in L^\infty(\Omega)$.

The proof follows as a particular case of Lemma 2.6.

LEMMA 2.8. *Assume that*

$$\frac{1}{r}-\frac{\beta}{N}>\frac{1}{p}-\frac{\gamma+1}{N}\quad\text{and}\quad 0\le(\beta-\gamma)\le 1. \tag{2.11}$$

Then, if $u\in E^{p,\gamma}_{\Sigma_1}(\Omega)$,

(a) $\||x|^{-\beta}u\|_{L^r(\Omega)}\le C(\Omega)\||x|^{-\gamma}|\nabla u|\|_{L^p(\Omega)}$,

(b) *the above inclusion is compact.*

Proof. The first part is a consequence of the Hölder and Caffarelli–Kohn–Nirenberg inequalities.

Let $\{u_k\}$ be a sequence in $E^{p,\gamma}_{\Sigma_1}(\Omega)$ such that $\||\nabla u_k||x|^{-\gamma}\|_{L^p(\Omega)} < M < \infty$. Then there exists a subsequence weakly convergent to a function $u \in E^{p,\gamma}_{\Sigma_1}(\Omega)$. By the classical Sobolev Theorem, for some subsequence,

$$\begin{aligned} u_{k_n} &\to u \quad \text{a.e. in } \Omega \setminus B_{1/n}(0), \\ u_{k_n} &\to u \quad \text{in } L^r_\beta(\Omega \setminus B_{1/n}(0)), \\ u_{k_n} &\to u \quad \text{in } L^r(\Omega \setminus B_{1/n}(0)). \end{aligned} \tag{2.12}$$

The diagonal subsequence u_{n_n} which will be denoted by u_n, converges a.e. in Ω.

We denote $\omega_n = u_n - u$, then by (2.12), $\omega_n \to 0$ a.e. in Ω, moreover,

$$\begin{aligned} &\int_\Omega |\omega_n|^r |x|^{-\beta r}\, dx \\ &\quad \le \bigg(\int_\Omega |\omega_n|^{p^*} |x|^{-p^*\gamma}\, dx \bigg)^{r/p^*} \bigg(\int_\Omega |x|^{r(\gamma-\beta)p^*/(p^*-r)}\, dx \bigg)^{(p^*-r)/p^*}. \end{aligned} \tag{2.13}$$

In the last inequalities we require that β, r will verify

(H) $r(\gamma-\beta)p^*/(p^*-r) > -N$.

But (H) is equivalent to the first inequality in (2.11), because

$$r[(\beta-\gamma)p^* + N] < Np^*$$

is equivalent to

$$\frac{1}{r} > \frac{\beta}{N} + \frac{1}{p^*} - \frac{\gamma}{N} = \frac{\beta}{N} + \frac{1}{p} - \frac{\gamma+1}{N}.$$

We use a Vitali type argument, by the Egorov Theorem and (H), we have that for all $\varepsilon > 0$ there exists a measurable set $A_\varepsilon \subset \Omega$ with $|\Omega \setminus A_\varepsilon| < \varepsilon$ such that $\omega_n \to 0$ uniformly in A_ε. Therefore

$$\begin{aligned} \int_\Omega |\omega_n|^r |x|^{-\beta r}\, dx &\le \int_{A_\varepsilon} |\omega_n|^r |x|^{-\beta r}\, dx + \int_{\Omega\setminus A_\varepsilon} |\omega_n|^r |x|^{-\beta r}\, dx \\ &\le \int_{A_\varepsilon} |\omega_n|^r |x|^{-\beta r}\, dx + C\bigg(\int_\Omega |\nabla \omega_n|^p |x|^{-p\gamma} dx \bigg)^{r/p} \\ &\quad \cdot \bigg(\int_{\Omega\setminus A_\varepsilon} |x|^{r(\gamma-\beta)p^*/(p^*-r)}\, dx \bigg)^{(p^*-r)/p^*}, \end{aligned} \tag{2.14}$$

where

$$\int_{A_\varepsilon} |\omega_n|^r |x|^{-\beta r} dx \to 0 \quad \text{as } n \to \infty$$

because $\omega_n \to 0$ as $n \to \infty$ uniformly in A_ε and $-r\beta > -N$ by hypothesis (2.11),

$$\int_\Omega |\nabla \omega_n|^p |x|^{-p\gamma}\, dx \le C \quad \text{uniformly in } n,$$

moreover, by (H) it is clear that there exists $s > 1$ such that $sr(\gamma-\beta)p^*/(p^*-r) > -N$, then by Hölder inequality,

$$\int_{\Omega\setminus A_\varepsilon} |x|^{r(\gamma-\beta)p^*/(p^*-r)}\,dx \le \left(\int_{\Omega\setminus A_\varepsilon} |x|^{sr(\gamma-\beta)p^*/(p^*-r)}\,dx\right)^{1/s} |\Omega\setminus A_\varepsilon|^{1-1/s} \le C\varepsilon^\delta$$

where $\delta = 1-1/s > 0$ and $C > 0$ is a constant which depends on Ω, r, $\gamma-\beta$ and N. Then we conclude that

$$0 \le \limsup_{\varepsilon\to0}\left(\limsup_{n\to\infty}\int_\Omega |\omega_n|^r|x|^{-\beta r}\,dx\right) \le C\limsup_{\varepsilon\to0}\varepsilon^\delta = 0. \qquad \square$$

COROLLARY 2.9. *Assume $\beta < (\gamma+1)$ and $-\infty < \gamma < (N-p)/p$. Then we have the following compact inclusions, $E^{p,\gamma}_{\Sigma_1}(\Omega) \hookrightarrow\hookrightarrow L^p_\beta(\Omega)$.*

PROOF. We divide the proof in two steps.

Step 1. If $\gamma \le \beta$ the proof is an immediate consequence of Lemma 2.8.

Step 2. If $\beta < \gamma$, we have that

$$\int_\Omega |x|^{-\beta r}|u|^r\,dx \le \int_\Omega |x|^{-\beta_0 r}|u|^r|x|^{(\beta_0-\beta)r}\,dx \le C\int_\Omega |x|^{-\beta_0 r}|u|^r\,dx$$

for some $\beta < \gamma < \beta_0 < (\gamma+1)$. Hence we conclude as before. $\square$

3. Construction of an eigenvalue sequence by minimax techniques and properties of the first one

3.1. Construction of an eigenvalue sequence by minimax techniques. In this subsection we will study the eigenvalue problem

$$(\mathrm{EP}^\beta_\gamma)\qquad \begin{cases} -\Delta_{p,\gamma}u = \lambda|x|^{-p\beta}|u|^{p-2}u & \text{in } \Omega,\\ B(u) = 0 & \text{on } \partial\Omega,\end{cases}$$

with Ω, γ as in the introduction and $\beta < (\gamma+1)$.

We will prove the existence of an eigenvalue sequence $\lambda_k(\beta) \subset \mathbb{R}^+$, with $\lambda_k(\beta) \to \infty$ as $k \to \infty$ for which problem $(\mathrm{EP}^\beta_\gamma)$ has nontrivial solution. To prove the positiveness, we consider the associated functional

$$\mathcal{J}(u) = \frac{1}{p}\int_\Omega |x|^{-p\gamma}|\nabla u|^p\,dx - \frac{\lambda}{p}\int_\Omega |x|^{-p\beta}|u|^p\,dx$$

defined in $E^{p,\gamma}_{\Sigma_1}(\Omega)$. If u_0 is a solution to problem $(\mathrm{EP}^\beta_\gamma)$ then $\mathcal{J}'(u_0) = 0$ and therefore,

$$0 = \langle \mathcal{J}'(u_0), u_0\rangle = \int_\Omega |x|^{-p\gamma}|\nabla u_0|^p\,dx - \lambda\int_\Omega |x|^{-p\beta}|u_0|^p\,dx.$$

Taking into account that for $\beta < (\gamma + 1)$, $E^{p,\gamma}_{\Sigma_1}(\Omega) \subset L^p_\beta(\Omega)$ it follows that

$$0 \geq \int_\Omega |x|^{-p\gamma} |\nabla u_0|^p \, dx - \lambda C \int_\Omega |x|^{-p\gamma} |\nabla u_0|^p \, dx,$$

as a consequence we have $\lambda \geq 1/C$, where $C > 0$ is the best constant in the Hardy–Sobolev inequality $\|u\|^p_{L^p_\beta(\Omega)} \leq C \||\nabla u|\|^p_{L^p_\gamma(\Omega)}$ that is a consequence of the Caffarelli–Kohn–Nirenberg inequalities [7].

Remark 3.1. The best constant of the above Hardy–Sobolev inequality in bounded regular domains and with mixed boundary conditions depends in general on Ω. See [24] and [10].

Once we have proved the L^∞-regularity and the compactness in the above section, we will briefly explain an adaptation to our framework of that in [12] to construct a sequence of eigenvalues, where the method used follows the ideas of Amann [3], i.e. the Lusternik–Schnirelman theory.

Consider the manifold

$$M_\alpha = \left\{ u \in E^{p,\gamma}_{\Sigma_1}(\Omega) : \frac{1}{p} \int_\Omega |x|^{-p\gamma} |\nabla u|^p \, dx = \alpha \right\}$$

and we define the functional

$$b(u) = \frac{1}{p} \int_\Omega |x|^{-p\beta} |u|^p \, dx.$$

Then we look the eigenvalues as critical points of the functional b restricted to the manifold M_α by using the mini-max method. Following the strategy of [12], we can formulate the following results on the existence of an eigenvalue sequence.

Theorem 3.2. *Consider* $C_k = \{C \subset M_\alpha : C$ *is compact,* $C = -C$ *and* $\gamma(C) \geq k\}$ *where* $\gamma(\,\cdot\,)$ *is the genus* (*see* [21] *for a definition and its properties*). *Define*

$$\nu_k = \sup_{C \in C_k} \min_{u \in C} b(u).$$

Then $\nu_k > 0$, *and there exists* $u_k \in M_\alpha$ *solution to* $(\mathrm{EP}^\beta_\gamma)$ *with* $b(u_k) = \nu_k$, $\lambda_k = \alpha/\nu_k$.

To finish this subsection, it can be proved that the sequence of critical values $\{\nu_k\}$ is infinite, namely that we have a sequence of eigenvalues.

Proposition 3.3. *Let* ν_k *defined as in Theorem* 3.2, *then* $\lim_{k\to\infty} \nu_k = 0$. *As a consequence,* $\lambda_k = \alpha \nu_k^{-1} \to \infty$ *as* $k \to \infty$.

The proofs of the results of this subsection are a little modification of that in [12] once we have the required compactness in Corollary 2.9. We omit it because of the extension of the paper.

3.2. Properties of the first eigenvalue to problem $(\mathrm{EP}^{\beta}_{\gamma})$**.** We consider the problem

$$(\mathrm{EP}^{\beta}_{\gamma}) \qquad \begin{cases} -\Delta_{p,\gamma} u = \lambda |x|^{-p\beta} |u|^{p-2} u & \text{in } \Omega, \\ B(u) = 0 & \text{on } \partial\Omega, \end{cases}$$

with $-\infty < \gamma < (N-p)/p$, $\beta < (\gamma + 1)$ and $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain with $0 \in \Omega$. The first eigenvalue is given by

$$\lambda_1(\beta) = \inf \bigg\{ \int_\Omega |x|^{-p\gamma} |\nabla u|^p \, dx : u \in E^{p,\gamma}_{\Sigma_1}(\Omega), \|u\|_{L^p_\beta(\Omega)} = 1 \bigg\}.$$

We prove that the classical properties to the first eigenvalue of the Dirichlet problem with the p-Laplace operator, i.e. $\lambda_1(\beta)$ is simple and isolated, are satisfied also for mixed Dirichlet–Neumann problem with the operator $\Delta_{p,\gamma}$.

THEOREM 3.4. *The first eigenvalue to problem* $(\mathrm{EP}^{\beta}_{\gamma})$, $\lambda_1(\beta)$, *is simple and isolated.*

The proof of this theorem follows as a consequence of the next four lemmas.

