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IRREDUCIBILITY, INFINITE LEVEL SETS, AND SMALL ENTROPY

Abstract

We investigate continuous piecewise affine interval maps with countably many laps that preserve the Lebesgue measure. In particular, we construct such maps having knot points (a point x where Dini's derivatives satisfy $D^+f(x) = D^-f(x) = \infty$ and $D_+f(x) = D_-f(x) = -\infty$) and estimate their topological entropy. Our main result is: for any $\varepsilon > 0$ we construct a continuous interval map $g = g_\varepsilon$ such that (i) g preserves the Lebesgue measure; (ii) knot points of g are dense in $[0,1]$ and for a G_δ dense set of z's, the set $g^{-1}(\{z\})$ is infinite; (iii) $h_{\text{top}}(g) \le \log 2 + \varepsilon$.

1 Introduction

A map $f\colon\ X \to X$ is called m-fold on $Y \subset X$, if for every $y \in Y$ a set $f^{-1}(y)$ contains at least m points. For a set X, we call a subset $Y \subset X$ cocountable if its complement $X \setminus Y$ is (at most) countable, and say that a map $f\colon\ X \to X$ is cocountably m-fold if it is globally 2-fold and m-fold on some cocountable subset $Y \subset X$.

In [5] the author proved the following estimate on topological entropy:

Theorem 1.1. *The topological entropy of any continuous cocountably m-fold map $f\colon\ [0,1] \to [0,1]$ satisfies $h_{\text{top}}(f) \ge \log m$.*

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This result is rather delicate, as there is a simple Raith's example of a continuous map $f\colon [0,1] \to [0,1]$ that is m-fold (for an arbitrarily chosen $m \in \mathbb{N}$) except at $y = 1$, which has a single preimage point, but its non-wandering set consists of the fixed endpoints, so that the entropy is zero (see [6] for more detailed information). It is folklore knowledge that analogous examples can be constructed on any n-dimensional manifold (orientable or non-orientable, also with boundary).

Moreover, in [7] the authors showed that the set of points, where the m-fold conditions fail in the hypotheses of Theorem 1.1, cannot be allowed to be uncountable, even if it is nowhere dense. Namely, for each integer $m > 0$ there exists a continuous map $f\colon [0,1] \to [0,1]$ such that f is globally 2-fold, f is m-fold on a set $Y = [0,1] \setminus K$, where K is a nowhere dense, closed (uncountable) set and at the same time $h_{\text{top}}(f) = \log 2$.

Despite Theorem 1.1 and related examples, the problem of understanding of relationship of two characteristics of an interval (or a tree) map - its topological entropy and cardinalities of level sets - is not completely solved. On the one hand the proofs used in [5], [7] are rather difficult with many technicalities, on the other hand all known (counter)examples work with a "poor" set of non-wandering points. Thus, one could expect some strengthened version of Theorem 1.1 stated for a class of irreducible interval maps (transitive, with a dense set of periodic points) proved by essentially simplified methods.

As a canonical expression of mentioned insufficient grasp of the subject we can introduce the following conjectures:

Conjecture 1.2. *Any continuous nowhere differentiable interval map preserving the Lebesgue measure has infinite topological entropy.*

We recall that by a knot point of function f we mean a point x where Dini's derivatives satisfy $D^+f(x) = D^-f(x) = \infty$ and $D_+f(x) = D_-f(x) = -\infty$.

Conjecture 1.3. *Any continuous interval map preserving the Lebesgue measure λ and with a knot point λ-a.e. has infinite topological entropy.*

Note that the existence of continuous interval maps used in the hypotheses has been proved in [3].

