RICHARD DEDEKIND
(1833-1916)*

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On February 12, 1916, Julius Wilhelm Dedekind died at his native Brunswick in Germany. He was one of the world's most distinguished workers at the theory of numbers, and in particular with Ernst Eduard Kummer and Leopold Kronecker at the theory of algebraic numbers; and most of his work is described in supplements to his editions of Dirichlet's *Vorlesungen über Zahlentheorie*.[1] In these supplements we can find references to his fundamental and enormously important ideas on the nature and meaning of numbers.

From the point of view of the fundamental principles of mathematics and the closely allied questions of logic and philosophy, the most important works of Dedekind are on the explanation of "continuity" by comparison with the system of real numbers, in which the irrational numbers were defined in a memorable way, and on the exceedingly subtle question of the definition, by logical concepts alone, of the integer numbers. Both of Dedekind's classical pamphlets: *Stetigkeit und irrationale Zahlen* of 1872 and *Was sind, und was sollen die Zahlen?* of 1888 have been translated into English by W.W. Beman under the title: *Essays on the Theory of Numbers: I. Continuity and Irrational Numbers; II. The Nature and Meaning of Numbers*.[2] It is to this translation that the notes below refer.

The ideas of Dedekind on the nature and meaning of numbers, which are here described (§ II) after his logically subsequent and historically earlier work on continuity (§I), led Dedekind to work out—apparently in complete independence of the previous work of De Morgan and the contemporary work of Charles Peirce—the greater part of what is now known as "the logic of relations." On another occasion I hope to give an account of later critical and constructive work on both these contributions of Dedekind to the principles of mathematics.

I.

In the autumn of 1858, Dedekind, who was then professor at the Polytechnic School of Zurich, had, for the first time in his life, to lecture on the elements of the differential calculus, and then felt more acutely than ever before the lack of a really scientific

*The correct year of Dedekind's birth is 1831, not 1833 (editor).

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[1]A short indication of Dedekind's mathematical works was given by G.B. Mathews in *Nature*, vol. XCVII, 1916, pp. 103-104.

[2]Chicago and London: The Open Court Publishing Co., 1901.

foundation of arithmetic. "In discussing," he said, "the notion of the approach of a variable magnitude to a fixed limiting value, and especially in proving the theorem that every magnitude which grows continually but not beyond all limits must certainly approach a limiting value, I had recourse to geometrical evidences. Even now I maintain that such an employment of geometrical intuition is, from a didactic standpoint, extraordinarily useful and indeed indispensable, if we do not wish to lose too much time. But no one will deny that this manner of introduction to the differential calculus can make no claim to scientific accuracy. In my own case this feeling of dissatisfaction was so overpowering that I made a firm resolve to meditate until I should find a purely arithmetical and completely rigorous foundation for the principles of infinitesimal analysis. People say so often that the differential calculus is occupied with continuous magnitudes, and yet nowhere is there given an explanation of this continuity; and even the most rigorous expositions of the differential calculus do not found their proofs on continuity, but appeal with more or less consciousness of the fact to geometrical notions or notions suggested by geometry, or rest on theorems which have never been proved arithmetically. To these belongs, for example, the above mentioned theorem, and a closer investigation convinced me that this or any equivalent theorem can be regarded, in a sense, as a sufficient foundation for infinitesimal analysis. So all reduced to the discovery of its real origin in the elements of arithmetic and thus to obtain at the same time an actual definition of the essence of continuity. I succeeded in doing this on November 24, 1858." Although Dedekind communicated his ideas and discussed them with some of his colleagues and pupils, he could not make up his mind for many years to let them be printed because "the exposition is not quite easy, and besides the matter itself is so unfruitful."[3] However, he had half determined to select that theme for a publication to be dedicated to his father on the celebration in April, 1872, of the fiftieth anniversary of his father's entry into office, when, in March of that year, he came across Heine's memoir in Vol. LXXIV of Crelle's *Journal*,[†] with which in essentials Dedekind agreed, "as indeed cannot be, otherwise," but the form of his own work appeared to him to be simpler and to emphasize more precisely the main point. Also Dedekind remarked the identity of his axiom of the continuity of the straight line with Cantor's axiom, of which he read when writing his preface, and that he could not recognize the utility of Cantor's distinction of real numbers of still higher kind, because of his conception of the real domain as complete in itself.