LEMMA 3.5. *Every eigenfunction u_1 corresponding to $\lambda_1(\beta)$ does not change sign in Ω, i.e. either $u_1 > 0$ or $u_1 < 0$ in Ω.*

PROOF. If v is an eigenfunction corresponding to the first eigenvalue, also $u = |v|$ is a solution of the minimization problem and then an eigenfunction. By the strong maximum principle (see [16]) $u > 0$ and then v has constant sign. □

LEMMA 3.6. *$\lambda_1(\beta)$ is simple, i.e. if u, v are two eigenfunctions corresponding to the first eigenvalue $\lambda_1(\beta)$, then $u = \alpha v$ for some $\alpha \in \mathbb{R}$.*

The proof of this lemma is similar to that in [17] with the appropriate changes to our framework.

LEMMA 3.7. *If v is an eigenfunction corresponding to the eigenvalue $\lambda > 0$, $\lambda \neq \lambda_1(\beta)$, then v changes sign in Ω, i.e. $v^+ \not\equiv 0 \not\equiv v^-$ and*

$$|\Omega^-|_{-p\beta} \geq (\lambda C)^{-(p+\varepsilon)/\varepsilon} \quad \textit{for some } \varepsilon = \varepsilon(\beta) > 0,$$

where $\Omega^- = \{x \in \Omega : v(x) < 0\}$.

PROOF. If we take v^- as a test function, by Hölder inequality for some $\varepsilon(\beta) > 0$,

$$\begin{aligned} \int_\Omega |x|^{-p\gamma} |\nabla v|^p \, dx &= \lambda \int_\Omega |x|^{-p\beta} (v^-)^p \, dx \\ &\leq \lambda \bigg(\int_\Omega |x|^{-p\beta} (v^-)^{p+\varepsilon} \, dx \bigg)^{p/(p+\varepsilon)} |\Omega^-|^{\varepsilon/(p+\varepsilon)}_{-p\beta}. \end{aligned}$$

Then by Hardy–Sobolev inequality and (3.1) we obtain $|\Omega^-|_{-p\beta} \geq (\lambda C)^{-(p+\varepsilon)/\varepsilon}$, where C is the optimal constant in the Hardy–Sobolev inequality. □

Lemma 3.8. *$\lambda_1(\beta)$ is isolated; that is, $\lambda_1(\beta)$ is the unique eigenvalue in $[0,a]$ for some $a > \lambda_1(\beta)$.*

Proof. Let $\lambda \geq 0$ be an eigenvalue and v a corresponding eigenfunction. $\lambda \geq \lambda_1(\beta)$ because $\lambda_1(\beta)$ is the infimum, then, $\lambda_1(\beta)$ is left-isolated. We argue by contradiction, i.e. we assume there exists a sequence of eigenvalues $\{\lambda_k\}$, $\lambda_k \neq \lambda_1(\beta)$ which converges to $\lambda_1(\beta)$. Let $\{u_k\}$ be a corresponding sequence of eigenfunctions with $\|u_k\|_{p,\gamma} = 1$. We can extract a subsequence, denoted again by $\{u_k\}$, weakly convergent in $E^{p,\gamma}_{\Sigma_1}(\Omega)$, strongly in $L^p_\beta(\Omega)$ and almost everywhere in Ω to a function $u \in E^{p,\gamma}_{\Sigma_1}(\Omega)$ (to simplify, we denote $E = E^{p,\gamma}_{\Sigma_1}(\Omega)$ and E^* its dual space). Then we have that p is subcritical, hence by Hölder inequality we get

$$\begin{aligned}
\|(|u_k|^{p-2}u_k & - |u|^{p-2}u)|x|^{-p\beta}\|_{E^*} \\
&= \sup_{\|\varphi\|_E=1} |\langle(|u_k|^{p-2}u_k - |u|^{p-2}u)|x|^{-p\beta}, \varphi\rangle| \\
&\leq C \sup_{\|\varphi\|_E=1} |\langle(|u_k|^{p-2}u_k - |u|^{p-2}u)|x|^{-p\beta}, \varphi\rangle| \\
&\leq C \sup_{\|\varphi\|_E=1} \|\varphi\|_{L^p_\beta(\Omega)} \|(|u_k|^{p-2}u_k - |u|^{p-2}u)\|_{L^r_\beta(\Omega)} \to 0 \quad \text{as } k \to \infty,
\end{aligned}$$

where $r = p/(p-1)$. Since $u_k = (-\Delta_{p,\gamma})^{-1}(\lambda_k|x|^{-p\beta}|u_k|^{p-2}u_k)$, and it is not difficult to see that the operator $(-\Delta_{p,\gamma})^{-1}$ is continuous from E^* to E, we have that the subsequence $\{u_k\}$ converges strongly to u in E, and subsequently, u is an eigenfunction corresponding to $\lambda_1(\beta)$ with $\|u\|_{p,\gamma} = 1$. Hence, by applying the Egorov Theorem ([6, Theorem IV.28]) to sequence $\{u_k\}$, we have that for all $0 < \varepsilon \ll 1$, there exists a measurable set $A_\varepsilon \subset \Omega$ such that $|\Omega \setminus A_\varepsilon| \leq \varepsilon$ and $\{u_k\}$ converges uniformly to u in A_ε. As a consequence for k large enough we have that u_k is positive in A_ε, that is a contradiction with the conclusion of Lemma 3.7. □

4. Bifurcation

Along this section we will consider $-\infty < \gamma < (N-p)/p$; $\beta, \mu \leq (\gamma+1)$, and the critical exponent $p^*_{\gamma,\mu}$ defined in (1.2). Hence the term $|x|^{-q\mu}u^{q-1}$ is "subcritical", if $q < p^*_{\gamma,\mu}$. Along this section we will assume that $\lambda_{\gamma,N,p}(\Sigma_1) = \lambda_{\gamma,N,p}$, i.e. $\lambda_{\gamma,N,p}(\Sigma_1)$ is not achieved in $E^{p,\gamma}_{\Sigma_1}(\Omega)$, see [10] for more details.

We start with a nonexistence result via a Pohozaev-type identity.

Theorem 4.1. *Assume $\gamma \leq \mu \leq (\gamma+1)$, $\Omega \subset \mathbb{R}^N$ is a bounded domain verifying $0 \in \Omega$, $\langle x, \nu\rangle > 0$ on Σ_1, $\langle x, \nu\rangle = 0$ on Σ_2; where ν is the outwards unitary normal to $\partial\Omega$. Then the double critical problem* (P_λ) *as in the introduction with*

$$f_\lambda(x,u) = \lambda \frac{u^{p-1}}{|x|^{p(\gamma+1)}} + |x|^{-p^*_{\gamma,\mu}\mu} u^{p^*_{\gamma,\mu}-1},$$

has not positive solution.

PROOF. We assume we have the necessary regularity in the following operation, if not, we can use an approximation argument as in [15]. Multiplying in the equation of (P_λ) by $\langle x, \nabla u\rangle$ and integrating by parts, we obtain

$$\begin{aligned}
(4.1)\quad \frac{N-p(\gamma+1)}{p}&\int_\Omega |x|^{-p\gamma}|\nabla u|^p\,dx + \frac{p-1}{p}\int_{\Sigma_1}\langle x,\nu\rangle|x|^{-p\gamma}|\nabla u|^p\,d\sigma\\
&-\frac{1}{p}\int_{\Sigma_2}\langle x,\nu\rangle|x|^{-p\gamma}|\nabla u|^p\,d\sigma\\
&=\lambda\frac{N-p(\gamma+1)}{p}\int_\Omega \frac{u^p}{|x|^{p(\gamma+1)}}\,dx + \frac{N-p^*_{\gamma,\mu}\mu}{p^*_{\gamma,\mu}}\int_\Omega \frac{u^{p^*_{\gamma,\mu}}}{|x|^{p^*_{\gamma,\mu}\mu}}\,dx\\
&-\int_{\Sigma_2}\langle x,\nu\rangle\bigg(\frac{\lambda u^p}{p|x|^{p(\gamma+1)}}+\frac{u^{p^*_{\gamma,\mu}}}{p^*_{\gamma,\mu}|x|^{p^*_{\gamma,\mu}\mu}}\bigg)\,d\sigma.
\end{aligned}$$

If u would be a solution to problem (P_λ) then

$$(4.2)\qquad \int_\Omega |x|^{-p\gamma}|\nabla u|^p\,dx = \lambda\int_\Omega \frac{u^p}{|x|^{p(\gamma+1)}}\,dx + \int_\Omega \frac{u^{p^*_{\gamma,\mu}}}{|x|^{p^*_{\gamma,\mu}\mu}}\,dx.$$

As a consequence, by (4.1) and (4.2) we conclude that

$$\begin{aligned}
(4.3)\qquad \frac{p-1}{p}&\int_{\Sigma_1}\langle x,\nu\rangle|x|^{-p\gamma}|\nabla u|^p\,d\sigma - \frac{1}{p}\int_{\Sigma_2}\langle x,\nu\rangle|x|^{-p\gamma}|\nabla u|^p\,d\sigma\\
&=\bigg(\frac{N-p^*_{\gamma,\mu}\mu}{p^*_{\gamma,\mu}}-\frac{N-p(\gamma+1)}{p}\bigg)\int_\Omega \frac{u^{p^*_{\gamma,\mu}}}{|x|^{p^*_{\gamma,\mu}\mu}}\,dx\\
&-\int_{\Sigma_2}\langle x,\nu\rangle\bigg(\frac{\lambda u^p}{p|x|^{p(\gamma+1)}}+\frac{u^{p^*_{\gamma,\mu}}}{p^*_{\gamma,\mu}|x|^{p^*_{\gamma,\mu}\mu}}\bigg)\,d\sigma.
\end{aligned}$$

Therefore the second member in (4.3) is zero and the first member is positive, that is a contradiction. □

In particular we deduce that there are no positive solutions of (P_λ) in the hypotheses of Theorem 4.1 in any appropriate cone with zero Neumann boundary condition in the lateral surface, and zero Dirichlet boundary condition in the complementary. This result is an extension of [18].

We remark that in problem (P_λ) if $f_\lambda(x,u)$ is at most p-linear in u, we have uniqueness of solution. For the sake of completeness we include the proof of a comparison result which imply directly the uniqueness.

LEMMA 4.2 (Comparison). *Assume that* $\mu \le (\gamma+1)$, $q<p$. *Let* v, u *be a subsolution and a supersolution respectively in* $E^{p,\gamma}_{\Sigma_1}(\Omega)$ *of the problem*

$$(4.4)\qquad \begin{cases} -\Delta_{p,\gamma}\omega = \lambda\dfrac{\omega^{p-1}}{|x|^{p(\gamma+1)}} + |x|^{-q\mu}\omega^{q-1} & \text{in } \Omega,\\ \omega>0 & \text{in } \Omega,\\ B(\omega)=0 & \text{on } \partial\Omega.\end{cases}$$

Then $v \le u$.