The goal of this paper is to provide more sophisticated examples related to Conjectures 1.2, 1.3. To this goal we investigate continuous piecewise affine interval maps with countably many laps and preserving the Lebesgue measure. We construct such maps having finitely many knot points and estimate their topological entropy. As the main result of this paper stated in Theorem 4.1 we obtain the following: for any $\varepsilon > 0$ we construct a continuous interval map $g = g_\varepsilon$ such that (i) g is nowhere monotone and preserves the Lebesgue measure (irreducibility); (ii) knot points of g are dense in $[0,1]$ and for a G_δ dense set of z's, the set $g^{-1}(\{z\})$ is infinite (infinite level sets); (iii) $h_{\mathrm{top}}(g) \leq \log 2+\varepsilon$ (small entropy). Two applications are presented in Corollary 4.2 and Theorem 4.3.

The paper is organized as follows. In Section 2 we give some basic notation, definitions and known results (Theorems 2.3, 2.4, 2.6). Section 3 is devoted to the both local and global perturbations and the map g cited above is constructed.

Finally, in Section 4 we prove the main results - Theorem 4.1 and its Corollary 4.2. We also present one application to the n-dimensional case - Theorem 4.3.

2 Definitions and known results

As general references one can use [13] or [9].

Let X be a compact metric space and $f\colon\ X \to X$ be a continuous map. By $\mathfrak{M}(X)$ we denote the set of all Borel normalized measures on X. The weak* topology on $\mathfrak{M}(X)$ is defined by taking the sets

$$V_\mu(f_1,\dots,f_k;\varepsilon_1,\dots,\varepsilon_k)=\Big\{\nu\colon\ \Big|\int f_j\,\mathrm{d}\mu-\int f_j\,\mathrm{d}\nu\Big|<\varepsilon_j,\ j=1,\dots,k\Big\}$$

as a basis of open neighborhood for $\mu \in \mathfrak{M}(X)$ with $\varepsilon_j > 0$ and f_j being a continuous function defined on X. The map f transports every measure $\mu \in \mathfrak{M}(X)$ into another measure $f_*\mu \in \mathfrak{M}(X)$. In what follows if we say "measure" we in fact mean Borel normalized measure and if we measure some set then we assume that it is measurable. The support of μ is the smallest closed set $S \equiv \mathrm{supp}\mu$ such that $\mu(S) = 1$.

If $\mu = f_*\mu$ then μ is said to be invariant (μ is preserved by f). It is equivalent to the condition $\mu(f^{-1}(S)) = \mu(S)$ for any measurable $S \subset X$. Let $\mathfrak{M}(f)$ be the set of measures preserved by f. A point $p \in X$ is said to be periodic if for some positive integer n, $f^n(p) = p$. The set of all periodic points of f is denoted by $\mathrm{Per}(f)$. A measure $\mu \in \mathfrak{M}(f)$ the $\mathrm{supp}\mu$ of which coincides with one periodic orbit (cycle) is said to be a CO-measure and the set of all CO-measures which are concentrated on cycles is denoted by $\mathfrak{P}(f)$.

We say that $S \subset X$ is f-invariant if $f(S) \subset S$. A measure $\mu \in \mathfrak{M}(f)$ is called ergodic if for any f-invariant set $S \subset X$ either $\mu(S) = 0$ or $\mu(S) = 1$. We denote the set of all f-invariant ergodic measures by $\mathcal{E}(f)$. If μ is ergodic then either $\mathrm{supp}\mu = \mathrm{orb}(p)$ for some periodic point $p \in \mathrm{Per}(f)$ or $\mathrm{supp}\mu$ is a perfect set.

For $\mu \in \mathfrak{M}(f)$, the measure-theoretic entropy of f is a quantity

$$h_\mu(f) = \sup_\zeta \lim_{n\to\infty} H_\mu(\zeta_n),$$

where the supremum is taken over all finite measurable partition ζ of X,

$$H_\mu(\zeta_n) = -\sum_{A\in\zeta_n} \mu(A)\log\mu(A)$$

and $\zeta_n = \{A_{i_0} \cap f^{-1}A_{i_1} \cap \cdots \cap f^{-(n-1)}A_{i_{n-1}} \colon\ A_{i_j} \in \zeta\}$. The topological entropy $h_{\text{top}}(f)$ of f can be defined as [13]

$$h_{\text{top}}(f) = \sup_\mu h_\mu(f),$$

where the supremum is taken over all μ from $\mathfrak{M}(f)$.