Comparing the system of rational numbers, in order of magnitude, with the points of a straight line L, we see that, if any origin be taken on L and a fixed unit of measurement, to any rational number a can be constructed a corresponding point; but there are points (those determined by incommensurable lengths measured from 0) to which no rational numbers correspond. Thus we can say that "L is infinitely richer in point-individuals than the domain domain R of rational numbers in number-individuals."[4] So if, as we wish,[5] all

[3]*Stetigkeit*, (2d ed., 1892), p. 2; cf. *Essays*, p. 2.

[†] We have corrected the misprint in the original, which read "Crede's *Journal*" (editor).

[4]*Stetigkeit*, p. 9; *Essays*, p. 9.

[5]"Was doch der Wunsch ist," *ibid.*

phenomena in the straight line are also to be followed out arithmetically[6] R must be refined by the creation of new numbers, and the domain of numbers raised to the same completeness—or "continuity"—as the straight line.

"The way in which irrational numbers are usually introduced is connected with the concept of magnitude—which itself is nowhere rigorously defined—and explains number as the result of the measurement of one such magnitude by another of the same kind.[7] Instead of this I demand that arithmetic shall be developed out of itself. That such connections with non-arithmetical notions have furnished the immediate occasion for the extension of the number-concept may, in general, be granted (though this was certainly not the case in the introduction of complex numbers); but this surely is no sufficient ground for introducing these foreign connections into arithmetic, the science of numbers. Just as negative and fractional rational numbers must and can be formed by a new creation, and as the laws of operation with these must and can be reduced to the laws of operation with positive integers, so we must we endeavor completely to define irrational numbers by means of the rational numbers alone. There only remains the question as to how to do this."[8]

Now the essence of this "continuity" of L was found by Dedekind[9] after long meditation to be: If all the points of L fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which generates this division. This, as Dedekind emphasized, will probably be considered as evidently true by every one; it cannot be proved, but is an axiom by means of which we first recognize the line of its continuity. If space has a real existence, it need not necessarily be continuous; many of its properties would remain the same if it were discontinuous;[10] and if we knew that it was discontinuous, nothing could prevent us, if we wished, making it continuous in thought by filling up its lacunae. Another simple logical transformation of the above axiom is not so obvious: there is one and only one point (of

[6]Cf. *Stetigkeit*, pp. 5-6, 10; *Essays*, pp. 4, 10.

[7]"The apparent advantage of this definition of number in point of generality vanishes the moment we think of complex numbers. In my view, the conception of the ratio to one another of two magnitudes of the same kind can be clearly developed only after the irrational numbers have been introduced."

[8]*Stetigkeit*, p. 10; *Essays*, pp. 9-10.

[9]*Stetigkeit*, p. 11; *Essays*, p. 11. This axiom has been frequently misunderstood; thus L. Couturat (*De l'infini mathématique*, Paris, 1896, p. 416) stated it: "If all the quantities of a kind can be divided into two classes such that all the quantities of the one precede (or follow) all those of the other, there exists a quantity of this kind which *both* follows all those of the inferior class *and* precedes all those of the superior class." Russell, in a review (*Mind*, Vol. VI, 1897, p. 117), rightly pointed out the mistake in this wording but wrongly advanced the same criticism against Dedekind's own axiom (*The Principles of Mathematics*, Vol. I, Cambridge, 1903, p. 279). In fact, we do not need, as Russell presumed, a "point left over to represent the section"; and Russell's (second) "emendation" (pp. 279-280) is Dedekind's original axiom.

[10]An example of this was given in the preface of *Was sind und was sollen die Zahlen?* (*Essays*, pp. 37-38). Choose any three points A, B, C, which do not lie in a straight line and which are such that the ratios of their distances AB, AC, BC are algebraic numbers; and regard as present in space only those points M for which the ratios of AM, BM, and CM to AB are algebraic numbers. The space consisting of the points M is everywhere discontinuous, but yet in it all the constructions in Euclid's *Elements* can be carried out just as well as in a continuous space.

the first class) which is on the extreme left of the first class, or one and only one of the second class on the extreme right of the second class, but not both.