Proof. We consider

$$\frac{-\Delta_{p,\gamma}u}{u^{p-1}} - \frac{-\Delta_{p,\gamma}v}{v^{p-1}} \ge |x|^{-q\mu}(u^{q-p} - v^{q-p}).$$

Multiplying by the test function $(v^p - u^p)_+$ and integrating in the last inequality we obtain:

$$\int_\Omega \bigg(\frac{-\Delta_{p,\gamma}u}{u^{p-1}} - \frac{-\Delta_{p,\gamma}v}{v^{p-1}}\bigg)(v^p - u^p)_+\,dx \ge \int_\Omega |x|^{-q\mu}(u^{q-p}-v^{q-p})(v^p-u^p)^+\,dx \ge 0.$$

Let $\omega = (v^p - u^p)_+$ be a test function with gradient

$$\nabla\omega = p(v^{p-1}\nabla v - u^{p-1}\nabla u)\chi_{[v>u]}$$

then we have

$$\begin{aligned}
&\int_\Omega \bigg(\frac{-\Delta_{p,\gamma}u}{u^{p-1}} - \frac{-\Delta_{p,\gamma}v}{v^{p-1}}\bigg)\omega\,dx \\
&= \int_\Omega |x|^{-p\gamma}\bigg(|\nabla u|^{p-2}\bigg\langle \nabla u, \nabla\bigg(\frac{\omega}{u^{p-1}}\bigg)\bigg\rangle - |\nabla v|^{p-2}\bigg\langle \nabla v, \nabla\bigg(\frac{\omega}{v^{p-1}}\bigg)\bigg\rangle\bigg)\,dx \\
&= \int_\Omega |x|^{-p\gamma}|\nabla u|^{p-2}\bigg\langle \nabla u, p\frac{v^{p-1}}{u^{p-1}}\nabla v - \bigg(p + \frac{p-1}{u^p}(v^p - u^p)\bigg)\nabla u\bigg\rangle\chi_{[v>u]}\,dx \\
&\quad + \int_\Omega |x|^{-p\gamma}|\nabla v|^{p-2}\bigg\langle \nabla v, p\frac{u^{p-1}}{v^{p-1}}\nabla u - \bigg(p - \frac{p-1}{v^p}(v^p - u^p)\bigg)\nabla v\bigg\rangle\chi_{[v>u]}\,dx \\
&= \int_\Omega |x|^{-p\gamma}\bigg(p\frac{v^{p-1}}{u^{p-1}}|\nabla u|^{p-2}\langle\nabla u, \nabla v\rangle - |\nabla u|^p\bigg(1 - (p-1)\frac{v^p}{u^p}\bigg)\bigg)\chi_{[v>u]}\,dx \\
&\quad + \int_\Omega |x|^{-p\gamma}\bigg(p\frac{u^{p-1}}{v^{p-1}}|\nabla v|^{p-2}\langle\nabla v, \nabla u\rangle - |\nabla v|^p\bigg(1 - (p-1)\frac{u^p}{v^p}\bigg)\bigg)\chi_{[v>u]}\,dx.
\end{aligned}$$

By Picone's identity, Lemma 2.1, we have that the last two terms in the above equality are nonpositive, as a consequence, $\mathrm{meas}([v > u]) = 0$. □

4.1. Bifurcation from infinity. In this subsection we study phenomena of bifurcation from infinity. We consider the problem

$$(\mathrm{P}_\gamma^{\omega_\beta}) \qquad \begin{cases} -\Delta_{p,\gamma}u = \lambda u^{p-1}\omega_\beta(x) + |x|^{-q\mu}u^{q-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ B(u) = 0 & \text{on } \partial\Omega, \end{cases}$$

with $1 < q < p$, $-\infty < \gamma < (N-p)/p$; $\beta, \mu < (\gamma+1)$, and the function ω_β defined by (1.1) at the introduction. We point out that $\omega_\beta(x)$ is a little modification of the potential $|x|^{-p\beta}$, for which we have proved, in Section 3, the existence of an infinite sequence of eigenvalues and the properties of the first one. The same computations can be done with $\omega_\beta(x)$ instead of $|x|^{-p\beta}$. Precisely, we prove in Theorem 4.6 that $\lambda_1(\omega_\beta)$ (which will denote the first eigenvalue to the associated eigenvalue problem to $(\mathrm{P}_\gamma^{\omega_\beta})$ that we will call $(\mathrm{EP}_\gamma^{\omega_\beta})$) is the unique bifurcation point from infinity of positive solutions to problem ($\mathrm{P}_\gamma^{\omega_\beta}$). Moreover, there exists a branch of solutions (λ, u_λ) to problem ($\mathrm{P}_\gamma^{\omega_\beta}$) such that $\|u_\lambda\|_\infty \to \infty$ as $\lambda \nearrow \lambda_1(\omega_\beta)$.

At the end of this subsection, as a consequence of the bifurcation to problem ($\mathrm{P}_\gamma^{\omega_\beta}$) and following [14] with the appropriate changes, we prove that $\lambda_{\gamma,N,p}$ is the unique bifurcation point from infinity of positive solutions to problem

$$(\mathrm{P}_\gamma)\quad \begin{cases} -\Delta_{p,\gamma} u = \lambda \dfrac{u^{p-1}}{|x|^{p(\gamma+1)}} + |x|^{-q\mu} u^{q-1} & \text{in } \Omega,\ 1 < q < p,\ \mu \le (\gamma+1), \\ u > 0 & \text{in } \Omega, \\ B(u) = 0 & \text{on } \partial\Omega. \end{cases}$$

We formulate the precise result.

THEOREM 4.3. *$\lambda_{\gamma,N,p}$ is the unique bifurcation point from infinity of positive solutions to problem* (P_γ). *Precisely, there exists a continuum of solutions $(\lambda, u_\lambda) \in (0, \lambda_{\gamma,N,p}) \times E^{p,\gamma}_{\Sigma_1}(\Omega)$ to problem* (P_λ)*, crossing the point $(u_0, 0)$, where u_0 is the unique solution to* (P_γ) *for $\lambda = 0$, and such that blows-up as $\lambda \nearrow \lambda_{\gamma,N,p}$.*

To study the bifurcation phenomena to problem ($\mathrm{P}_\gamma^{\omega_\beta}$), we use the change $v = \lambda^{1/(p-q)} u$, then v satisfies the equation $-\Delta_{p,\gamma} v = \lambda(\omega_\beta(x) v^{p-1} + |x|^{-q\mu} v^{q-1})$ with $v > 0$ in Ω and $B(v) = 0$ on $\partial\Omega$. In the sequel we denote $f(x, s) = \omega_\beta(x) s^{p-1} + |x|^{-q\mu} s^{q-1}$. In these terms we can follow the strategy of [4]. We will work in the Banach space $Y = \{u \in \mathcal{C}(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\}$, endowed with the norm $\|\cdot\|_\infty$; we also set $B_r = \{u \in Y : \|u\|_\infty < r\}$. Consider the map

$$\Phi_\lambda : Y \to Y, \quad \Phi_\lambda(u) = u - (-\Delta_{p,\gamma})^{-1}(\lambda f(x, u)).$$

Taking into account the regularity obtained in Section 5, Φ_λ is a compact perturbation of the identity. To study the bifurcation from infinity of solutions to problem (P_γ^β) we set $z = u/\|u\|_\infty^2$, $u \not\equiv 0$, and consider

$$\Psi_\lambda(z) = z - \|z\|_\infty^2 \left[(-\Delta_{p,\gamma})^{-1} \left(\lambda f\left(x, \frac{z}{\|z\|_\infty^2} \right) \right) \right], \quad z \neq 0, \quad \Psi_\lambda(0) = 0.$$

Then λ is a bifurcation point from the trivial solution for $\Psi_\lambda(z) = 0$ if and only if λ is a bifurcation point from infinity for $\Phi_\lambda(u) = 0$.

The following lemmas are in order.

LEMMA 4.4. *For $\lambda \in (0, \lambda_1(\omega_\beta))$ we have $i(\Psi, 0, 0) = 1$.*

PROOF. Given $0 < \lambda^* < \lambda_1(\omega_\beta)$ there exists $R > 0$ such that $\Psi_\lambda(u) \neq 0$ provided $\lambda \in [0, \lambda^*]$ and $u \in Y$ with $\|u\|_\infty \ge R$. Otherwise, there exists a sequence $\{(u_k, \lambda_k)\}$ such that $\|u_k\|_\infty \to \infty$, $\lambda_k \to \overline{\lambda}$ and $\Psi_{\lambda_k}(u_k) = 0$. Letting $v_k = u_k \|u_k\|_\infty^{-1}$, by the properties of f and the elliptic regularity given in Theorem 5.1 allow us to conclude that, up to a subsequence, $v_k \to \overline{v}$ uniformly. And, taking into account that $u_k = -\Delta_{p,\gamma}^{-1}(\lambda_k f(x,u_k))$ we obtain

$$-\Delta_{p,\gamma}(v_k) = \lambda_k(\omega_\beta(x)v_k^{p-1} + |x|^{-q\mu}v_k^{q-1}\|u_k\|_\infty^{q-p}).$$

By passing to the limit as $k \to \infty$ we get $-\Delta_{p,\gamma}\overline{v} = \overline{\lambda}\omega_\beta(x)\overline{v}^{p-1}$. Moreover, $\overline{v} \geq 0$ and $\|\overline{v}\|_\infty = 1$. Hence by Lemmas 3.5 and 3.7, $\overline{\lambda} = \lambda_1(\omega_\beta) \notin [0,\lambda^*]$, that is a contradiction.

It follows that $\Psi_{t\lambda}(z) \neq 0$, for all $t \in [0,1]$ and all z with $0 < \|z\|_\infty \leq R^{-1}$. Therefore, for any $0 < \varepsilon \leq R^{-1}$ and the invariance by homotopy yields that

$$\deg(\Psi_\lambda, B_\varepsilon, 0) = \deg(Id, B_\varepsilon, 0) = 1. \qquad \square$$

Lemma 4.5. *For all $\lambda > \lambda_1(\omega_\beta)$ we have $i(\Psi,0,0) = 0$.*

Proof. We claim that if $\lambda > \lambda_1(\omega_\beta)$, there exists $R > 0$ such that

$$\begin{cases} -\Delta_{p,\gamma}(u) = \lambda f(x,u) + \tau & \text{in } \Omega, \\ B(u) = 0 & \text{on } \partial\Omega \end{cases}$$

has not positive solution with $\|u\|_\infty \geq R$, for all $\tau \geq 0$.

Assume by contradiction that there exist two sequences $\{u_k\}$ and $\{\tau_k\} \subset \mathbb{R}^+$ such that $\|u_k\|_\infty \to \infty$ as $k \to \infty$, and $-\Delta_{p,\gamma}(u_k) = \lambda f(x,u_k) + \tau_k$. Then $v_k = u_k\|u_k\|_\infty^{-1}$ verifies

$$-\Delta_{p,\gamma}v_k = \lambda(\omega_\beta(x)v_k^{p-1} + |x|^{-q\mu}v_k^{q-1}\|u_k\|_\infty^{q-p}) + \tau_k\|u_k\|_\infty^{1-p}. \tag{4.5}$$

Passing to a subsequence, we can assume that $v_k \to \overline{v}$ uniformly and either

(a) $\tau_k\|u_k\|_\infty^{1-p} \to c \geq 0$,

or

(b) $\tau_k\|u_k\|_\infty^{1-p} \to \infty$.

In the case (a), the right hand side of (4.5) remains bounded in Y, and we can assume, up to a subsequence, that $v_k \to \overline{v} \in Y$ uniformly. Hence, taking limits,

$$-\Delta_{p,\gamma}\overline{v} = \lambda\omega_\beta(x)\overline{v}^{p-1} + c \geq (\lambda_1 + \varepsilon)\omega_\beta(x)\overline{v}^{p-1} \geq 0 \quad \text{for all } 0 < \varepsilon \leq \lambda - \lambda_1(\omega_\beta).$$

Then, taking $\rho\phi_1$ with $0 < \rho \ll 1$, ϕ_1 a positive eigenfunction with $\|\phi_1\|_\infty = 1$, and by the iteration method with $\rho\phi_1$, $\overline{v}$ we arrive to a contradiction with the isolation of λ_1, see Theorem 3.4.

In the case (b), for k large enough we have

$$-\Delta_{p,\gamma}v_k \geq \lambda\omega_\beta(x)v_k^{p-1} \geq 0,$$

and we conclude as in the case (a).