In particular, when $X = [0,1]$ and $f\colon\ [0,1] \to [0,1]$ is continuous the map f will be called an interval map.

Theorem 2.1. *[2] Let $f\colon\ [0,1] \to [0,1]$ be an interval map preserving the Lebesgue measure. The set $\mathfrak{P}(f)$ is dense in $\mathfrak{M}(f)$ (in the weak* topology). In particular, the set of all periodic points of f is dense in $[0,1]$.*

□

Proposition 2.2. *Let $f\colon\ [0,1] \to [0,1]$ have a dense set of periodic points and let f be 2-fold on $Y \subset X$, where $[0,1] \setminus Y$ is finite. Then the set $\{x \in [0,1] \colon\ x \notin Per(f)\ \&\ f(x) \in Per(f)\}$ is dense in $[0,1]$.*

PROOF. Choose an interval $J \subset [0,1]$. By our assumption there are closed intervals K, J_1, J_2 such that

$$J_1 \subset J\ , J_1 \cap J_2 = \emptyset,\ f(J_1) = f(J_2) = K \subset f(J).$$

Since the set $\text{Per}(f)$ is dense, there is a periodic point $p \in J_2$ and also a non-periodic point $x \in J_1 \subset J$ for which $f(x) = f(p) \in \text{Per}(f)$. □

We will need following ergodic decomposition.

Theorem 2.3. *[12] Let $\mu \in \mathfrak{M}(f)$. Then there is a measure m on $\mathcal{E}(f)$ such that $\mu(S) = \int_{\mathcal{E}(f)} \lambda(S)\,\mathrm{d}m$ for any measurable set S.*

□

Fix $f\colon\ [0,1] \to [0,1]$ and $x \in [0,1]$. The Lyapunov exponent, $\lambda_f(x)$, is given by

$$\lambda_f(x) = \lim_{n\to\infty} \frac{1}{n} \log |(f^n)'(x)|$$

if the limit exists. The Lyapunov characteristic $\chi\colon\ [0,1] \to [0,\infty]$ is defined as

$$\chi_f(x) = \begin{cases} \lambda_f(x), & \lambda_f(x) > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

The following known theorem (its one-dimensional version) will be one of the key tools when proving Theorem 4.1.

Theorem 2.4. (*the Margulis-Ruelle inequality*) (see [10, pp. 281-285]). *Let $f\colon\ [0,1] \to [0,1]$ be a piecewise Lipschitz map, let μ be an invariant measure for f, and assume that f is differentiable μ-a.e. Then*

$$h_\mu(f) \le \int_{supp\mu} \chi_f \,\mathrm{d}\mu.$$

$\square$

For a pair (T, g) with $T \subset \mathbb{R}$ closed and continuous $g\colon\ T \to T$, $g_T\colon\ \mathrm{conv}T \to \mathrm{conv}T$ (by conv$T$ we mean the convex hull of T) is a piecewise affine "connect-the-dots" interval map given by (T, g). An interval map $f\colon\ [0,1] \to [0,1]$ has a subsystem (T, g) if $T \subset [0,1]$ is closed, $g = f|T$ and $g(T) \subset T$. A subsystem (T, g) of f is piecewise monotone, respectively strictly ergodic if g_T is piecewise monotone, respectively if there is exactly one measure $\mu \in \mathfrak{M}(f)$ such that supp$\mu = T$ and no other measure has its support as a subset of T.