The purely arithmetical definition of new numbers among those of the system R so as to make it a continuous system was now brought about on a basis analogous to that of the above axiom. Any rational number a brings about a division of the system R into two classes A_1, A_2, such that any number of A_1 is smaller than any number of A_2; a is either the greatest of A_1 or the least of A_2. If now we have any division of R into classes A_1, A_2, such that any member of A_1 is smaller than any member of A_2, we call such a division a "section" or "cut" (*Schnitt*, *coupure*), and denote it by (A_1, A_2). We can then say that any rational number a generates a section, or strictly speaking two sections (which, however, we will not regard as essentially different).[11] But there are an infinity of sections—such as that where A_1 consists of all the rational numbers r such that $r^2 < D$ is a positive non-square integer, and A_2 of the rest—which are not generated by rational numbers,—that is to say, neither has A_1 a maximum nor A_2 a minimum; and in this consists the incompleteness or discontinuity of R. Now, whenever we have a section (A_1, A_2) generated by no rational number, we create *(erschaffen)* a new, an "irrational number," which we regard as completely defined by the section (A_1, A_2) and is said to generate it.[12]

By comparing two sections, (A_1, A_2) and (B_1, B_2), as to the inclusion or not of any term of A_1 in B_1, or *vice versa*, we arrive at a basis for determining the order any two real (rational or irrational) numbers α and β as symbolized by

$$\alpha = \beta, \alpha > \beta, \text{ or } \alpha < \beta;$$ [13]

and also definitions of new sections whose generators may be represented by

$$\alpha + \beta, \alpha - \beta, \alpha . \beta \text{ and } \alpha^{\beta},$$

may be given.[14]

We will now indicate the use of the conception of a section to prove the theorems on limits mentioned above.[15] A variable x is said to have a fixed limiting value α, when $x - \alpha$ ultimately sinks, numerically speaking, below any positive, non-zero number; and our first theorem is that, if x increases continually, but not beyond all values, it approaches a *definite* limit. By the supposition, we have numbers α_2 such that we always have $x < \alpha_2$; denote the totality of these numbers by A_2, and that of the other real numbers by A_1. Any member (α_1) of A_1 has the property that in course of the process $x \gtreqless \alpha_1$, and so every member of A_1, is smaller than any member of A_2, so that (A_1, A_2) is a section. Its generator (α) is either the greatest in A_1 or the least in A_2; the former cannot be the case, because x never ceases to increase. Thus α is the least member of A_2, and it is a limit of

[11] *Stetigkeit*, p. 12; cf. *Essays*, p. 13.
[12] *Stetigkeit*, p. 14; *Essays*, p. 15.
[13] *Stetigkeit*, pp. 15-19; *Essays*, pp. 15-21.
[14] *Stetigkeit*, pp. 19-22; *Essays*, pp. 21-24.
[15] *Stetigkeit*, pp. 22-24; *Essays*, pp. 24-27.

the x's, for, whatever member of A_1 the number α_1 may be, we ultimately have $\alpha_1 < x < \alpha$.

Still more often used is the equivalent of this theorem: If, in the process of variation of x, for any positive δ (however small) a corresponding place can be given from which one x varies by less than δ, then x approaches a limiting value. This can easily be derived from the foregoing theorem, or directly, as we do here, from the principle of continuity.

If $x = a$ at the instant referred to in the theorem, ever afterwards $x > a-\delta$ and $x < a+\delta$. On this fact we found a double separation of the system of real numbers. Put every number α_2 such that, in the course of the process, we have $x \leqq \alpha_2$, in a class A_2, and let A_1 consist of all the other numbers; so that, if α_1 is such a number it will happen infinitely often, however far the process may have progressed, that $x > \alpha_1$. Since any α_1 is less than any α_2, there is a definite generator α of the section (A_1, A_2), which we will call the upper limiting value of x. Similarly, a second section (B_1, B_2) of the system of real numbers is brought about by x, if any number β_1 (such as $a-\delta$) such that in the course of the process $x \geqq \beta_1$ is put in B_1; and the generator β is called the lower limit of x. The two numbers α and β are also evidently characterized by the property that, if ε is taken positive and arbitrarily small, we always have $x < \alpha+\varepsilon$ and $x > \beta-\varepsilon$, but never finally $x < \alpha-\varepsilon$ and never finally $x > \beta+\varepsilon$. Now, two cases are possible: if α and β are different from one another (so that $\alpha > \beta$), x oscillates, and suffers, however far the process may have progressed, variations whose amount exceeds $(\alpha-\beta)-2\varepsilon$. But the original supposition, which is now first used, excludes this, and so there only remains the case $\alpha=\beta$; and we see that x approaches the limiting value α.