In particular we can conclude that

$$-\Delta_{p,\gamma}u = \lambda f(x,u) + t\|u\|_\infty^{2(p-1)}$$

has not positive solution if $\|u\|_\infty \geq R$, for all $t \in [0,1]$. Letting $z = u\|u\|_\infty^{-2}$, one infers that

$$-\Delta_{p,\gamma}(z) = \|z\|_\infty^{2(p-1)} \lambda f(x, z\|z\|_\infty^{-2}) + t$$

has not positive solution if $0 < \|z\|_\infty \leq R^{-1}$, for all $t \in [0,1]$. Hence the homotopy $H\colon [0,1] \times Y \to Y$,

$$H(t,z) = z - (-\Delta_{p,\gamma}^{-1}(\|z\|_\infty^{2(p-1)} f(x, z\|z\|_\infty^{-2}) + t))$$

verifies $H(t,z) \neq 0$ for all $z \in Y$ with $0 < \|z\|_\infty \leq R^{-1}$ and for any $t \in [0,1]$. Therefore, for all $0 < \varepsilon \leq R^{-1}$ we get

$$\deg(\Psi_\lambda, B_\varepsilon, 0) = \deg(H(0,\cdot), B_\varepsilon, 0) = \deg(H(1,\cdot), B_\varepsilon, 0) = 0. \qquad \square$$

Theorem 4.6. *Let $\lambda_1(\omega_\beta)$ be the first eigenvalue to problem* $(\mathrm{EP}_\gamma^{\omega_\beta})$. *Then, $\lambda_1(\omega_\beta)$ is the unique bifurcation point from infinity of positive solutions to problem* $(\mathrm{P}_\gamma^{\omega_\beta})$. *Precisely, there exists a branch of solutions Σ_β to problem* $(\mathrm{P}_\gamma^{\omega_\beta})$ *such that $\|u_\lambda\|_\infty \to \infty$ as $\lambda \nearrow \lambda_1(\omega_\beta)$.*

Proof. The arguments are similar to those in [4] and we will be sketchy. We have that, by Lemma 4.4, $i(\Psi_\lambda) = 1$ for all $0 < \lambda < \lambda_1(\omega_\beta)$, while, by Lemma 4.5, $i(\Psi_\lambda) = 0$ for all $\lambda > \lambda_1(\omega_\beta)$, where $i(\cdot)$ means the Leray–Schauder index with respect to 0. This change of index permit to show that the solutions of $\Psi_\lambda = 0$ contains a continuum branching off $(\lambda_1(\omega_\beta), 0)$, which corresponds a branch of solutions to problem $(\mathrm{P}_\gamma^{\omega_\beta})$ emanating from ∞ at $\lambda = \lambda_1(\omega_\beta)$. $\square$

Now we compare the best constant in the Hardy–Sobolev inequality with the approximating eigenvalue problems, $(\mathrm{EP}_\gamma^{\omega_\beta})$. (See [13]).

Lemma 4.7. *Let $\lambda_1(\omega_\beta)$ be the first eigenvalue to problem* $(\mathrm{EP}_\gamma^{\omega_\beta})$ *as before. Then $\lambda_1(\omega_\beta) \geq \lambda_{\gamma,N,p}$, and moreover, $\lim_{\beta \nearrow (\gamma+1)} \lambda_1(\omega_\beta) = \lambda_{\gamma,N,p}$.*

Proof. The first inequality follows immediately from the definition of the first eigenvalue by the Rayleigh quotient. Also it is easy to see that $\{\lambda_1(\omega_\beta)\}$ is a nonincreasing sequence. Then we have to prove that the limit cannot bigger than $\lambda_{\gamma,N,p}$. Assume by contradiction that $\lim\limits_{\beta \nearrow (\gamma+1)} \lambda_1(\omega_\beta) = \lambda_{\gamma,N,p} + \rho$. Then, we can choose $\varphi \in E_{\Sigma_1}^{p,\gamma}(\Omega)$ such that

$$\frac{\int_\Omega |x|^{-p\gamma} |\nabla \varphi|^p \, dx}{\int_\Omega |x|^{-p(\gamma+1)} \varphi^p \, dx} \leq \lambda_{\gamma,N,p} + \frac{\rho}{2}.$$

Therefore,

$$\lambda_1(\omega_\beta) \leq \frac{\int_\Omega |x|^{-p\gamma} |\nabla \varphi|^p \, dx}{\int_\Omega \omega_\beta(x) \varphi^p \, dx},$$

that is a contradiction, because the last member in this inequality has to be smaller than $\lambda_{\gamma,N,p} + \rho$ for $(\gamma+1) - \beta > 0$ sufficiently small. $\square$

Next lemma is in order to prove Theorem 4.3.

Lemma 4.8. *Let $\{(u_\beta,\mu_\beta)\}$ be the sequence of solutions to problems* $(\mathrm{P}_\gamma^{\omega_\beta})$, *with $\lambda=\mu_\beta$ such that*

(a) $(u_\beta,\mu_\beta)\in\Sigma_\beta$.
(b) $\mu_\beta\to\lambda_0\in(0,\lambda_{\gamma,N,p})$ *as* $\beta\nearrow(\gamma+1)$.
(c) $\|u_\beta\|_{E^{p,\gamma}_{\Sigma_1}(\Omega)}\le C$ *for all* $\beta<(\gamma+1)$.

Then there exists a subsequence $u_{\beta_n}=u_n$ that is a Palais–Smale sequence to the functional

$$J(u,\lambda_0)=\frac{1}{p}\int_\Omega |x|^{-p\gamma}|\nabla u|^p dx-\frac{\lambda_0}{p}\int_\Omega \frac{u^p}{|x|^{p(\gamma+1)}}\,dx-\frac{1}{q}\int_\Omega |x|^{-q\mu}u^q\,dx,$$

i.e. $\lim_{n\to\infty}u_n=u$, *a solution to* (P_γ) *with* $\lambda=\lambda_0$.

Proof. By (c) there exists a subsequence $u_{\beta_n}=u_n$ such that $u_n\rightharpoonup u$ in $E^{p,\gamma}_{\Sigma_1}(\Omega)$, $u_n\to u$ in $L^q(\Omega)$ and $u_n\to u$ a.e. in Ω. We will denote $\omega_{\beta_n}=\omega_n$, then, since

$$\begin{aligned}0&=\lim_{n\to\infty}\langle J'(u_n,\mu_n),\phi\rangle\\&=\lambda_0\lim_{n\to\infty}\int_\Omega(|x|^{-p(\gamma+1)}-\omega_n(x))u_n^{p-1}\phi\,dx+\langle J'(u,\lambda_0),\phi\rangle.\end{aligned}$$

By Fatou's Lemma it follows that

$$\begin{aligned}&\lim_{n\to\infty}\bigg|\int_\Omega(|x|^{-p(\gamma+1)}-\omega_n(x))u_n^{p-1}\phi\,dx\bigg|\\&\qquad\le\int_\Omega\limsup_{n\to\infty}||x|^{-p(\gamma+1)}-\omega_n(x)||u_n|^{p-1}|\phi|\,dx=0.\end{aligned}$$

Then we have proved that $J'(u,\lambda_0)=0$. As a consequence of that and the hypotheses,

$$\begin{aligned}0&=\langle J_n'(u_n)-J'(u),u_n-u\rangle\\&=\lim_{n\to\infty}\bigg\{\int_\Omega|x|^{-p\gamma}\langle|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u,\nabla(u_n-u)\rangle\,dx\\&\quad+\int_\Omega(\lambda_0|x|^{-p(\gamma+1)}u^{p-1}-\mu_n\omega_n(x)u_n^{p-1})(u_n-u)\,dx\\&\quad-\int_\Omega|x|^{-q\mu}(u_n^{q-1}-u^{q-1})(u_n-u)\,dx\bigg\}.\end{aligned}$$

If $p\ge2$, that limit is greater or equal than

$$\begin{aligned}&o(1)+c_p\lim_{n\to\infty}\int_\Omega|\nabla(u_n-u)|^p\,dx\\&\qquad+\lim_{n\to\infty}\int_\Omega(\lambda_0|x|^{-p(\gamma+1)}u^{p-1}-\mu_n\omega_n(x)u_n^{p-1})(u_n-u)\,dx.\end{aligned}$$

Moreover, since $\mu_n \to \lambda_0$ as $n \to \infty$ and by Fatou's Lemma

$$\begin{aligned}\lim_{n\to\infty} \int_\Omega (\omega_n(x) u_n^{p-1} & - |x|^{-p(\gamma+1)} u^{p-1})(u_n - u)\,dx \\ &= \lim_{n\to\infty} \int_\Omega (\omega_n(x) - |x|^{-p(\gamma+1)}) u^{p-1}(u_n - u)\,dx \\ &\quad + \lim_{n\to\infty} \int_\Omega \omega_n(x)(u_n^{p-1} - u^{p-1})(u_n - u)\,dx = 0\end{aligned}$$

then we have proved in the case $p \ge 2$ that

$$\begin{aligned}0 &= \lim_{n\to\infty} \langle J'(u_n,\lambda_0) - J'(u,\lambda_0)\rangle \\ &\ge \left(1 - \frac{\lambda_0}{\lambda_{\gamma,N,p}}\right) \lim_{n\to\infty} \int_\Omega |x|^{-p\gamma} |\nabla(u_n - u)|^p\,dx + o(1).\end{aligned}$$

If $1 < p < 2$ we argue in a similar way as before but using in this case that

$$\langle |x|^{p-2}x - |y|^{p-2}y, x - y\rangle \ge c_p(|x| + |y|)^{p-2}|x - y|^2. \qquad \square$$

PROOF OF THEOREM 4.3. By Lemma 4.8 we can pass to the limit, since the limit inf of the set of the branches is not empty, because, by uniqueness, all these branches cross the axis $\lambda = 0$ at the same point.

To finish, we prove that the limit branch is non degenerated. First we prove that the approximated branches do not collapse into the vertical axis. By uniqueness for $\lambda = 0$: if we have a sequence $\{u_{\beta_n}\}$ of solutions to the approximated problems such that $u_{\beta_n} \to u_0$, then the Palais–Smale condition implies $u_{\beta_n} \to u_0$ the solution corresponding to $\lambda = 0$. On the other hand, it is easily seen that the approximated branches are bounded away from the horizontal axis. It suffices to see that if u_n is a solution to the corresponding λ_n, then

$$\begin{aligned}J_{\beta_n}(u_n,\lambda_n) &= \min J_{\beta_n}(u,\lambda_n) \\ &\le \min\left(\frac{1}{p}\int_\Omega |x|^{-p\gamma}|\nabla u|^p\,dx - \frac{1}{q}\int_\Omega |x|^{-q\mu}u^q\,dx\right) = -C_q < 0.\end{aligned}$$

Hence the branches do not collapse at the horizontal axis.

Now we will prove that $\lambda_{\gamma,N,p}$ is the unique bifurcation point from infinity to problem (P_λ). By a rescaling argument, suppose that there exists another one, namely, λ_0, and a sequence $\{(u_k,\lambda_k)\}$ with $\|u_k\|_{p,\gamma} = M_k \to \infty$ and $\lambda_k \to \lambda_0$. It is clear that $\lambda_0 \le \lambda_{\gamma,N,p}$, we define $v_k = M_k^{-1}u_k$, therefore,

$$-\Delta_{p,\gamma} v_k = \lambda_k \frac{v_k^{p-1}}{|x|^{p(\gamma+1)}} + M_k^{q-p}|x|^{-q\mu}v_k^{q-1}.$$

As a consequence,

$$\begin{aligned}\lambda_0 = \lim_{k\to\infty}\lambda_k &= \lim_{k\to\infty}\frac{\int_\Omega |x|^{-p\gamma}|\nabla v_k|^p\,dx - M_k^{q-p}\int_\Omega |x|^{-q\mu}v_k^q\,dx}{\int_\Omega (v_k^p/|x|^{p(\gamma+1)})\,dx}\\ &\geq \lim_{k\to\infty}\frac{1-M_k^{q-p}C}{1/\lambda_{\gamma,N,p}} = \lambda_{\gamma,N,p},\end{aligned}$$

where the inequality is consequence of Hölder and Hardy–Sobolev inequalities, and C is a positive constant which depends only on q, Ω. □

4.2. Bifurcation from zero. In this subsection we consider the problem

$$(\mathrm{P}^{\omega_\beta}_{s,\lambda})\qquad \begin{cases} -\Delta_{p,\gamma}(u) = \lambda\omega_\beta(x)u^{s-1} + |x|^{-q\mu}u^{q-1} & \text{in } \Omega,\\ u>0 & \text{in } \Omega,\\ B(u)=0 & \text{on } \partial\Omega,\end{cases}$$

with $-\infty<\gamma<(N-p)/p$, $1<s\leq p<q<p^*_{\gamma,\mu}$ (see (1.2)), $\beta\leq(\gamma+1)$, $\mu<(\gamma+1)$ and $\omega_\beta(x)$ as in the previous subsection. We prove that there exists a continuum of solutions to problem $(\mathrm{P}^{\omega_\beta}_{s,\lambda})$ bifurcating from $(\lambda,\|u\|_\infty)=(0,0)$ if $s<p$. In the case $s=p$ we can follow the strategy of Theorem 4.6 to prove a global bifurcation result for each problem $(\mathrm{P}^{\omega_\beta}_{p,\lambda})$, finding an unbounded branch, Γ_β, bifurcating to the left from $(\lambda_1(\beta),0)$. We prove the next result.