Proposition 2.5. *Let $f\colon\ [0,1] \to [0,1]$ be piecewise affine possibly with countably many laps and having a piecewise monotone strictly ergodic subsystem (T, g) supporting an invariant measure μ with $h_\mu(f) > 0$. Then for each $x \in T$,*

$$\lambda_f(x) = \int_{[0,1]} \log |f'| \,\mathrm{d}\mu \in (0,\infty).$$

PROOF. We have

$$\frac{1}{n} \log |(f^n)'| = \frac{1}{n} \log \Big(\prod_{j=0}^{n-1} |f'(f^j)| \Big) = \frac{1}{n} \sum_{j=0}^{n-1} \log |f'(f^j)|$$

and the right-hand sums converge on the set T uniformly to a constant $\lambda_\mu = \int_{[0,1]} \log |f'| \,\mathrm{d}\mu$ - see [13, Theorem 6.19, p.160]. The value λ_μ is positive by

(1), our assumption $h_\mu(f) > 0$ and Theorem 2.4. Since (T, g) is piecewise monotone, the number λ_μ is less than ∞. □

The Variational principle represents a basic relationship between measure-theoretic and topological entropy. In the context of interval maps one can restrict attention to the subset of strictly ergodic piecewise monotone pairs and corresponding invariant measures.

Theorem 2.6. *[4]. Let f be an interval map. Then*

$$h_{\mathrm{top}}(f) = \sup_{(T,g)} h_\mu(f),$$

where the supremum is taken over all strictly ergodic piecewise monotone subsystems (T, g) of f and corresponding invariant measures μ.

□

3 Constructions

3.1 Local perturbation.

In the first subsection of this section we describe a specific local perturbation of an interval map, i.e. a change of definition of a map on a "small" subset of its domain. All is summarized in Definition 3.1.

For $n \geq 1$, the maps α_5 are "connect-the-dots" maps with the dots (see Figure 1(a))

$$\{(0,0), (1/5,1), (2/5,0), (3/5,1), (4/5,0), (1,1)\}.$$

In order to describe how we will perturb maps we start with a map $\kappa\colon [0,1] \to [0,1]$ defined as the uniform limit of a sequence $\{\kappa_n\}_{n\geq 1}$: fix a sequence $\{\delta_n\}_{n\geq 1}$ of positive real numbers with $\delta_1 = 1/2$ and such that $10\delta_{n+1} < \delta_n$; then the intervals $K_n = [1/2 - \delta_n, 1/2 + \delta_n]$ satisfy

$$[0,1] = K_1 \supset K_2 \supset K_3 \cdots, \ 10\lambda(K_{n+1}) < \lambda(K_n). \tag{2}$$

We construct maps $\kappa_n\colon [0,1] \to [0,1]$ inductively:
$(n = 1)$: $\kappa_1 = \alpha_5$.
$(n > 1)$: If the map κ_{n-1} is already defined, we put (see Figure 1(b) for $n = 3$) $\kappa_n = \kappa_{n-1}$ on $[0,1] \setminus K_n$ and $\kappa_n = h \circ \alpha_5 \circ h_n^{-1}$ on K_n, where h_n, respectively h is affine, preserves orientation and maps the unit interval onto K_n, respectively $\kappa_{n-1}(K_n)$.

