ℵ Modern Logic ω

THE STATUS OF CANTORIAN NUMBERS

MARIA J. FRÁPOLLI

Departamento de Filosofía
Universidad de Granada
18071 Spain

Abstract. A critical evaluation of Cantor's number conception is undertaken against which the interpretations by Wang and Hallett of Cantorian set theory are measured. Wang takes Cantor's theory to tend to be a theory of numbers rather than a theory of sets, while Hallett takes Cantor as proposing an ordinal theory of cardinal numbers which however permits Cantor to accept ordinal numbers as given without defining them. The evidence presented, however, shows that Cantor conceived numbers, both cardinals and ordinals, as extensional objects, and while either Wang's or Hallett's interpretations eliminate certain difficulties of Cantorian set theory, neither one of them is an accurate depiction of Cantor's theory.

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0. Introduction

In Cantor's work the status and characterization of finite and transfinite numbers is far from clear. Wang [*1974*; in Benecerraf & Putnam *1983*, 539] asserts that Cantor's tends to be more a theory of numbers as concepts than a theory of sets. He claims that, for the European mathematician, numbers are not sets but universals, and are better seen as *urelements*. Hallett [*1984*], on the other hand, supports the view that Cantor proposes an ordinal theory of cardinal numbers which leads him not to define ordinal numbers but accept them as given in some sense, while he identifies powers with number-classes. If Wang is right, Cantor's theory deals with sets as well as objects which are not sets, including numbers, and his theory, therefore, is not a pure set theory. If, however, Hallett's interpretation is correct, Cantor's might be a pure set theory which is immune to Frege's well-known criticism on the strange status of "ones."

I agree with Hallett that Cantor proposes an extensional perspective of numbers, although not that ordinal numbers have a privileged status. His is *not* an ordinal account of powers. I take Cantor's to be a naive approach to numbers as sets of "ones" to which he remained faithful all his life. This approach, of course, is not free of problems. Besides psychologism and the difficulties stressed by Frege, Cantor's view has yet another disadvantage: it needs to suppose without proof that (1) if M and N are equivalent sets, the power of M is *identic*al, and not merely equivalent, to the power of N and (2) if M' and N' are similar sets, their ordinal numbers are not merely similar but coincide. Hallett's approach is designed to overcome (1) by taking ordinals as primitives and postulating (2) (see [Hallett *1984*, 133-142]).

From the *Grundlagen* [*1883*; in Cantor [*1932*] onwards, Cantor distinguishes between *transfinite ordinal numbers* and *powers*. In this work he defines ordinal numbers (*Anzahlen*) for the first time in relation to infinite sets. Unlike ordinals, Cantor does not call powers "numbers" from the outset. It is not evident, therefore, that he always considered both as the same kind of entity.

Before addressing the problem of the status of numbers, let us look at a map of the relations between powers and some germane notions in Cantor's work.

1. Powers, alephs and cardinal numbers

To represent the size of sets, Cantor uses four different terms: *power* (*Mächtigkeit*), *cardinal number* (*Kardinalzahl*), *valence* (*Valenz*) and *aleph* (*Alef*). He begins by using *power*, a term introduced by him in 1878, and does not call powers "cardinal numbers" until 1887 ([*1887-8*]; see [*1932*, 387]), in a text in which he introduces also "valence" as synonymous with the other two. In his manuscript *Principien* (published by [Grattan-Guinness *1970*]), Cantor also uses "valence" as equivalent to "power." Sometimes he defines powers using number-classes of ordinal numbers, as in *Grundlagen* [*1883*], sometimes through the mental procedure of abstraction, as in *Beiträge* [*1895-7*] in which he inaugurates his discourse on "alephs." So it is pertinent to ask whether this diverse terminology hides any differences between the things in question.

I shall argue that there is no difference in Cantor's first set theory either between powers and transfinite cardinal numbers or between these and alephs even in the sense that some of these terms could name different steps in the development of the concept of transfinite cardinal number. (Cantor does not often use the term "valence" and when he does, it is clear that it adds nothing to the notion of power. Therefore, I shall henceforth ignore that term completely). I understand "Cantor's first set theory" to be that expounded in both *Grundlagen* [*1883*] and the *Beiträge* [*1895-7*]. These claims may be opposed on any one of the following accounts:

(A) We can see that there was a shift in Cantor's approach from the *Grundlagen* to the *Beiträge*, for in the former work he uses the term "power" to refer to a concept distinct from "number," i.e., he does not consider powers to be entities in the same sense as ordinal numbers;

(B) Though every cardinal number is a power, the converse is not true, i.e., there are powers which do not number well-ordered sets; or

(C) The operation "power set of" starting in sets of aleph-zero elements defines the series of alephs, which is not identical to the series of cardinal numbers defined on number-classes.