Theorem 4.9. *Consider the problem $(\mathrm{P}^{\omega_\beta}_{s,\lambda})$ with $s=p<q$; $\beta,\mu<(\gamma+1)$. Then $\lambda_1(\omega_\beta)$ is the unique bifurcation point from the trivial solution. Precisely, there exists an unbounded branch of solutions emanating from $(\lambda,\|u\|_\infty)=(\lambda_1(\omega_\beta),0)$.*

Proof. By similar arguments to Lemma 4.4 one shows that $i(\Phi_\lambda)=1$ for all $0\leq\lambda<\lambda_1(\omega_\beta)$, while as in Lemma 4.5 it can be proved that $i(\Phi_\lambda)=0$ for all $\lambda>\lambda_1(\omega_\beta)$. Then the conclusion follows as in the proof of Theorem 4.6. □

In the sequel we assume $s<p$. In this case we prove the next result.

Theorem 4.10. *Assume that $1<s<p<q$ and $\beta,\mu<(\gamma+1)$. Then there exists a continuum of solutions Σ to problem $(\mathrm{P}^{\omega_\beta}_\gamma)$ bifurcating from $(\lambda,\|u\|_\infty)=(0,0)$. Moreover,*

(a) *If $(\lambda,u)\in\Sigma$ and $\lambda>0$, then $u\not\equiv 0$.*
(b) *There exists a constant $\rho_0>0$ such that if $\rho\in(0,\rho_0]$ and $(\lambda,u)\in\Sigma$, with $\|u\|_\infty=\rho$, then $\lambda\geq\lambda(\rho)>0$.*

There are two difficulties that may not use the global bifurcation theorem by Rabinowitz, [20]. The first one is that $s<p$, then the term u^{s-1} could have infinite derivative at zero. The second difficulty is the presence of the potential $|x|^{-p\gamma}$ in the operator.

We can solve the first difficulty proceeding as in [5], we consider the truncature function $h_\delta(t) = \delta^{s-p}|t|^{p-2}t$ if $t \le \delta$ and $h_\delta(t) = t^{s-1}$ if $t > \delta$. We define the approximated problems $(\mathrm{P}^\delta_{s,\lambda})$ as $(\mathrm{P}^{\omega_\beta}_{s,\lambda})$ with $h_\delta(u)$ instead of u^{s-1}. To solve the second difficulty we can proceed as in Theorem 4.9 and we have that there exists a continuum $\mathcal{C}_\delta$ of solutions to $(\mathrm{P}^\delta_{s,\lambda})$ bifurcating from $(\lambda_1^\delta, 0)$ with $\lambda_1^\delta = \delta^{p-s}\lambda_1(\omega_\beta)$, and $\lambda_1(\omega_\beta)$ is the first eigenvalue to problem

$$-\Delta_{p,\gamma}\varphi = \lambda\omega_\beta(x)\varphi^{p-1} \quad \text{in } \Omega, \quad \varphi \in E^{p,\gamma}_{\Sigma_1}(\Omega).$$

T o finish we use a topological lemma given by Whyburn in [25]:

LEMMA 4.11. *Let $\{\Sigma^n\}_{n\in\mathbb{N}}$ be a sequence of connected sets in a complete metric space E. Assume that*

(a) $\bigcup \Sigma^n$ *is precompact in E, and*
(b) $\liminf \Sigma^n \neq \emptyset$.

Then, $\limsup \Sigma^n$ *is not empty, closed and connected.*

For $R > 0$ and T_R denoting the ball of radius R in $E = \mathbb{R} \times X$, with $X = E^{p,\gamma}_{\Sigma_1}(\Omega) \cap \mathcal{C}(\overline{\Omega})$. Let we denote Σ^δ the connected component of $\mathcal{C}_\delta \cap T_R$ that contains $(\lambda_1^\delta, 0)$. For a sequence $\delta_n \to 0$ and $\Sigma^n = \Sigma^{\delta_n}$ we have that $\bigcup \Sigma^n$ is precompact by the next lemma.

LEMMA 4.12. *Assume the hypotheses of Theorem* 4.10 *and let $\{(\lambda_k, u_k)\}$ be a bounded sequence of solutions to $(\mathrm{P}^\delta_{s,\lambda})$, then there exists a subsequence which converges to (λ_0, u_0), a solution to $(\mathrm{P}^{\omega_\beta}_{s,\lambda})$ with $\lambda = \lambda_0$.*

PROOF. Up to a subsequence, we have that $\lambda_k \to \lambda_0$, $u_k \rightharpoonup u_0$ (weakly) in $E^{p,\gamma}_{\Sigma_1}(\Omega)$ and $u_k \to u_0$ (strongly) in $L^s_\beta(\Omega)$ by Lemma 2.8, therefore, $u_k \to u_0$ in $E^{p,\gamma}_{\Sigma_1}(\Omega)$ because

$$\begin{aligned}\int_\Omega &|x|^{-p\gamma}|\nabla u_k|^{p-2}\langle\nabla u_k, \nabla(u_k - u_0)\rangle\,dx \\ &= \lambda_k\int_\Omega u_k^{s-1}(u_k - u_0)\omega_\beta(x)\,dx + \int_\Omega |x|^{-q\mu}u_k^{q-1}(u_k - u_0)\,dx \to 0,\end{aligned}$$

and by the classical inequalities, for $x, y \in \mathbb{R}^N$,

$$\langle |x|^{p-2}x - |y|^{p-2}y, x - y\rangle \ge c_p\begin{cases} |x-y|^p & \text{for } p \ge 2, \\ (|x|+|y|)^{p-2}|x-y|^2 & \text{for } 1 < p < 2, \end{cases} \tag{4.6}$$

we conclude that

$$\int_\Omega |x|^{-p\gamma}|\nabla(u_k - u_0)|^p dx \longrightarrow 0 \quad \text{as } k \to \infty. \qquad \square$$

END OF PROOF TO THEOREM 4.10. Since $\lambda_1^{\delta_n} \to 0$ as $\delta_n \to 0$ it follows that $(0,0) \in \liminf \Sigma^n$, then Lemma 4.11 applies to Σ^n. The conclusion is that $\mathcal{C}_R = \limsup \Sigma^n = \limsup(\mathcal{C}_{\delta_n} \cap T_R) \neq \emptyset$ is connected and closed. Moreover, it is clear that $\mathcal{C}_R$ meets T_R for all $R > 0$. We set $\mathcal{C} = \bigcup_{R>0} \mathcal{C}_R$ then we have proved is a continuum in E with $(0,0) \in \mathcal{C}$.

To prove (a) and (b) we argue as follows:

(a) We claim that there exists $c(\lambda)$ small such that if $(\lambda, u) \in \Sigma$ then $\|u\|_\infty \geq c(\lambda)$, this is enough to prove (a). Since $\lambda_1^\delta \to 0$ as $\delta \to 0$, given $\lambda > 0$, there exists $\delta_0 > 0$ such that

$$\lambda h(\delta_0)\delta_0^{1-p} > \lambda_1(\omega_\beta) \quad \text{if } \delta < \delta_0.$$

Let $\varepsilon > 0$ small, we can find $c(\lambda) > 0$ such that

$$\lambda h_\delta(t) + t^{q-1} > (\lambda_1(\omega_\beta) + \varepsilon)t^{p-1} \quad \text{for all } t \in (0, c(\lambda)].$$

Hence if $\|u\|_\infty < c(\lambda)$, we have that

$$-\Delta_{p,\gamma} u \geq (\lambda_1(\omega_\beta) + \varepsilon)\omega_\beta |u|^{p-2}u.$$

Moreover, we can find $\alpha > 0$ small such that for φ_1 a positive eigenfunction associated to the first eigenvalue $\lambda_1(\omega_\beta)$ we find $\alpha\varphi_1 \leq u$. As a consequence, by the iteration method, we find a solution to the eigenvalue problem

$$-\Delta_{p,\gamma} u = (\lambda_1 + \delta)\omega_\beta(x)|u|^{p-2}u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \Sigma_1.$$

But this is a contradiction with the isolation of $\lambda_1(\omega_\beta)$ (Lemma 3.8).

To prove (b), we argue by contradiction, i.e. we assume that there exists $(\lambda_n, u_n) \in \Sigma^n$ with $\lambda_n \to 0$, $u_n \to u$ and $\|u_n\|_\infty = \rho \leq \rho_0$. Then we have that $u_n \rightharpoonup u$ in $E^{p,\gamma}_{\Sigma_1}(\Omega)$, hence $u \not\equiv 0$ is a weak solution of

$$-\Delta_{p,\gamma} u = |u|^{q-1}u \quad \text{with } \|u\|_\infty \leq \rho_0,$$

but this is a contradiction because if we define $v_n = u_n/\|u_n\|_\infty$, and u_n is a sequence of solutions to $-\Delta_{p,\gamma} u = |u|^{q-u}u$ in $E^{p,\gamma}_{\Sigma_1}(\Omega)$ with $\|u_n\|_\infty \to 0$, then there exists a subsequence weakly convergent in $E^{p,\gamma}_{\Sigma_1}(\Omega)$ verifying $-\Delta_{p,\gamma} v_n = v_n^{q-1}\|u_n\|_\infty^{q-p}$ and for some subsequence,

$$\int_\Omega |\nabla v_n|^{-p\gamma} dx = \int_\Omega v_n^{q-1}\|u_n\|_\infty^{q-p}\, dx \to 0,$$

that is a contradiction. □

Remark 4.13. Theorem 4.10 may be generalized to obtain bifurcation in energy from $(0,0)$ when $\beta = (\gamma + 1)$. The proof is obtained following similar arguments of [1], since the exponent $s < p$ gives margin to use Lemma 2.8 in order to obtain the required compactness.

5. Hölder continuity

In this section we are going to prove Hölder continuity for the solutions to problem

$$(5.1)\qquad \begin{cases} -\Delta_{p,\gamma} u = f & \text{in } \Omega, \\ B(u) = 0 & \text{on } \partial\Omega, \end{cases}$$

where we assume that $\Omega \subset \mathbb{R}^N$ is a smooth domain with $0 \in \Omega$, $f \in L^r_\eta$ with $r > N/p$, $\eta = -p^*\gamma(r-1)/r$, $-\infty < \gamma < (N-p)/p$ and the boundary conditions $B(\,\cdot\,)$ as in the introduction. The main theorem of this section is the following.

Theorem 5.1. *Let $u \in E^{p,\gamma}_{\Sigma_1}(\Omega)$ be a solution to problem* (5.1). *Then $u \in \mathcal{C}^\kappa(\overline{\Omega})$ for some $0 < \kappa < 1/2$.*

The proof of this theorem is an immediate consequence of Theorems 5.7 and 5.15.

We start with next two lemmas by Stampacchia (see [23]), that we will use in the sequel.

Lemma 5.2. *Let α be a positive constant and let $\{x_k\}$ be a sequence verifying*

$$x_{k+1} \le C B^k x_k^{(1+\alpha)}, \quad \text{con}\, C > 0,\ B > 1.$$

Then, if $x_0 \le C^{-1/\alpha} B^{-1/\alpha^2}$, it follows $x_k \le B^{-k/\alpha} x_0$ and as a consequence $\lim_{k\to\infty} x_k = 0$.

Lemma 5.3. *Let $\phi\colon [0,\infty) \to [0,\infty)$ be an increasing function and suppose that there exists θ, $0 < \theta < 1$ such that for all $R > R_0$*

$$(5.2)\qquad \phi(\theta R) \le \theta^\eta \phi(R) + BR^\beta, \quad \text{con}\, 0 < \beta < \eta.$$

Then there exists $C = C(\theta,\eta,\beta)$ such that for all $\rho < R > R_0$ it has,

$$\phi(\rho) \le C\bigg\{\bigg(\frac{\rho}{R}\bigg)^\beta \phi(R) + B\rho^\beta\bigg\}.$$

5.1. Interior Hölder continuity. In this subsection we will prove hölder continuity of solutions to problem (5.1) in a neighbourhood of the origin, to do that, we will work inside of a small ball $B_{R_0}(0)$ verifying $B_{R_0}(0) \subset\subset \Omega$. We will note $A^+(k,R) = \{x \in B_R(0) | u^+(x) > k\}$.