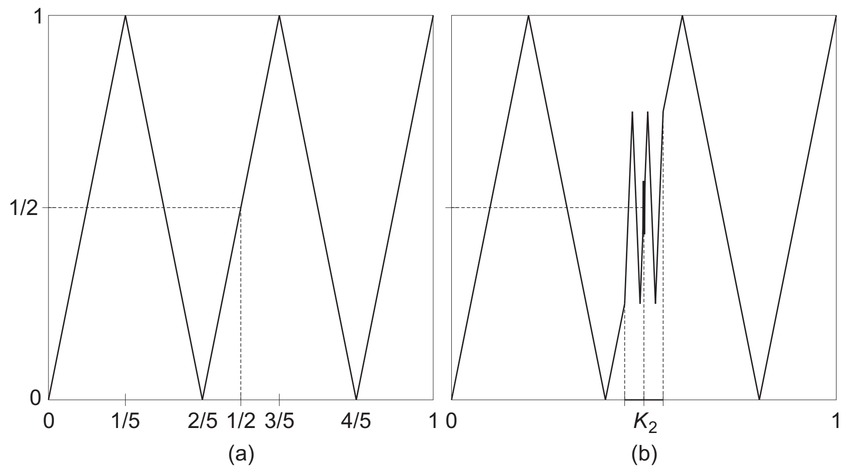

Figure 1: Figure 1. (a) The map α_5; (b) The map κ_3.

Clearly, each κ_n is continuous and it preserves the Lebesgue measure. Moreover, by our construction and (2)

$$\sup_{x\in[0,1]} |\kappa_n(x) - \kappa_{n-1}(x)| \le 5^n \lambda(K_n) < \frac{5^n}{10^{n-1}} = \frac{5}{2^{n-1}},$$

hence the map $\kappa = \lim_n \kappa_n$ exists, it is continuous and the Lebesgue measure preserving again. Since the map κ depends on the sequence $\Delta = \{\delta_n\}_{n\ge1}$, we will sometimes use the notation $\kappa = \kappa[\Delta]$.

Let $f\colon\ [0,1] \to [0,1]$ be an interval map, consider a point $x \in (0,1)$ and a $\beta > 0$ such that $0 \le x - \beta < x + \beta \le 1$ and $f(x-\beta) < f(x+\beta)$, let $\kappa[\Delta]$ be as above for some Δ.

Definition 3.1. By an increasing (x,β,Δ)-perturbation of f we mean a continuous map $\tilde{f}\colon\ [0,1] \to [0,1]$ given by $\tilde{f} = f$ on $[0,1] \setminus [x-\beta, x+\beta]$ and $\tilde{f} = r_{x,\beta} \circ \kappa[\Delta] \circ d_{x,\beta}^{-1}$ on $[x-\beta, x+\beta]$, where $d_{x,\beta}$, respectively $r_{x,\beta}$ is affine, preserves orientation and maps the unit interval onto $[x-\beta, x+\beta]$, respectively $[f(x-\beta), f(x+\beta)]$. If $f(x-\beta) > f(x+\beta)$, a decreasing (x,β,Δ)-perturbation of f is defined analogously by using the map $1 - \kappa[\Delta]$ instead of $\kappa[\Delta]$.

3.2 Global perturbation.

In the second subsection we apply above local perturbation repeatedly to obtain a global change of definition of a map on a dense subset of its domain.

For a piecewise affine map f (possibly with countably many laps) let $W(f)$ be the set consisting of all points in which f is not differentiable and endpoints $0,1$. Let $\{J_m\}_{m\geq 1}$ be the sequence of all rational subintervals of $(0,1)$. Consider the full tent map $f\colon [0,1]\to[0,1]$ given by $f(x)=1-|1-2x|$, $x\in[0,1]$.

Fix an $\varepsilon>0$. We inductively define maps g_m:
($m=0$): $g_0=f$, $x_0=1$, $p_0=0$.
($m>0$): Since by Theorem 2.1 the map g_{m-1} has a dense set of periodic points and each point from $[0,1)$ has at least two g_{m-1}-preimages, by Proposition 2.2 there is a point x_m such that

$$x_m\in J_m,\ x_m\notin \operatorname{Per}(g_{m-1}),\ g_{m-1}(x_m)=p_m\in\operatorname{Per}(g_{m-1}), \tag{3}$$

$$p_m\notin\bigcup_{j=1}^{m-1}\operatorname{orb}(p_j),\ x_m\notin W(g_{m-1})\cup\{x_0,\dots,x_{m-1}\}; \tag{4}$$