Dauben seems to support (A). In his view, Cantor begins to speak of powers without realizing that they are one of the extensions into the Infinite of the finite whole numbers. In [*1883*] Cantor acknowledges that the ordinals of infinite well-ordered sets are real (actual) numbers. But in Dauben's opinion [*1979*, 179; *1980*, 203-205], Cantor did not come to the realization that powers were also numbers until [*1891*]. In the text from [*1891*] mentioned by Dauben we can read:

> 'Powers' represent the only and necessary generalization of finite 'cardinal numbers,' they are simply the actually infinitely large cardinal numbers, and they are of the same reality and certainty *(Bestimmtheit)* as the former... ,

the original German of which reads

> Die 'Mächtigkeiten' repräsentieren die einzige und notwendige Verallgemeinerung der endlichen 'Kardinalzahlen', sie sind nichts anderes als die aktual-unendlich-grossen Kardinalzahlen, und es kommt ihnen dieselbe Realität und Bestimmtheit zu wie jenen [*1891*, 280]

However, as we have already shown, this is not the first time Cantor identified powers with cardinal numbers. Moreover, the thesis that powers constitute an extension of finite numbers dates from [*1878*, 119], so it is clear that from the beginning Cantor considered powers to be the concept corresponding to finite whole number in the domain of Infinity. The same idea is repeated in [*1883*, 181] where Cantor explains the splitting up of the notion of finite whole number into two different notions, power and transfinite ordinal number, when we go up into the realm of the Infinite and the convergence of these two notions in the concept of finite number when we come down again. While Cantor does not call powers "numbers" in [*1883*] as he does transfinite ordinals, his characterization of the relationships between finite numbers, on one hand, and powers and orderings, on the other, makes it clear that he does not distinguish between their respective statuses. Powers and transfinite ordinal numbers belong, with the familiar whole numbers, to the same ontological category.

Thesis (B) says that there are sets which cannot be well ordered. Cantor himself rules out this possibility in his correspondence to Dedekind, appealing to an alleged proof of the well-ordering principle [*1932*, 443]. He never did find a completely satisfactory proof, as we know, but he believed at least from *Grundlagen* onwards that every set could be well ordered. Thus, he says in the *Grundlagen*, for example

> Dass es immer möglich ist, jede *wohldefinierte* Menge in die *Form* einer *wohlgeordneten* Menge zu bringen, auf dieses, wie mir scheint, ... besonders merkwürdige Denkgesetz werde ich in einer späteren Abhandlung zurückkommen. (see [*1932*, 169])

In the sense that, if every well-defined set has a power, it must be one of the numbers defined through the number-classes of ordinals, the plausibility of (C) also vanishes. Therefore I will henceforth take "powers," "transfinite cardinal numbers" and "alephs" as synonymous.

2. Definitions of Power

It is possible to distinguish three different approaches to powers in Cantor. We may call *nominalist definitions* those in which Cantor does not consider powers anything but an alternative way of talking about equivalence between sets. We find a nominalist definition in [*1878*]:

> When two well-defined multiplicities *M* and *N* can be related to one another completely and univocally, element by element, [...] then let this henceforth be stated by the expression that these multiplicities *have the ssme power*, or also that they are *equivalent*.

The original text says:

> Wenn zwei wohldefinierte Mannigfaltigkeiten *M* und *N* sich eindeutig und vollständig, Element für Element, einander zu-

> ordnen lassen [...], so möge für das Folgende die Ausdruckweise gestattet sein, dass diese Mannigfaltigkeiten *gleiche Mächtigkeit* haben, oder auch, dass sie *äquivalent* sind. [*1878*, 119]

When Cantor defines powers in this way, he does not even suggest what kind of entities they might be, nor does he ever speak of them entities. Only up to 1883 does he use nominalist definitions. As we are interested here in Cantor's first set theory, we can leave aside this nominalist period from now on.