Dedekind remarked[16] that, while the lengthiness in the definitions of the elementary operations can partly be overcome by the use of auxiliary concepts such as that of an "interval" (a system of rational numbers such that, if a and a' are any members of it, all the numbers between a and a' are also members of it)[17] and of its limits, yet "still lengthier considerations seem to loom up when we wish to transfer the innumerable theorems of the arithmetic of rational numbers, as, for example, the theorem $(a+b)c = ac + bc$, to any real numbers. However, this is not so, for we soon convince ourselves that here all reduces to proving that the arithmetical operations themselves have a certain continuity. What I mean by this I will put in form of a general theorem: If the number λ is the result of a calculation undertaken with the numbers $\alpha, \beta, \gamma, \ldots$, and if λ lies inside the interval L, then intervals A, B, C,..., inside which $\alpha, \beta, \gamma, \ldots$, respectively lie, can be given such that the result of the same calculation in which $\alpha, \beta, \gamma, \ldots$ are replaced by any numbers of A, B, C,... respectively, is always a number lying inside L. The forbidding clumsiness, however, which marks the enunciation of such a theorem convinces us that here something must be done to aid language. This is done in the most satisfactory way by introducing the concepts of *variable magnitudes*, *functions*, and *limiting values*; and indeed the most convenient

[16]*Stetigkeit*, pp. 20-22; *Essays*, pp. 22-24.
[17]Both the classes of any section are "intervals."

thing is to base the definitions of the simplest arithmetical operations on these concepts, but this cannot be carried out farther here."[18]

II.

The last few words contain an indication of the fundamental concepts upon which Dedekind's theory of integers was based. The notion of an aggregate or "system" of things is, of course, the most fundamental, and also we utilize, in counting, the capability of the mind to *refer* things to things, to let a thing *correspond* to a thing, or to *imagine* (*abzubilden)* a thing by a thing. Without this capability no thought is possible, and on this single, but quite indispensable, foundation must, in Dedekind's view, the whole science of numbers be erected.[19] This idea of correspondence is the idea of functionality or, in other words, of establishing a *relation* between things.

Dedekind's views on the nature of numbers may be expressed as follows. Arithmetic, including Algebra and Analysis, "is a part of logic, and the number-concept is quite independent of the notions or intuitions of space and time, and is an immediate consequence of the pure laws of thought." Toward the beginning of his *Stetigkeit*,[20] he wrote: "I regard the whole of arithmetic as a necessary or at least natural consequence of the simplest arithmetical act, that of counting, and counting itself is nothing else than the successive creation of the infinite series of positive integers, in which each individual is defined by the one immediately preceding; the simplest act is the passing from an already formed individual to the consecutive new one to be formed. The chain of these numbers forms even by itself an exceedingly useful instrument for the human mind; it presents an inexhaustible wealth of remarkable laws obtained by the introduction of the four fundamental operations of arithmetic. Addition is the combination of any repetitions we wish of the above mentioned simplest act into a single act; from it in a similar way arises multiplication. While the performance of these two operations is always possible, that of the inverse operations, subtraction and division, proves to be limited. Whatever the immediate occasion may have been and whatever comparisons or analogies with experience or intuition may have led us, it is certainly true that just this limitation in performing the indirect operations has in each case been the real motive for a new creative act. Thus negative and fractional numbers have been created by the human mind; and in the system of all rational numbers there has been gained an instrument of infinitely greater

[18]*Stetigkeit*, pp. 21-22; cf. *Essays*, pp. 23-24.