for a sequence $\{k_n^m\}_{n\geq 1}$ of positive integers fulfilling

$$\sum_{n=1}^{\infty}\frac{\log(|5^n g'_{m-1}(x_m)|)}{k_n^m+1}<\frac{\varepsilon}{2^m}, \tag{5}$$

there is a sequence $\Delta_m=\{\delta_n^m\}_{n\geq 1}$ (of sufficiently small delta's, shortly, sufficiently small Δ_m) and a corresponding (increasing or decreasing) (x_m,β_m,Δ_m)-perturbation g_m of g_{m-1} such that for each $j\in\{1,\dots,m\}$ and $n\geq 1$ ($K_n^j=[1/2-\delta_n^j,1/2+\delta_n^j]$),

$$x\in d_{x_j,\beta_j}(K_n^j)\implies \{g_m^i(x)\}_{i=1}^{k_n^j}\cap d_{x_j,\beta_j}(K_n^j)=\emptyset, \tag{6}$$

$$\max\{\lambda(g_m^i(d_{x_j,\beta_j}(K_n^j))\colon\ i=0,\dots,k_n^j\}<1/n \tag{7}$$

and, in particular, for $[x_m-\beta_m,x_m+\beta_m]=d_{x_m,\beta_m}([0,1])$,

$$\lambda(g_m([x_m-\beta_m,x_m+\beta_m]))<1/m. \tag{8}$$

We will argue the properties (6), (7) in more details.

Claim 3.2. *If (6), (7) is true for $j\in\{1,\dots,m-1\}$ and g_{m-1} then the sequence $\Delta_m=\{\delta_n^m\}_{n\geq 1}$ fulfilling (6), (7) for $j\in\{1,\dots,m\}$ and corresponding g_m also exists.*

Proof. Since by (4)

$$\operatorname{orb}(p_m)\cap\bigcup_{j=1}^{m-1}\operatorname{orb}(p_j)=\emptyset,$$

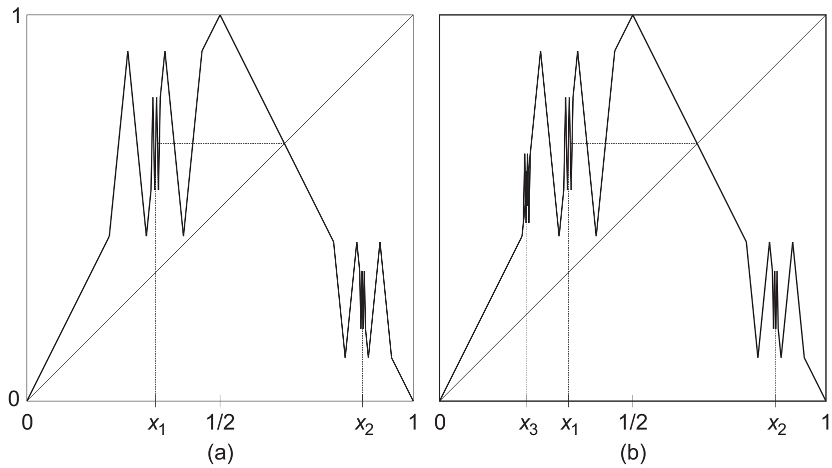

Figure 2: Figure 2. (a) The map g_2; (b) The map g_3.

the (7) applied on g_{m-1} means that for a sufficiently small $\tilde{\Delta}_m$ and corresponding $\tilde{g}_m$ the properties (6), (7) remain true for $\tilde{g}_m$ up to finitely many n's. Taking appropriately Δ_m smaller than $\tilde{\Delta}_m$ (if necessary), we obtain the map g_m fulfilling (6), (7) for $j \in \{1, \dots, m\}$ and every n. $\square$

Claim 3.3. *For any $m \in \mathbb{N}$ and any invariant measure $\mu \in \mathfrak{M}(g_m)$,*

$$\int_{[x_m-\beta_m, x_m+\beta_m]} \log |g_m'| \, \mathrm{d}\mu \leq \sum_{n=1}^{\infty} \frac{\log(|5^n g_{m-1}'(x_m)|)}{k_n^m + 1}.$$