Theorem 5.4 (Cacciopoli type inequality). *Let $u \in E^{p,\gamma}_{\Sigma_1}(\Omega)$ be a solution to*

$$-\Delta_{p,\gamma} u = f \quad \textit{in } \Omega$$

with $f \in L^r_\eta(\Omega)$ *for some* $r > N/p$, *and* $\eta = -p^*\gamma(r-1)/r$. *Then, for all* $0 < \rho < R < R_0$ *and all* $k \in \mathbb{R}$, *we have*

$$\tag{5.3} \begin{aligned} &\int_{A(k,\rho)} |x|^{-p\gamma}|\nabla u|^p\,dx \\ &\qquad \le \frac{H}{(R-\rho)^p}\int_{A(k,R)} |x|^{-p\gamma}|u-k|^p\,dx + H\mathcal{X}^p|A(k,R)|^{1-p/N+\varepsilon}_{-p^*\gamma}, \end{aligned}$$

where

$$\varepsilon = \frac{p}{p-1}\left(\frac{p}{N}-\frac{1}{r}\right) > 0.$$

Proof. First we observe that $\varepsilon > 0$ is a consequence of $r > N/p$. We define $t^+_k(u) = u - T^k(u)$ then

$$\frac{\partial t^+_k}{\partial x_i}(u(x)) = \begin{cases} 0 & \text{a.e. in } A^-(k,\rho), \\ u_{x_i}(x) & \text{a.e. in } A^+(k,\rho). \end{cases}$$

Let $g(r)$ be a nonnegative function such that $g \in \mathcal{C}^1([0,\infty))$, $g(r) = 1$ for $r \le \rho$ and $g(r) = 0$ for $r \ge R$. It is clear that the function $v(x) = t^+_k(u)g(|x-y|) \in E^{p,\gamma}_{\Sigma_1}(\Omega)$. Taking v as a test function, we get

$$\tag{5.4} \begin{aligned} &\int_{A^+(k,R)} g(r)|x|^{-p\gamma}|\nabla u|^p\,dx \\ &+\int_{A^+(k,R)} (u-k)|x|^{-p\gamma}|\nabla u|^{p-2}\left\langle \nabla u, \frac{x-y}{r}\right\rangle g'(r)\,dx = \int_{A^+(k,R)} fg(r)(u-k)\,dx. \end{aligned}$$

Now we estimate the terms in the above equality. By Hölder and Young inequalities,

$$\tag{5.5} \begin{aligned} &\left|\int_{A^+(k,R)} g'(r)(u-k)|x|^{-p\gamma}|\nabla u|^{p-2}\langle \nabla u, \frac{x-y}{r}\rangle\,dx\right| \\ &\le \frac{p-1}{p}\int_{A^+(k,R)} g(r)|x|^{-p\gamma}|\nabla u|^p\,dx + \frac{1}{p}\int_{A^+(k,R)} \frac{|g'(r)|^p}{|g(r)|^{p-1}}|x|^{-p\gamma}|u(x)-k|^p\,dx \end{aligned}$$

$$\tag{5.6} \begin{aligned} \left|\int_{A^+(k,R)} fg(r)(u-k)\,dx\right| &\le \left(\int_{A^+(k,R)} \frac{(u-k)^{p^*}g(r)^{p^*}}{|x|^{p^*\gamma}}\,dx\right)^{1/p^*} \\ &\quad\cdot\left(\int_{A^+(k,R)} |f|^r|x|^{(\gamma+\varepsilon)r}\right)^{1/r}\left(\int_{A^+(k,R)} \frac{dx}{|x|^{\varepsilon s}}\right)^{1/s} \end{aligned}$$

where $1/p^* + 1/r + 1/s = 1$, $\varepsilon > 0$. We observe that we can take $\varepsilon s = p^*\gamma$, $(\gamma+\varepsilon)r = p^*\gamma(r-1)$. Therefore by Caffarelli–Kohn–Nirenberg (Remark 2.3) and Young inequalities it follows that

$$
\begin{aligned}
(5.7)\quad \bigg|\int_{A^+(k,R)} fg(r)(u-k)\,dx\bigg| &\le c\delta\int_{A^+(k,R)} |x|^{-p\gamma}|\nabla u|^p\,dx \\
&+\frac{1}{\delta}\bigg(\int_{A^+(k,R)} |f|^r|x|^{\gamma p^*(r-1)}dx\bigg)^{p'/r}|A^+(k,R)|_{-p^*\gamma}^{p'(1/r'-1/p^*)},
\end{aligned}
$$

for some positive constants c, δ. Taking into account that

$$
p'\bigg(\frac{1}{r'}-\frac{1}{p^*}\bigg)=1-\frac{p}{N}+\varepsilon
$$

and inserting (5.5) and (5.7) in (5.4) with the function

$$
g(r)=\begin{cases} 1 & \text{for } 0\le r\le\rho,\\ \dfrac{(R-r)^p(R+pr-(p+1)\rho)}{(R-\rho)^{p+1}} & \text{for } \rho<r<R,\\ 0 & \text{for } r\ge R,\end{cases}
$$

we obtain (5.3) with $A^+(k,r)$. The same computations with the obvious changes can be done with $A^-(k,r)$, and as a consequence for $A(k,r)$. □

LEMMA 5.5. *Let $u\in L^\infty$ be a positive function verifying (5.3) for all $k\in\mathbb{R}$. Then, if $k_0+\sup|u|<M$, we have*

$$
\begin{aligned}
(5.8)\quad \sup_{Q_{R/2}} u-k_0 &\le c\bigg(\int_{A(k_0,R)} |x|^{-p\gamma}(u-k_0)^p\,dx\bigg)^{1/p} \\
&\cdot|A(k_0,R)|_{-p^*\gamma}^{\alpha/p}R^{-(N-p^*\gamma)\varepsilon/(\alpha p)}R^{-\gamma(p^*-p)/(\alpha p)}+\chi R^\beta
\end{aligned}
$$

where $p\beta=(N-p^*\gamma)\varepsilon$ and $\alpha^2+\alpha=\varepsilon$, $\alpha>0$.

PROOF. Without loss of generality, we take $k_0=0$. By (5.3), a simple computation permit us to estimate

$$
\begin{aligned}
(5.9)\quad &\int_{A(k,\rho)} |x|^{-p\gamma}(u-k)^p\,dx \\
&\le c|A(k,r)|^{p/N}\bigg\{\frac{1}{(r-\rho)^p}\int_{A(k,r)} |x|^{-p\gamma}(u-k)^p\,dx+\chi^p|A(k,r)|_{-p^*\gamma}^{1-p/N+\varepsilon}\bigg\}
\end{aligned}
$$

for all $0<\rho<r<R$. Si $h<k$ para todo $\rho<r$ tenemos

$$
(5.10)\qquad |A(k,\rho)|_{-p\gamma}\le\frac{1}{(h-k)^p}\int_{A(h,r)} |x|^{-p\gamma}(u-h)^p\,dx.
$$

Defining $U(k,t)=\int_{A(k,t)}|x|^{-p\gamma}(u-k)^p dx$, the inequalities (5.9) and (5.10) imply

$$
\begin{aligned}
(5.11)\quad U(k,\rho)&\le c(r-\rho)^{-p}U(h,r)|A(h,r)|^{p/N} \\
&+c\chi^p(h-k)^{-p}U(k,r)|A(k,r)|_{-p^*\gamma}^{1-p/N+\varepsilon}|A(k,r)|_{-p\gamma}^{-1}|A(k,r)|^{p/N}.
\end{aligned}
$$

Since $|A(k,r)|_{-p^*\gamma} \le cr^{\gamma(p-p^*)}|A(k,r)|_{-p\gamma}$, it follows that

$$
\begin{aligned}
(5.12)\quad U(k,\rho) &\le c(r-\rho)^{-p}U(h,r)|A(h,r)|^{p/N} \\
&\quad + c\chi^p(h-k)^{-p}U(k,r)|A(k,r)|^{\varepsilon-p/N}_{-p^*\gamma}r^{-p^*\gamma/(N-p)}|A(k,r)|^{p/N} \\
&\le c\bigg\{\frac{r^p}{(r-\rho)^p}+\bigg(\frac{\chi r^\beta}{h-k}\bigg)^p\bigg\}r^{-(N-p^*\gamma)\varepsilon}U(h,r)|A(h,r)|^{\varepsilon}_{-p^*\gamma}.
\end{aligned}
$$

We define

$$\phi(k,t) = U(k,t)|A(k,t)|^{\alpha}_{-p^*\gamma}.$$

Then taking power α in (5.10) and multipying in (5.12) by $|A(k,\rho)|^{\alpha}_{-p\gamma}$ we obtain

$$\phi(k,\rho) \le c\bigg[\bigg(\frac{r}{r-\rho}\bigg)^p+\bigg(\frac{\chi r^\beta}{k-h}\bigg)^p\bigg]\frac{r^{-(N-p^*\gamma)\varepsilon}}{(k-h)^{p\alpha}}r^{-\gamma(p^*-p)}\phi^{1+\alpha}(h,r).$$

We define $d = \chi R^\beta + CR^{-(N-p^*\gamma)\varepsilon/(\alpha p)}R^{-\gamma(p^*-p)}\phi_0^{1/p}$ with C a positive constant that we will elect. We define the sequences

$$k_i = d\bigg(1-\frac{1}{2^i}\bigg),\quad r_i = \frac{R}{2}\bigg(1+\frac{1}{2^i}\bigg),\quad i=0,1,\ldots$$

For $\phi_i = \phi(k_i,r_i)$, the last inequality implies

$$\phi_{i+1} \le cd^{-p\alpha}2^{p(1+\alpha)i}R^{-(N-p^*g)\varepsilon-\gamma(p^*-p)}\phi_i^{1+\alpha},\quad i=0,1,\ldots$$

Taking C sufficiently large, we have the hypotheses of Lemma 5.2, then as a consequence

$$\lim_{i\to\infty}\phi_i = 0,\quad \text{i.e. } \phi(d,R/2)=0.$$

Therefore we conclude that

$$
\begin{aligned}
\sup_{Q_{R/2}} u \le d = c\bigg(\int_{A(0,R)}|x|^{-p\gamma}u^p\,dx\bigg)^{1/p} \\
\cdot |A(0,R)|^{\alpha/p}_{-p^*\gamma}R^{-((N-p^*\gamma)\varepsilon-\gamma(p^*-p))/(\alpha p)}+\chi R^\beta.
\end{aligned}
$$

The proof finish substituting u by $u-k_0$. □

Lemma 5.6. *Let $u\in L^\infty$ be a function verifying* (5.3) *for all $k\in\mathbb{R}$. Let us take $2k_0 = M(2R)-m(2R)$. Assume that for some $0<\eta<1$, we have*

$$|A(k_0,R)| \le \eta|Q_R|.$$

Then, if there exists $n\in\mathbb{N}$ such that

$$\omega(u,2R)\ge 2^{n+1}\chi R^\beta,\quad \textit{for } k_n = M(2R)-2^{-(n+1)}\omega(u,2R),$$

we obtain

$$|A(k_n,R)| \le Cn^{-(N(p-1))/(p(N-(\gamma+1)))}|Q_R|^{(N-p\gamma-1)/(p(N-(\gamma+1)))}.$$

Proof. For $k_0 < h < k$ we define the truncature function

$$G(s) = \begin{cases} k-h & \text{if } s \geq k, \\ s-h & \text{if } h < s < k, \\ 0 & \text{if } s \leq h. \end{cases}$$

Hence $G(u) = 0$ in $Q_R \setminus A(k_0, R)$ and by hypothesis $|Q_R \setminus A(k_0, R)| \geq (1-\eta)|Q_R|$. By the Hardy–Sobolev inequality,

$$\begin{aligned} &\bigg(\int_{A(k_0,R)} G(u)^{N/(N-(\gamma+1))}\, dx \bigg)^{1-(\gamma+1)/N} \\ &\qquad\leq C \int_{Q_R} |x|^{-\gamma} |\nabla G(u)|\, dx = C \int_{A(h,R)-A(k,R)} |x|^{-\gamma} |\nabla u|\, dx \end{aligned}$$

we define $\Delta(h,k) = A(h,R) \setminus A(k,R)$, it follows

$$\begin{aligned} (k-h)|A(k,R)|^{1-(\gamma+1)/N} &\leq \bigg(\int_{Q_R} G(u)^{N/(N-(\gamma+1))}\, dx \bigg)^{1-(\gamma+1)/N} \\ &\leq c|\Delta(h,k)|^{1-1/p} \bigg(\int_{\Delta(h,k)} |x|^{-p\gamma} |\nabla u|^p \bigg)^{1/p}. \end{aligned}$$