A review of Frege's work *Grundlagen der Arithmetik* [*1885*] introduces a new kind of definition, which I will refer to as *Fregean definitions.* Here Cantor calls powers "general concepts" and defines the power of a set *M* to be the general concept under which all sets equivalent to *M* and nothing else fall (see, for example, Cantor's *Principien* in [Grattan-Guinness *1970*, 85-86]). In Cantor's words,

> Ich nenne 'Mächtigkeit eines Inbegriffs oder einer Menge von Elementen' (wobei letztere gleich- oder ungleichartig, einfach oder zusammengesetzt sein können) denjenigen Allgemeinbegriff, unter welchen alle Mengen, welche der gegebenen Menge äquivalent sind, und nur diese fallen. [*1885*; 441] p. 441.

An identical definition can be found in *Principien* [Grattan-Guinness *1970*, 85]. Cantor wrote this paper in 1884.

An example of this kind from *Principien* is as follows:

> By the *power* or *valence* of a given set *M* I understand the *general concept* (concept of genus, category) under which falls every set equivalent to set *M* (and therefore also set *M* itself) and only these. [Grattan-Guinness *1970*, 85]

The original German version is:

> Unter *Mächtigkeit* oder *Valenz* einer gegebenen Menge *M* verstehe ich den *Allgemeinbegriff* (Gattungsbegriff, Kategorie), *unter welchen alle der Menge M äquivalenten Mengen und nur diese, (und daher auch die Menge M selbst) fallen.*

He uses this kind of definition around 1885 and in it the influence of Frege's work is obvious. In this period he considers powers as concepts and distinguishes them from *power classes* (*Mächtigkeitsclasse*), which are the extensions of powers, i.e., sets of equivalent sets (see, for example *Principien* [Grattan-Guinness *1970*, 85-86]).

From 1887 onwards and clearly in [*1895*] his position becomes openly extensional and we find what may be called *extensional definitions.* One of the best-known is:

> By the 'power' or 'cardinal number' of *M* we mean the general concept which with the help of our active faculty of thought arises from the set *M* by abstracting from the qualities of its various elements *m* and from the order inherent in its presentation. (see also [Jourdain *1955*, 86])

In Cantor's original German version (in Cantor [*1932*, 282]) this is:

> 'Mächtigkeit' oder 'Kardinalzahl' von *M* nennen wir den Allgemeinbegriff, welcher mit Hilfe unseres aktiven Denkvermögens dadurch aus der Menge *M* hervorgeht, dass von der Beschaffenheit ihrer verschiedenen Elemente *m* und von der Ordnung ihres Gegebenseins abstrahiert wird.

Although he continues to call powers "general concepts," he says:

> Since out of every single element *m*, when one disregards its qualities, comes a "one," the cardinal number $\overline{\overline{\mathrm{M}}}$ is itself a certain set consisting of pure ones that has existence in our mind as an intellectual representation or projection *of* the given set *M.*

In Cantor's original German text (as given in [*1932*, 283]), this is:

> Da aus jedem einzelnen Elemente *m*, wenn man von seiner Bgschaffenheit absieht, eine 'Eins' wird, ist die Kardinalzahl $\overline{\overline{M}}$ selbst eine bestimmte aus lauter Einsen zusammengesetzte Menge, die als intellektuelles Abbild oder Projektion der gegebenen Menge *M* in unserm Geiste Existenz hat.

This is not the only characterization of this kind, for in the *Mitteilungen* [*1887-8*] we read:

> Both the cardinal numbers and the order types are *simple* formations of concepts; each of them a *true unity*, because in them a multiplicity and diversity of *ones* is joined in *unity*.

The original German text (see [Cantor *1932*, 380]) says:

> Die Kardinalzahlen sowohl, wie die Ordnungstypen sind *einfache* Begriffsbildungen; jede von ihnen ist eine *wahre Einheit* (μονάσ), weil in ihr eine Vielheit und Mannigfaltigkeit von *Einsen einheitlich* verbunden ist.

It is evident, therefore, that as early as 1887 Cantor considers cardinal and ordinal numbers, which are order types of well-ordered sets, to be the same kind of simple objects, namely, sets of "ones" abstracted from the sets they number. For Cantor ordinal types are sets obtained after abstracting the constitution of their elements from other sets. If order is also abstracted away we obtain powers. Order types share order and cardinality with the sets from which they arise while powers share only cardinality. Thus, cardinals and ordinals are equivalent to the sets of which they are numbers (see Cantor [*1932*, 284] and Jourdain [*1955*, 88]).

