Real Analysis Exchange
Vol. 22(1), 1996-97, pp. 433–436

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KUBOTA'S AD-INTEGRAL IS MORE GENERAL THAN BURKILL'S AP-INTEGRAL

Abstract

A valid proof that Kubota's AD-integral is more general than Burkill's AP integral is given.

Recently Russell Gordon ([2]) has pointed out defects in several proofs for the statement of the title here and left unresolved whether this is a true statement. To show that it is true, we will use the following results.

Proposition 1 *Let f be a function defined on the compact interval $[a,b]$. Then f is AP-integrable on $[a,b]$ if and only if for each $\epsilon > 0$ there exist an AP-major function M and an AP-minor function m of f on $[a,b]$ satisfying the following extra conditions:*

(i) $M(b) - m(b) < \epsilon$,

(ii) both M and m are approximately differentiable nearly everywhere on $[a,b]$.

Proposition 2 *If F is approximately continuous on $[a,b]$ and is approximately differentiable nearly everywhere on $[a,b]$, then F is generalized continuous on $[a,b]$.*

To avoid the possibility of being ambiguous, some terms used above are explained below.

A function is *approximately differentiable* at a point x if the approximate derivative of the function at x exists in the extended real number system $[-\infty, +\infty]$, which is a little bit different from the usual definition requiring that it exists in the real number system $(-\infty, +\infty)$.

For M to be an AP-major function of f on $[a,b]$ means that the following hold:

Mathematical Reviews subject classification: Primary: 26A39
Received by the editors August 12, 1996

(a) M is approximately continuous on $[a,b]$ and $M(a)=0$,

(b) $\underline{M}'_{ap} \geq f$ almost everywhere on $[a,b]$

(c) $\underline{M}'_{ap} > -\infty$ nearly everywhere on $[a,b]$.

For m to be an AP-minor function of f means that $-m$ is an AP-major function of $-f$. A function f is AP-integrable on $[a,b]$ (in the sense of Burkill) if for each $\epsilon > 0$ there exist an AP-major function M and an AP-minor function m of f on $[a,b]$ such that $M(b)-m(b)<\epsilon$.

Proposition 1 is an extension of a result for the ordinary Perron integral established by McGregor in [4], and is a consequence of Theorem 5 in [1], where McGregor's result was extended to a certain abstract Perron integral. To be more accessible, a proof of this proposition will be given at the end.

A function F is *generalized continuous* on $[a,b]$ if $[a,b]$ can be written as a union of a sequence $\{E_n\}$ of closed sets such that $F|_{E_n}$ is continuous on E_n for each n. Note that, by an application of the Baire category theorem, every generalized continuous function on $[a,b]$ is a B_1^* function on $[a,b]$ in the sense (see [2]) that every nonempty perfect set E in $[a,b]$ contains a perfect portion P such that $F|_P$ is continuous on P.

Proposition 2 is the same statement as Theorem 4 in [2], of which the proof there remains valid even taking approximate differentiability to mean what we have mentioned above.

To see what we mean by Kubota's AD-integral, let us recall the following term first. A function F is ACG_c on $[a,b]$ if F is approximately continuous on $[a,b]$ and $[a,b]$ can be written as a union of countably many closed sets on each of which the function F is AC. A function f is AD-integrable on $[a,b]$ (in the sense of Kubota) if there exists an ACG_c function F on $[a,b]$ such that $F'_{ap} = f$ almost everywhere on $[a,b]$. The notation ACG_c was introduced by Gordon in [2], where the AD-integral is termed as AK_c-integral.

Now, we prove our claim.

Theorem 1 *Every AP-integrable function is AD-integrable.*

PROOF. Let f be AP-integrable on $[a,b]$ with F as its indefinite AP-integral satisfying the condition $F(a)=0$. It is clear that our proof will be complete if we show that the function F is ACG_c on $[a,b]$. To this end, let M be an AP-major function of f on $[a,b]$ which is approximately differentiable nearly everywhere on $[a,b]$, the existence of such an M being guaranteed by Proposition 1. Then M is generalized continuous on $[a,b]$ by Proposition 2. Thus there exists a sequence $\{E_n\}$ of closed sets such that $\cup E_n = [a,b]$ and $M|_{E_n}$ is continuous on E_n for each n. As M is an AP-major function of f on $[a,b]$, we also know that $M - F$ is monotone there. Then, being approximately continuous, $M - F$ must be continuous on $[a, b]$. In particular, $(M - F)|_{E_n}$ is continuous on the closed set E_n for each n. As $F|_{E_n} = M|_{E_n} - (M - F)|_{E_n}$, we conclude that $F|_{E_n}$ is continuous on E_n for each n. Thus F is generalized continuous on $[a, b]$. This implies that F is B_1^* on $[a, b]$. Then, by an application of Theorem 3 in [2], F is ACG_c on $[a, b]$, and the proof is done. □

Remark 1 *The theorem says that Kubota's AD-integral is more general than Burkill's AP-integral considered here. However, as pointed out in* [2]*, whether it is more general than the AP-integral considered in Gordon's book* [3] *(where the AP-major and AP-minor functions are not assumed to be approximately continuous) still remains to be seen.*

PROOF OF PROPOSITION 1. The "if" part being trivial, only the "only if" part requires proof. Suppose that f is AP-integrable on $[a, b]$ with F as its indefinite AP-integral satisfying the condition $F(a) = 0$ and let $\epsilon > 0$. Then there exist an AP-major function P and an AP-minor function p such that $P(b) - p(b) < \epsilon/2$. It is well-known (e.g. see [3, Chapter 17]) that such P and p are BVG and thus have finite approximate derivative almost everywhere on the interval. Then, letting E denote the set of all x in the interval at which at least one of P and p fails to have a finite approximate derivative, we conclude that the measure of E is zero. Then there exists a G_δ set S of measure zero such that $E \subset S \subset [a, b]$. Thus, there exists (see, [4] or [5] (For a proof of a similar but weaker result see page 214 in Natanson's book "Theory Of Functions Of A Real Variable, Vol I", or page 369 in Titchmarsh's book "The Theory Of Functions".)) an absolutely continuous function w on $[a, b]$ such that $w'(x) = +\infty$ for all $x \in S$, $0 \leq w'(x) < +\infty$ for all $x \in [a, b] \setminus S$, $w(a) = 0$ and $w(b) < \epsilon/4$. Let $M = P + w$ and $m = p - w$. Then one sees easily that M and m have the property we want in Proposition 1. □

I would like to take the opportunity to thank Professor Gordon for many useful comments through an e-mail, with which an error in the original version was eliminated and the proof of Proposition 1 shortened.

References

[1] P. S. Bullen and C. M. Lee, *On integrals of Perron Type*, Trans. AMS, **182** (1973), 481–501.

[2] R. A. Gordon, *Some comments on an approximately continuous Khintchine integral*, Real Analysis Exchange, **20** (1994-5), 831–841.

[3] R. A. Gordon, *The integrals of Lebesgue, Denjoy, Perron, and Henstock*, Graduate Studies in Mathematics, **4**, American Mathematical Society, Providence, 1994.

[4] J. C. McGregor, *An integral of Perron type*, Thesis, University British Columbia, Vancouver, B. C., Canada, 1951.

[5] Z. Zahorski, *Über die Menge der Punkte in welchen die Ableitung unendlich ist*, Tôhoku Math. J., **48** (1941), 321–330.