[19]In the eleventh appendix of Dedekind's edition of Dirichlet's *Vorlesungen über Zahlentheorie* (3rd ed., 1879, §163, p. 470), Dedekind said: "It happens very frequently in mathematics and other sciences that, if we have a system Ω of things or elements ω, every definite element ω is replaced according to a certain law by a definite element ω' corresponding to it. We are accustomed to call such an act a substitution and say that by this substitution the element ω passes over into the element ω' and the system Ω into a system Ω' of elements ω'. The expression of this is somewhat more convenient if we.....conceive this substitution as a transformation (*Abbildung*) of the system Ω." To this he added the note: "On this ability (*Fähigkeit*) of mind to compare a thing ω with a thing ω', or to refer ω to ω', or to let ω correspond to ω', without which thought is impossible, rests, as I will try to prove in another place, the whole science of numbers."

[20]Pp. 5-6; cf. *Essays*, p. 4.

perfection. Numbers are free creations of the human mind; they serve as a means to grasp the difference of things more easily and distinctly. Only by means of the purely logical structure of the science of numbers and the continuous number-region obtained in it are we in a position accurately to investigate our notions of space and time, by referring them to this number-domain created in our mind."

Dedekind had the intention of showing the development of the conception of the natural (integral) numbers from the purely logical conceptions of aggregate and "representation" (*Abbildung*), before the publication (1872) of his work on continuity, but it was only after the appearance of this work that, from 1872 to 1878, he wrote out a sketch of his system containing all its essential ideas, and showed it to and discussed it with many mathematicians. In 1887 a careful exposition was carried out and published in the next year under the title *Was sind und was sollen die Zahlen?*[21] The motive for the publication was the appearance of the essays of Kronecker and von Helmholtz. His own work, as he said, though similar in many respects to those essays, was in its foundations essentially different, and he had formed his own view "many years before and without influence from any side."

Dedekind regarded the maxim that "in science anything which can be proved is not to be accepted without proof"[22] as unfulfilled even in the most recent methods of laying the foundations of arithmetic. And Dedekind's answer to this want was one of the first examples of that tendency of modern mathematics to extend exactness of treatment to the very principles, that has been gradually carried out by mathematical logicians like Frege, Peano and Russell.

As we should expect, the tract at first excited the derision of those unperceiving mathematicians who thought that Dedekind was merely taking an unnecessarily long time to prove obvious things like the commutative law in arithmetic. That such things seem to be immediately obvious will at once be granted, but the logical problem which interested Dedekind and many others since about the middle of the nineteenth century was whether or not such theorems are logically implied by those (logical) principles which hold for all true thought without exception, and are not of merely empirical validity. If we are in sympathy with efforts to solve the problems of the nature of our knowledge, we ought not to complain that the detailed writing out of logical steps takes up a large space. Besides, such a complaint is irrelevant.

Dedekind considered what he called "systems," which are what logicians call "classes" and mathematicians now usually call "aggregates," and then the idea of a correspondence of the elements of a system with elements of another system or with one another. He viewed such a correspondence as a "transformation"; and, when he came to consider "similar [or one-to-one] transformations of a system into a part of itself," he arrived at defining an "infinite" system[23] and thus fell upon much the same ideas that Georg Cantor

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[21]Brunswick, 1888; second unaltered edition, 1893 [prefaces dated Oct. 5, 1887 and Aug. 24, 1893]; *Essays*, pp. 31-115.

[22] *Was sind und was sollen die Zahlen?*, p. vii; cf. *Essays*, p. 31.

[23]*Essays*, pp. 63, 41-42.

independently did.[24] A special infinite system is the "simply infinite system" N which is such that there exists a similar transformation φ such that $\varphi(N)$ is a part of N, and N is the common part of all systems S which contain a definite element of N which is not of $\varphi(N)$, and for which $\varphi(S)$ is a part of S.[25] We can see without much difficulty that N consists of an element a, its transform a', the transform a'' of a', and so on; but it is to be noticed that Dedekind defines his infinite systems as wholes and does not use the vague words "and so on."

The ordinal numbers then appear as mental abstractions from such systems as N,[26] the theorem of complete induction is proved for them,[27] and the various other fundamental arithmetical concepts and theorems established. In particular, Dedekind considered cardinal numbers to be logically subsequent to ordinal numbers.[28]

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[24] *Contributions to the Founding of the Theory of Transfinite Numbers*, Chicago and London, 1915, p. 41.

[25]Cf. *Essays*, pp. 67, 56-58.

[26]*Ibid*., p. 68.

[27]*Ibid*., pp. 69-70, 60-62, 32-33, 42-43.

[28]*Ibid*., pp. 109-110, 32.