On the other hand, by (5.3) with R and $2R$,

$$\begin{aligned} &\int_{A(h,R)} |x|^{-p\gamma} |\nabla u|^p\, dx \\ &\qquad \leq \frac{c}{R^p} \int_{A(h,2R)} |x|^{-p\gamma} (u-h)^p\, dx + c\chi^p |A(k,2R)|_{-p^*\gamma}^{1-p/N+\varepsilon} \\ &\qquad \leq cR^{N-p(\gamma+1)} (M(2R)-h)^p + c\chi^p R^{(N-p^*\gamma)(1-p/N)} (M(2R)-h)^p. \end{aligned}$$

Where we have used that if $h \leq k_n$, $M(2R) - h \geq M(2R) - k_n \geq \chi R^\beta$, and $p\beta = (N - p^*\gamma)\varepsilon$, then we conclude that,

$$\int_{A(h,R)} |x|^{-p\gamma} |\nabla u|^p\, dx \leq cR^{N-p(\gamma+1)} (M(2R)-h)^p.$$

Hence,

$$(k-h)|A(k,R)|^{1-(\gamma+1)/N} \leq C|\Delta(h,k)|^{1-1/p} R^{(N-p(\gamma+1))/p} (M(2R)-h). \tag{5.13}$$

Taking the levels

$$k_i = M(2R) - \frac{1}{2^{i+1}} \omega(u, 2R),$$

then (5.13) applied to $h = k_{i-1}$, $k = k_i$ implies that

$$|A(k_i,R)|^{1-(\gamma+1)/N} \leq c|\Delta(k_{i-1},k_i)|^{1-1/p} R^{(N-p(\gamma+1))/p}$$

taking the power $p/(p-1)$,

$$\begin{aligned} |A(k_n,R)|^{(p(N-(\gamma+1)))/(N(p-1))} &\leq |A(k_i,R)|^{(p(N-(\gamma+1)))/(N(p-1))} \\ &\leq c|\Delta(k_{i-1},k_i)| R^{(N-p(\gamma+1))/(p-1)}. \end{aligned}$$

Adding from $i = 1$ to $i = n$ we obtain

$$\begin{aligned} n|A(k_n, R)|^{(p(N-(\gamma+1)))/(N(p-1))} &\le cR^{(N-p(\gamma+1))/(p-1)} \sum_{i=1}^{n} |\Delta(k_{i-1}, k_i)| \\ &\le cR^{(N-p(\gamma+1))/(p-1)} |A(k_0, R)| \end{aligned}$$

and finally we have

$$\begin{aligned} |A(k_n, R)| &\le Cn^{-(N(p-1))/(p(N-(\gamma+1)))} \\ &\quad \cdot |Q_R|^{(N-p(\gamma+1))/(p(N-(\gamma+1)))+(p-1)/(p(N-(\gamma+1)))} \\ &= Cn^{-(N(p-1))/(p(N-(\gamma+1)))} |Q_R|^{(N-p\gamma-1)/(p(N-(\gamma+1)))}. \qquad \square \end{aligned}$$

Now we can prove the main theorem of this subsection that is an extension to our framework of the classical De Giorgi Theorem.

Theorem 5.7. *Let $u \in L^\infty$ be a function verifying* (5.3) *for all $k \in \mathbb{R}$. Then u is Hölder continuous in a neighbourhood of the origin.*

Proof. Let as take $2k_0 = M(2R) - m(2R)$ as before. We can assume

$$|A(k_0, R)| \le \frac{1}{2} |Q_R|.$$

We take $k_n = M(2R) - 1/2^{n+1} \omega(u, 2R)$, then $k_n > k_0$. By using (5.8) for k_n instead of k_0, we have,

$$\begin{aligned} \sup_{Q_{R/2}} (u - k_n) &\le c \bigg(\int_{A(k_n,R)} |x|^{-p\gamma} (u - k_n)^p \, dx \bigg)^{1/p} \\ &\quad \cdot |A(k_n, R)|^{\alpha/p}_{-p^*\gamma} R^{-(N-p^*\gamma)\varepsilon/(\alpha p)} R^{\gamma(p-p^*)/(\alpha p)} + \chi R^\beta \\ &\le c \sup_{Q_R} (u - k_n) |A(k_n, R)|^{\alpha/p}_{-p^*\gamma} |A(k_n, R)|^{1/p}_{-p\gamma} \\ &\quad \cdot R^{-(N-p^*\gamma)\varepsilon/(\alpha p)} R^{(\gamma(p-p^*))/\alpha p} + \chi R^\beta. \end{aligned} \tag{5.14}$$

Claim. For n big enough we have

$$c|A(k_n, R)|^{\alpha/p}_{-p^*\gamma} |A(k_n, R)|^{1/p}_{-p\gamma} R^{-(N-p^*\gamma)\varepsilon/(\alpha p)} R^{\gamma(p-p^*)/(\alpha p)} < 1/2.$$

As a consequence,

$$\omega\left(u, \frac{R}{2}\right) \le \omega(u, 2R) \left(1 - \frac{1}{2^{n+2}}\right) + c\chi R^\beta. \tag{5.15}$$

Then either

$$\omega(u, 2R) \le 2^{n+1} \chi R^\beta \quad \text{or} \quad \omega(u, 2R) > 2^{n+1} \chi R^\beta,$$

and as a consequence, it satisfies (5.15. In both cases we have

$$\omega\left(u, \frac{R}{2}\right) \le \omega(u, 2R) \left(1 - \frac{1}{2^{n+2}}\right) + c\chi R^\beta.$$

By applying Lemma 5.3 with $\theta = 1/4$ and $\eta = \log_\theta(1 - 1/2^{n+2})$. Then

$$\omega(u,\rho) \le C\left(\left(\frac{\rho}{R}\right)^{\beta}\omega(u,R) + \chi\rho^{\beta}\right), \quad \text{for all } \rho < R < R_0.$$

To finish we only need to prove the claim. To do that we use the Lemma 5.6. If

$$\begin{aligned}\omega(u,2R) &\ge 2^{n+1}\chi R^{\beta},\\ |A(k_n,R)| &\le Cn^{-(N(p-1))/(p(N-(\gamma+1)))}|Q_R|^{(N-p\gamma-1)/(p(N-(\gamma+1)))}\end{aligned}$$

then we obtain

$$\begin{aligned}c|A(k_n,R)|_{-p^*\gamma}^{\alpha/p}&|A(k_n,R)|_{-p\gamma}^{1/p}R^{-(N-p^*\gamma)\varepsilon/(\alpha p)}R^{\gamma(p-p^*)/(\alpha p)}\\ &\le cn^{-N(p-1)/(p(N-(\gamma+1)))}R^{-\gamma}R^{(N(N-p\gamma-1)/(p(p(N-(\gamma+1))))}\\ &\quad\cdot R^{-(N-p^*\gamma)/p-(\gamma(p^*-p))/(\alpha p)} < 1/2\end{aligned}$$

for n sufficiently large, because the exponent on R is nonnegative for α or equivalently ε small enough. Hence we have proved the claim and then the Theorem 5.7. □

5.2. Hölder continuity at the boundary. In this subsection we will prove the Hölder continuity of solutions to problem (5.1) in neighbourhoods of the points where the boundary conditions change. Hence we will work at subsets of the form $B_\eta(x_0) \cap \Omega$ with $x_0 \in \overline{\Sigma}_1 \cap \overline{\Sigma}_2 = \Gamma$ and $\eta > 0$ small. Taking into account that the weight $|x|^{-p\gamma}$ is regular at the boundary, we observe that it is enough to prove that regularity for solutions to problem

$$\text{(P)} \qquad \begin{cases} -\Delta_p u = f & \text{in } \Omega,\\ B(u) = 0 & \text{on } \partial\Omega,\end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $f \in L^r$ with $r > N/p$. Then by simplicity we prove it to solutions to problem (P), where the boundary conditions $B(\,\cdot\,)$ are as in the introduction, but without the weight $|x|^{-p\gamma}$. In this subsection we denote $E^{p}_{\Sigma_1}(\Omega) = E^{p,\gamma}_{\Sigma_1}(\Omega)$ with $\gamma = 0$.

Before to start, we remark that in this kind of problems there is a lack of regularity for these mixed Dirichlet–Neumann problems in Γ. The known results about these questions are detailed in [22] for an elliptic second order operator with fixed boundary conditions, i.e. the solutions are in $\mathcal{C}^{\kappa}(\overline{\Omega})$ for some $0 < \kappa < 1/2$ and in [8] for the laplace operator with moving boundary conditions, i.e. the solutions are in $\mathcal{C}^{\kappa}(\overline{\Omega})$ for some $0 < \kappa < 1/2$ with κ independent of $\mathcal{H}_{N-1}(\Sigma_1) \in [0, \partial\Omega]$. Here we extend the results about Hölder continuity of [22], [8] to our framework.

Some notations are in order. For $u \in W^{1,p}(\Omega)$ we denote $A^+(k) = [u > k] = \{x \in \Omega : u(x) > k\}$, $k \in \mathbb{R}$. And for $y \in \overline{\Omega}$, $\rho > 0$ we denote $\Omega(y,\rho) = B_\rho(y) \cap \Omega$, $A^+(k,\rho) = A^+(k) \cap \Omega(y,\rho)$.

For $y \in \Gamma$, there exists a positive constant τ and some $\widetilde{\rho}$ such that for all $0 < \rho < \widetilde{\rho}$,

$$\|u\|_{L^{p^*}(\Omega(y,\rho))} \le \tau \|\nabla u\|_{L^p(\Omega(y,\rho))} \quad \text{for all } u \in E^p_{\Sigma_1}(\Omega). \tag{5.16}$$

Lemma 5.8. *We can choose $\tau \in (C(N), \infty)$ independent of $\mathcal{H}_{N-1}(\Sigma_1) \in (0, |\partial\Omega|)$ such that* (5.16) *holds.*

The proof of this lemma can be obtained with the appropriate changes of Lemma 6.1 in [8].

Remark 5.9. We point out that this lemma is the key to treat phenomena of convergence of the boundary conditions to have some compactness properties, although we will not treat here this problem, (see [8], [9] and [10]).

Theorem 5.10. *Let u be a function in $E^p_{\Sigma_1}(\Omega)$, then we have that*

$$|A^+(h,\rho)|^{(pN-p)/N} \le (|A^+(k,\rho)| - |A^+(h,\rho)|)^{p-1} \frac{\tau^p}{|h-k|^p} \int_{A^+(k,\rho)} |\nabla u|^p \, dx. \tag{5.17}$$

This result is proved in [22] for $p = 2$, the extension for $p \ne 2$ is a straightforward computation, and we omit it.

Theorem 5.11 (Cacciopoli type inequality). *Let $u \in E^p_{\Sigma_1}(\Omega)$ be a solution to problem* (P). *Then there exist two positive constants ζ, Λ independents of τ such that for $y \in \overline{\Omega}$, $0 < \rho < R < \widetilde{\rho}(y)$ and all $k > 0$ we have*

$$\begin{aligned}\int_{A^+(k,\rho)} |\nabla u|^p \, dx \le{}& \frac{\zeta}{(R-\rho)^p} \int_{A^+(k,R)} |u-k|^p \, dx \\ &+ \Lambda \bigg(\int_{A^+(k,R)} |f|^r \, dx \bigg)^{p/((p-1)r)} |A^+(k,R)|^{p^2(1-N/(pr))/(p-1)N + (1-p/N)}.\end{aligned} \tag{5.18}$$

The proof of this theorem follows in a similar to the proof of Theorem 5.4.