3. Cantor's ontological commitment to transfinite numbers

As we know, Cantor introduced the term "power" in ([*1878*]; see [*1932*, 119]) and for the first time he explicitly identifies powers with transfinite cardinal numbers in a letter to Professor Lasswitz, published in the *Mitteilungen* ([*1887-8*]; see [*1932*, 387]) All the while, Cantor's ontological commitment to the new entity was changing, as I hope to demonstrate forthwith.

We ought not confuse the idea that (i) Cantor adhered to different concepts of powers throughout his life with the idea that (ii) his ontological commitment to powers (and to ordinal numbers) evolves from a vague *nominalist* conception into a strong realism. I support (ii) but not (i). As I see it, it is unlikely that an idea of powers was clear in Cantor's mind in [*1878*] and that later he rejected it in favour of another well-defined notion of power. I think that Cantor began to speak of powers (and also of *symbols of infinity)* without having a precise picture of the meaning and scope of these new terms. The move from a rather misty concept of infinite sizes and orderings to a mature and clear-sighted concept of transfinite numbers developed along with and depended upon Cantor's idea of mathematical existence.

Cantor's ontological commitment is not seen uniformly throughout his work. In fact, we should distinguish between his support for (a) mathematical entities definable in a finite domain, such as integers and rationals, and for (b) mathematical entities whose definition requires actual infinities, such as transfinite numbers and irrationals. He was always a realist as regards (a), but not (b). Just as there are nominalist definitions of powers, we also find nominalist definitions of irrational numbers (see for instance ([Cantor *1872*]; see *1932*, 93]) as well as of transfinite ordinals in Cantor's work. He begins to talk of *signs*, such as "ω", "ω+1", to refer to the different derivative aggregates of infinite order of some set of points, calling them *symbols of infinity* (*Unendlichkeitssymbole*). Only from [*1883*] onwards does he consider these symbols to be real *ordinal numbers*, and he grants them this privileged status after incorporating the notion of actual infinity explicitly in his work. One aim of [*1883*] is to present actual Infinity in a suitable form for mathematical work. Once we have a correct account of the *proper* Infinite as opposed to the instability of the *improper* Infinite as it has always been used in analysis, there is no reason to treat infinite orderings as something qualitatively different from finite numbers, and the same may be said of powers. In [*1883*] Cantor acknowledges that his thesis is evolving and claims that he has been dealing with infinite real whole numbers without realizing their meaning as concrete (ordinal) numbers; he writes:

> Die unendlichen realen ganzen Zahlen, welche ich im folgenden definieren will und zu denen ich schon vor einer längeren Reihe von Jahren geführt worden bin, ohne dass es mir zum deutlichen Bewusstsein gekommen war, in ihnen konkrete Zahlen von realer Bedeutung zu besitzen, [...] gehören also zu den Formen und Affektionen des Eigentlich-unendlichen. ([*1883*]; see [*1932*, 166])

He expresses this development clearly in a letter to Dedekind in 1882, in which he accounts for the move from his former term *symbol of infinity* to the new *real whole number* by the fact that these transfinite numbers stand in certain relationships which may be reduced to basic (arithmetical) operations (see [Noether-Cavailles *1937*, 57]). In Cantor's words:

> Vielleicht wundern Sie sich über meine Künhheit, die Dinge, +1, ..., α, ... auch *ganze* Zahlen, und zwar die *ganzen, realen* Zahlen der zweiten Classe zu nennen, während ich sie doch bisher, wo ich mich ihrer [...] bediene, bescheiden: Unendlichkeitssymbole gennant habe.
>
> Doch erklärt sich diese meine Freiheit aus der Bemerkung, dass unter den Gedankendingen a, die ich ganze reale Zahlen der *zweiten* Classe nenne, Beziehungen vorhanden sind, die sich auf die Grundoperationen zurückführen lassen.

All this process fits quite well in the way that new mathematical terms acquire a real meaning as Cantor asserts in two notes on [*1883*] ([*1883*, 207, notes 7 and 8]). In a realist context such as this, saying that a term becomes a concept with real mathematical meaning amounts to an acknowledgement that it names an existent entity.

The history of powers parallels, therefore, that of ordinal numbers. (The same can also be said of irrational numbers. Cantor considered transfinite numbers as irrationals in some sense. See Cantor [*1932*, 395 and 406].) Powers and ordinals were born as names for sizes and for places in an ordering and only afterwards were they accepted as new numbers, once Cantor had developed a suitable concept of infinity which allowed him to treat infinite multiplicities as finite sets. With the introduction of actual infinities, Cantor provided a uniform account for finite and transfinite numbers as well as for rationals and irrationals.