PROOF. By the representation Theorem 2.3 it is sufficient to assume that μ is ergodic. Let $x \in \operatorname{supp}\mu$ be a generic point for μ (see [13]). Putting $L_n = d_{x_m,\beta_m}(K_n^m)$, from (6) we get

$$\mu(L_n) \leq \frac{1}{k_n^m + 1}; \tag{9}$$

by our definition of (x_m, β_m, Δ_m)-perturbation (g_m of g_{m-1})

$$|g_m'| = |5^n g_{m-1}'(x_m)| \text{ on } L_n \setminus L_{n+1}. \tag{10}$$

Since $[x_m - \beta_m, x_m + \beta_m] = \bigcup_{n=1}^{\infty}(L_n \setminus L_{n+1})$, from (9) and (10) we obtain

$$\int_{[x_m-\beta_m, x_m+\beta_m]} \log|g'_m| \,\mathrm{d}\mu = \sum_{n=1}^{\infty} \int_{L_n \setminus L_{n+1}} \log|g'_m| \,\mathrm{d}\mu \le$$
$$\le \sum_{n=1}^{\infty} \int_{L_n} \log|g'_m| \,\mathrm{d}\mu \le \sum_{n=1}^{\infty} \frac{\log(|5^n g'_{m-1}(x_m)|)}{k_n^m + 1}.$$

□

Notice that each g_m preserves the Lebesgue measure and by (8)

$$\sup_{x \in [0,1]} |g_m(x) - g_{m-1}(x)| < 1/m;$$

the reader can easily see that

$$g = \lim_m g_m \tag{11}$$

is defined well and it preserves the Lebesgue measure again.

4 The main result

Theorem 4.1. *The continuous interval map g defined by* (11) *has the following properties:*

(i) g is nowhere monotone and preserves the Lebesgue measure;

(ii) knot points of g are dense in $[0,1]$ *and for a G_δ dense set Z of z's, the set $g^{-1}(\{z\})$ is infinite;*

(iii) $h_{\mathrm{top}}(g) \le \log 2 + \varepsilon$.

Proof. The property (i) directly follows from our construction of g.

Let us prove (ii). It follows from (3) and our choice of the intervals J_m that the sequence $\{x_m\}$ is dense in $[0,1]$. We will show that g has a knot point at every x_m. By the property (4) of our construction, for every $k \ge m$ hold true equalities

$$g(x) = g_k(x) = g_m(x) \text{ for every } x \in \{x_m\} \cup d_{x_m,\beta_m}(W(\kappa[\Delta_m])). \tag{12}$$

Since the map $\kappa[\Delta_m]$ has a knot point at $1/2$ and the maps r_{x_m,β_m}, d_{x_m,β_m} are affine, Definition 3.1 and (12) give us that also each of the maps g_k, g, $k \ge m$ has a knot point at $x_m = d_{x_m,\beta_m}(1/2)$. It means that each of the sets

$$S_m := \{z \in [0,1]\colon\ \#g^{-1}(\{z\}) > m\}^{\circ}$$

is open and dense in $[0,1]$ hence $Z = \bigcap_m S_m$ is G_δ dense.

(iii) Let us fix g_m.