Theorem 5.12. *Let $u \in E^p_{\Sigma_1}(\Omega)$ be a solution of* (P). *Fixed $0 < \sigma < 1$ and given $k_1 > 0$, there exists a constant $\theta(\sigma) > 0$ such that fixed $y \in \overline{\Omega}$, $\rho < \widetilde{\rho}(y)$, for all $k > k_1$ that verifies*

$$|A^+(k,\rho)| \le \theta |\Omega(y,\rho)| \tag{5.19}$$

then

$$|A^+(k + \sigma d, \rho - \sigma\rho)| = 0, \tag{5.20}$$

where

$$d^p \ge \frac{1}{\theta \rho^N} \int_{A^+(k,\rho)} |u-k|^p \, dx + \rho^{p^2(1-N/(pr))/(p-1)} \bigg(\int_{A^+(k,\rho)} |f|^r \, dx \bigg)^{p/((p-1)r)}.$$

See [22] for a proof in the case $p = 2$, the proof of this theorem is a little modification. As a consequence we can prove the next result.

Theorem 5.13. *Let $u \in E^p_{\Sigma_1}(\Omega)$ be a solution to problem* (P)*. Fixed $y \in \overline{\Omega}$ and $4\rho < \widetilde{\rho}(y)$ we define*

$$l_1 = \sup_{x\in\Omega(y,4\rho)} u(x), \quad l_2 = \sup_{x\in\Omega(y,4\rho)} u(x)$$

and $\omega = \operatorname{osc}(u, 4\rho) = l_1 - l_2$. Let $0 < \eta_1 < 1$ be such that $l_1 - \eta_1\omega > 0$,

$$|A^+(l_1 - \eta_1\omega; 2\rho)| \le \theta|\Omega(y, 2\rho)|$$

where θ is the number related to $\sigma = 1/2$. Then there exist two positive numbers $\overline{\eta} < 1$ and $\overline{N} = \overline{N}(\tau, \|f\|_{L^r})$ independent of ρ and y such that

$$\operatorname{osc}(u, \rho) \le \overline{\eta}\omega + \overline{N}\rho^{p(1-N/(pr))/(p-1)}. \tag{5.22}$$

Proof. It is clear that there exists $\eta_1 = \eta_1(\lambda_1) > 0$ sufficiently small and probably dependent of y and ρ such that $k_1 = l_1 - \eta_1\omega > 0$ and

$$|A^+(l_1 - \eta_1\omega; 2\rho)| \le \theta|\Omega(y, 2\rho)|.$$

If we take

$$M^{r(p-1)/p} \ge \int_\Omega |f|^r\,dx$$

we have

$$\begin{aligned}\frac{1}{\theta}(2\rho)^N &\int_{A^+(l_1-\eta_1\omega;2\rho)} |u - l_1 - \eta_1\omega|^p\,dx + \rho^{p^2(1-N/(pr))/(p-1)} \\ &\cdot\bigg(\int_{A^+(l_1-\eta_1\omega;2\rho)} |f|^r dx\bigg)^{p/((p-1)r)} \\ &\le (\eta_1\omega + c_2M^{1/p}\rho^{p(1-N/(pr))/(p-1)})^p = d^p\end{aligned}$$

then by Theorem 5.12 we get

$$u(x) \le l_1 - \eta_1\omega + \frac{1}{2}\eta_1\omega + \frac{1}{2}c_2M^{1/p}\rho^{p(1-N/(pr)/(p-1))} \quad \text{a.e. in } \Omega(y, \rho)$$

thus we obtain

$$\operatorname{osc}(u, \rho) \le \sup_{\Omega(y,\rho)} u - l_2 \le \left(1 - \frac{1}{2}\eta_1\right)\omega + \frac{1}{2}c_2M^{1/p}\rho^{p(1-N/(pr))/(p-1)}$$

and as consequence we have proved (5.22) with $\overline{\eta} = 1-\eta_1/2$ and $\overline{N} = c_2M^{1/p}/2$.

Now we are going to see that in fact we can take $\overline{\eta} < 1$ independent of $y \in \overline{\Omega}$ and ρ. To do that we consider the sequences

$$\eta_j = \frac{1}{2^{j+1}}; \quad j = 0, 1, \ldots$$

and we take in correspondence $h = l_1 - \eta_{j+1}\omega$, $k = l_1 - \eta_j\omega$. By (5.17) it follows that

$$\begin{aligned}|A^+(l_1 - \eta_{j+1}\omega; 2\rho)|^{(pN-p)/N} &\le \frac{c_1 2^{p(j+2)}}{\omega^p}\bigg(\int_{A^+(l_1-\eta_1\omega;2\rho)} |\nabla u|^p\,dx\bigg)\\ &\qquad\cdot (|A^+(l_1-\eta_j\omega;2\rho)| - |A^+(l_1-\eta_{j+1}\omega;2\rho)|)^{p-1},\end{aligned}$$

by Theorem 5.11 it has

$$\begin{aligned}&|A^+(l_1 - \eta_{j+1}\omega; 2\rho)|^{(pN-p)/N} \le \frac{c_1(\tau,p)2^{p(j+2)}\zeta}{\omega^p\rho^p}\\ &\quad\cdot\bigg(\int_{A^+(l_1-\eta_j\omega;4\rho)} (u - l_1 + \eta_j\omega)^p\,dx + \omega_N \Lambda M(4\rho)^{p^2(1-N/(pr))/(p-1)}\bigg)\\ &\quad\cdot (|A^+(l_1-\eta_j\omega;2\rho)| - |A^+(l_1-\eta_{j+1}\omega;2\rho)|)^{p-1}.\end{aligned}$$

If $j \le n$ we obtain

$$\begin{aligned}&|A^+(l_1 - \eta_n\omega; 2\rho)|^{(pN-p)/N} \le C\omega_N\rho^{N-p}\\ &\quad\cdot\bigg(c_1\zeta 4^{N-1} + c_1\Lambda M 4^{p(1-N/(pr))/(p-1)} 2^{p(N+2)}\bigg(\frac{\rho^{p(1-N/(pr))/(p-1)}}{\omega}\bigg)^p\bigg)\\ &\quad\cdot (|A^+(l_1-\eta_j\omega;2\rho)| - |A^+(l_1-\eta_{j+1}\omega;2\rho)|)^{p-1}\end{aligned}$$

if we sum in the above inequality with respect to $j = 1, \dots, n$ we get

$$\begin{aligned}&n|A^+(l_1-\eta_n\omega;2\rho)|^{(pN-p)/N}\\ &\quad\le \omega_N^p\bigg(c_1\zeta + c_1\Lambda M 4^{p(1-N/(pr))/(p-1)}\bigg(2^{n+2}\frac{\rho^{p(1-N/(pr))/(p-1)}}{\omega}\bigg)^p\bigg)\rho^{p(N-1)}.\end{aligned}$$

Now we define

$$R = c_1\frac{\omega_N^p}{2^N}[c_1\zeta 4^{N-1} + \Lambda M 4^{p(1-N/(pr))/(p-1)}].$$

Let $\overline{n}$ be such that $R/\overline{n}\omega_N \le \theta^{p(N-1)/N}$, to do that, it is sufficient to take $\overline{n} \ge C\tau^s$ for some positive constants C, s. Therefore we have two alternatives:

(1) If $\omega(\rho) \le 2^{\overline{n}+2}\rho^{p(1-N/(pr))/(p-1)}$, then it satisfies (5.22).

(2) If $\omega(\rho) > 2^{\overline{n}+2}\rho^{p(1-N/(pr))/(p-1)}$ then

$$|A^+(l_1 - \eta_{\overline{n}})\omega; 2\rho| \le \bigg(\frac{R}{\overline{n}\omega_N}\bigg)^{N/(p(N-1))}\omega_N(2\rho)^N \le \theta|\Omega(y,2\rho)|$$

thus we obtain $l_1 - \eta_{\overline{n}} > 0$ and

$$\operatorname{osc}(u,\rho) \le \overline{\eta}\operatorname{osc}(u,4\rho) + \overline{N}\rho^{p/(p-1)(1-N/(pr))}. \qquad \square$$

THEOREM 5.14. *Let u be a solution to problem* (P) *and suppose the hypotheses of Theorem* 5.13. *Then, there exist two constants $H > 0$ and $0 < \kappa = \kappa(\beta) < 1/2$ such that for all $y \in \Gamma$, $\rho < \delta(y)$ we obtain*

$$\operatorname{osc}(u, \rho) \leq H\rho^{\kappa}. \tag{5.23}$$

PROOF. Let $r(y) = \min\{1, \widetilde{\rho}(y)\}$. By Theorem 5.13, we have that for all $\rho < r(y)$, $\omega(\rho) \leq \overline{\eta}\omega(4\rho) + \overline{N}\rho^{p(1-N/(pr))/(p-1)}$. Given $\overline{\eta}$ as in Theorem 5.13, $\overline{\eta} = 1 - \eta_{\overline{n}}$, we take $\overline{\kappa}$ such that $4^{\overline{\kappa}}\overline{\eta} = a < 1$ and we define

$$\kappa = \min\left\{\overline{\kappa}, \frac{p}{p-1}\left(1 - \frac{N}{pr}\right)\right\}.$$

We have observed, in the proof of Theorem 5.13, that

$$\overline{n} \geq \frac{R}{\omega_N \theta^{p(N-1)/N}} = c\tau^s$$

for some positive constants c, s, and we recall that, by Lemma 5.8, $\tau \in (c(N), \infty)$ does not depend on $\mathcal{H}_{N-1}(\Sigma_1)$. As a consequence, $\overline{n}$ is independent of $\mathcal{H}_{N-1}(\Sigma_1)$. Since $4^{\overline{\kappa}}\overline{\eta} = a < 1$, $\kappa = \min\{\overline{\kappa}, 1 - N/p\}$ then $4^{\overline{\kappa}} < 2^{\overline{n}+1}/2^{\overline{n}+1} - 1$ and taking logarithms, we have

$$\kappa < \frac{1}{\log 4} \log\left(\frac{2^{\overline{n}+1}}{2^{\overline{n}+1} - 1}\right)$$

is independent of $\mathcal{H}_{N-1}(\Sigma_1)$.

Let T be a positive constant such that $\omega(\rho) \leq T\rho^{\kappa}$ for $r(y)/4 \leq \rho \leq r(y)$, then by (5.22) we deduce that

$$\omega(\rho) \leq \overline{\eta}4^{\kappa}T\rho^{\kappa} + \overline{N}\rho^{\kappa} \quad \text{for } \frac{r(y)}{4^2} \leq \rho \leq \frac{r(y)}{4},$$

in general we get

$$\omega(\rho) \leq \left\{(4^{\kappa}\overline{\eta})^i T + \overline{N}\sum_{s=0}^{i-1}(4^{\kappa}\overline{\eta})^s\right\}\rho^{\kappa} \leq \left(Ta^i + \frac{\overline{N}}{1-a}\right)\rho^{\kappa} \quad \text{for } \frac{r(y)}{4^{i+1}} \leq \rho \leq \frac{r(y)}{4^i}.$$

Since we can take $\overline{i}$ big enough such that $Ta^{\overline{i}} < 1$, we have

$$H = 1 + \frac{N}{1-a} \quad \text{for } \rho < \delta(y) = \frac{r(y)}{4^{\overline{i}}}. \qquad \square$$

THEOREM 5.15. *Let u be a solution to problem* (P). *Then, $u \in \mathcal{C}^{\kappa}(\overline{\Omega})$ for some $0 < \kappa < 1/2$ independent of $\mathcal{H}_{N-1}(\Sigma_1) \in (0, |\partial\Omega|)$.*

Proof. The Hölder regularity of solutions to (P) in a subset $\Omega' \subset\subset \Omega \cup \Sigma_i$ for $i = 1, 2$ is classical. Let we consider $x_1, x_2 \in \Omega(y, \rho)$, $y \in \Gamma$ and $0 < \rho < \widetilde{\rho}$. Then

(1) if $|x_1 - x_2| \geq \delta$ then
$$\frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|^\kappa} \leq 2\frac{\max u(x)}{\delta^\kappa},$$

(2) if $|x_1 - x_2| < \delta$, since $|u(x_1) - u(x_2)| \leq \omega(|x_1 - x_2|) \leq H|x_1 - x_2|^\kappa$ it follows that
$$\frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|^\kappa} \leq H.$$

The same arguments work in $\Omega(y, \rho)$ for $y \in \overline{\Omega} \setminus \Gamma$ (see [23] for example). In any case, we obtain that $\|u\|_{\mathcal{C}^\kappa(\overline{\Omega})} \leq \overline{M}$ where $\overline{M} = \max\{2(\max u(x))\delta^\kappa, H\}$. $\square$

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