I wish to stress that all this does not mean, however, that we must distinguish different concepts of power. Powers were numbers for Cantor as were ordinal numbers. Furthermore, they acquire this status together, namely, at the moment when Cantor decides to unify the domains of finity and infinity, in about 1883. Before this date, we find only the prehistory of the concept of transfinite number.

4. What are Cantorian Powers?

Hallett (for example at [*1984*, 119]) points to three plausible alternative interpretations of Cantorian powers:

(D) powers are the number-classes, and therefore sets of ordinals;

(E) they are not sets but another kind of primitive entity; or

(F) they have to be identified with equivalence classes, as in the Frege-Russell view.

As I see it, there is yet a fourth possibility:

(G) Powers (and ordinals) are special types of sets, equivalent to the sets from which they are abstracted but different from number-classes.

Furthermore, in view of the different kinds of definition in Cantor, it would not be unreasonable to maintain that Cantor shifted from one alternative to another. I shall argue that in Cantor's first set theory there is a unique conception of powers not mentioned in the possibilities listed by Hallett. To my mind, the correct interpretation of Cantorian powers is (6).

Wang's position would be (E) and as we have seen therefore incompatible with Cantor's approach in *Mitteilungen* and *Beiträge*. In my view, Wang's position does not fit in with the spirit but perhaps it does agree to a certain extent with the letter of Cantor's previous works. Thus (E) might be seen as Cantor's early position which changed later to a set-theoretical approach.

Hallett declares his support for (D). But although Wang's and Hallett's views are very far apart, I hope it is not unfair to say that Hallett would not reject completely the idea of Cantor's moving from an intentional concept of numbers towards the extensional theory of the *Beiträge*. Indeed he suggests some different degrees of "set theoretical reductionism," appearing in a weak form in the *Grundlagen* and the *Mitteilungen* and strengthened in the *Beiträge* and *Principien* (see, e.g., [Hallett *1984*, 125, 128). According to Hallett, in [*1883*] Cantor relates numbers to sets in the sense that the existence of numbers depends on the existence of sets, and that numbers are nothing but numbers of sets, although there is no reduction of one realm into the other. In other words, for Hallett, in [*1883*] Cantor does not delete the realm of numbers as entities of a special kind in favour of the domain of pure set theory. In a nutshell, Hallett's view is that in [*1883*] the link between sets and numbers is forged by the former providing objectivity to the latter.

The idea of Cantor shifting from (E) to an extensional account such as (D) or (F) would be plausible if the difference between *Fregean* and *extensional definitions* concealed two conceptions of powers, in other words, if he used the term *general concept* to differentiate it from the term *set.* Actually *Fregean definitions* seem to be compatible with (F) and a fifth possibility:

(H) powers are a special type of set abstracted from equivalence classes and equivalent to them.

Among the authors who share the view that Cantor adheres to more than one concept of power we may also count Meschkowski [*1967*, 71-72]. Unlike Wang or Hallett, he maintains that Cantor always favoured an extensional view. Meschkowski considers that while in [*1885*] Cantor favoured (F), i.e., the Frege-Russell view, by [*1895-7*] Cantorian powers may be interpreted as (G). He supposes therefore that Cantor shifted from (F) to (G).

Nevertheless, the apparent compatibility between *Fregean definitions* and (F) and (H) disappears if we take the former in their contexts. In the *Principien*, for instance, after the Fregean-type definition cited in section 2 above, we read:

> I also say of *equivalent* sets that they belong to *one* and *the same power class:* the *class* of a set *M* is thus nothing but the *extension [...]* of the *general concept belonging to set M,* which [the general concept] I called the *power of set M.* [Grattan-Guinness *1970*, 85-86]

The original German version is:

> Von *äquivalenten* Mengen sage ich auch, dass sie *zu einer* und *derselben Mächtigkeitsclasse* gehören: die *Classe* einer Menge *M* ist also nichts Anderes, als der *Umfang* ... des zur Menge *M* gehörigen *Allgemeinbegriffs*, welchen ich die *Mächtigkeit der Menge M* gennant habe.