Using Theorem 2.6 let us fix a continuous strictly ergodic invariant measure $\mu \in \mathfrak{M}(g_m)$ with $h_\mu(g_m) > 0$, denote $S = \operatorname{supp}\mu$. Then $(S, \iota = g_m|S)$ is an infinite minimal subsystem of g_m and each point of S is (uniformly) recurrent. The map g_m is piecewise affine with countably many laps accumulated exactly in points $x_1, \dots, x_m$. By (3), $S \cap \{x_1, \dots, x_m\} = \emptyset$ hence the set S is a subset of finitely many laps of g_m. It implies that the map ι_S is Lipschitz and since the measure with respect to μ of any countable set is zero, both the piecewise affine maps g_m, ι_S are differentiable μ-a.e. Applying Theorem 2.4, Proposition 2.5 and (1) we get

$$0 < h_\mu(g_m) = h_\mu(\iota_S) \le \int_{[0,1]} \lambda_{\iota_S}\, \mathrm{d}\mu = \int_{[0,1]} \log|g_m'|\, \mathrm{d}\mu.$$

Putting $J = \bigcup_{j=1}^m [x_j - \beta_j, x_j + \beta_j]$, Claim 3.3 and the properties (3)-(6) imply

$$\int_{[0,1]} \log|g_m'|\, \mathrm{d}\mu \le \sum_{j=1}^m \int_{[x_j-\beta_j, x_j+\beta_j]} \log|g_j'|\, \mathrm{d}\mu + \int_{[0,1]\setminus J} \log|g_m'|\, \mathrm{d}\mu \le$$

$$\le \left(\sum_{j=1}^m \sum_{n=1}^\infty \frac{\log(|5^n g_{j-1}'(x_j)|)}{k_n^j + 1} \right) + \log 2 \le \sum_{j=1}^m \frac{\varepsilon}{2^j} + \log 2,$$

i.e., using Theorem 2.6 and the Variational principle (see [13]),

$$h_\mu(g_m) \le h_{\mathrm{top}}(g_m) \le \sum_{k=1}^m \frac{\varepsilon}{2^k} + \log 2. \tag{13}$$

Since the topological entropy is lower semicontinuous on the space of all continuous interval maps equipped with the supremum norm (see [11]) and $g = \lim_m g_m$, the conclusion $h_{\mathrm{top}}(g) \le \log 2 + \varepsilon$ follows from (13). □

It can be rather easily shown (and we leave it to the reader) that the map g satisfies: for every open subsets U, V of $[0,1]$ there is an $n_0 \in \mathbb{N}$ such that $g^n(U) \cap V \ne \emptyset$ whenever $n \ge n_0$ (g is topologically mixing).

Corollary 4.2. *There is a continuous interval map $f\colon\ [0,1] \to [0,1]$ such that*

(i) f is topologically mixing;

(ii) *for some G_δ dense $Y \subset [0,1]$ of the full Lebesgue measure, $f^{-1}(\{y\})$ is infinite for each $y \in Y$;*

(iii) $h_{\mathrm{top}}(f) \le \log 2 + \varepsilon$.

Proof. Let Z be a G_δ dense set having the property (ii) of Theorem 4.1. Obviously one can consider an F_σ set $\tilde{Z} \subset Z$ which is c-dense in $[0,1]$ hence by [8] for some homeomorphism $h\colon\ [0,1] \to [0,1]$

$$1 = \lambda(\tilde{Y} = h(\tilde{Z})) = \lambda(Y = h(Z)).$$

Then for $f = h \circ g \circ h^{-1}$ and each $y \in Y$ we get

$$\#f^{-1}(\{y\}) = \#h((g^{-1}(h^{-1}(h(z))) = \infty,$$

i.e. the property (ii) is fulfilled. The properties (i), (iii) remain preserved for the conjugated map f. □

As a direct consequence of Theorem 4.1 we will leave to the reader the proof of the following natural generalization.

Theorem 4.3. *Let us consider the map $G\colon\ [0,1]^n \to [0,1]^n$ defined as the product map $G = \underbrace{g \times g \times \cdots \times g}_{n-times}$. The map G fulfills:*

(i) *G is topologically mixing and preserves the Lebesgue measure;*

(ii) *for a G_δ dense set of z's, the set $G^{-1}(\{z\})$ is infinite;*

(iii) $h_{\mathrm{top}}(G) \le n \log 2 + \varepsilon$.

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