Cantor himself eliminates possibility (F) as he does not identify the general concept, i.e., the power of a given set M, with the equivalence class, i.e., the class of all sets equivalent to M. This is its extension, which he calls *the power class* of *M.* Neither would he accept option (H) for he goes on to say:

> The *power* of a set *M* is determined hereafter as the *representation common* to *all* sets *equivalent* to set *M* and *only these* and therefore also to set *M* itself; [...] It seems to me to be the *most original*, the *simplest root-concept*, both *psychologically* as well as *methodologically*, resulting from *the abstraction* of every *particularity* which a set of a determinate class can offer and overlooking the *constitution* of its *elements,*

> *as well as with regard to* relations and *orderings*,[...]. Once one *reflects* only *what is common to all sets belonging to one and the same class* results the concept of *power* or *valence*. ([Grattan-Guinness [*1970*, 86]; the underlining here is mine)

It is clear from the way Cantor expresses himself here in 1884 that he abstracts powers from every single set, from any one of the sets belonging to a given power class and not from the power class as a whole. This procedure is patent in the extensional definitions as seen in the *Beiträge*, but even when he uses a Fregean style as in the *Principien*, thus before developing the curious theory of "ones," powers are still equivalent to the sets whose size they indicate.

It is my opinion, therefore, that Cantor's concept of powers remains unaltered from the *Grundlagen* on. Once he accepts transfinite numbers as real numbers, he concedes that they are sets. Both the power of a given set *M* and its ordinal number, if it is well-ordered, are the skeleton, the very structure of *M* in Cantor's first set theory. Numbers are like x-rays of sets. The difference between what I call Fregean and *extensional definitions* is only one of emphasis. As we have seen, Cantor always calls powers *general concepts so* that this apparently intentional terminology does not permit us to decide between, on one hand, Wang's or Hallett's and, on the other, my own interpretation. Cantor does not affirm anywhere in his works that powers are equivalent to power classes, though he explicitly states that the power of a set *M* stems from *M* and is equivalent to it. Even more important is that he *never* claims that we obtain power by abstraction from equivalence classes – rather, we abstract them from every well-defined set. With single sets, he uses the idea of abstraction while combining the power-class discourse with the idea of sets falling under general concepts. It is obvious that the notion of power is defined by abstraction, and thus through equivalence classes: a power is what all equivalent sets and only they share. But this must not be confused with the psychological procedure of abstraction, which has nothing to do with power classes. In *Fregean definitions* one sense of the function "power of" is stressed: one power represents the size of infinitely many sets. In *extensional definitions* the other is stressed: a set can possess only one cardinal number. In the former case Cantor claims that all equivalent sets fall *under* one power, in the latter that a power is *abstracted* from a set. They are two sides of the same coin.

Unlike Meschkowski, Hallett rejects the view, as I do myself, that Cantor supports a Frege-Russell conception of cardinal numbers. It is well known that in his review of Frege's *Grundlagen der Arithmetik* Cantor criticizes the Frege's account of cardinal numbers as they would be indefinite in size, as extensions of concepts usually are (see [Cantor *1932*, 440]). Hallett [*1984*, 126-128] interprets Cantor's criticism as a consquence of what he considers to be Cantor's theory of infinity, i.e., a theory of limitation of size from the outset in which the Absolute takes the role of upper limit. According to Hallett, Cantor always distinguishes between sets and inconsistent multiplicities considering at the same time that every set has a number and that numbers are themselves sets. Thus, given that some extensions of concepts might be as big as the whole Universe, i.e., inconsistent multiplicities, Frege's theory of number would not guarantee every concept to have a number. I agree with Hallett in that numbers are sets and that every set has a number but not that Cantor supported a theory of limitation of size throughout his life nor that his criticism of Frege's view was based on his own theory of infinity. The most plausible explanation to my mind is that Cantor misunderstood Frege, as Zermelo points out (see [Cantor *1932*, 441-442]). Cantor appears to believe that Frege's proposal is to identity numbers with extensions of concepts without going through equivalence classes. In Cantor's view, concepts do not usually have fixed extensions. This would present a great difficulty, not for a theory of limitation of size that Cantor did not endorse at that time in any case, but to his "x-ray" account of numbers. If extensions are not fixed, we cannot abstract powers out of them.

Cantor conceives numbers, both powers and ordinals, as extensional entities – peculiar sets which have the ghostly "ones" as elements and share size and sometimes order with their origin-sets. Frege's criticism of previous theories of "ones" reaches, therefore, the heart of Cantor's view of numbers as an unintended target. The "ones" theory, moreover, is guilty of psychologism in spite of his intentions. Both Wang's and above all Hallett's interpretations eliminate the difficulties arising from the status of "ones" and solve the problem of the uniqueness of numbers proceeding from equivalent and similar sets. But neither of them is Cantor's theory.

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