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INTEGRAL REPRESENTATIONS FOR A CLASS OF OPERATORS ON L^1_E

Abstract

Let $(X, \mathcal{A}, \mu)$ be a finite measure space, E a locally convex Hausdorff space, L^1_E the space of functions $f : X \to E$ which are μ-integrable by semi-norms, $P(\mu, E)$ the space of Pettis integrable functions and $P_1(\mu, E)$ those elements of $P(\mu, E)$ which are measurable by semi-norms. We prove that a linear continuous mapping $T : L^1_E \to E$ is of the form $T(f) = \int gf d\mu$ ($g \in L^\infty$) if and only if $h(T(f)) = 0$ whenever $h \circ f = 0$ for any $f \in L^1_E, h \in E'$. Similar results are proved for $P(\mu, E)$ and $P_1(\mu, E)$.

1 Introduction and notation

In this paper R stands for the set of real numbers, K will denote the field of real or complex numbers (we will call them scalars), $(X, \mathcal{A}, \mu)$ a finite measure space and E a locally convex space space over K with topology generated by an increasing and closed under multiplications with positive real numbers family of semi-norms $\|.\|_p$, $p \in P$; E' will denote the topological dual of E. If $x \in E$ and $f \in E'$ then $f(x)$ will also be denoted by $< x, f >$ or $< f, x >$. For a $p \in P$, $V_p = \{x \in E : \|x\|_p \leq 1\}$; polars will be taken in the duality $< E, E' >$. For locally convex spaces, the notation and results of [6] will be used. For measure theory, notation and results of ([2], [1], [7], [8], [3] are used. All locally convex spaces are assumed to be Hausdorff and over K. L^1 will denote the space of μ-integrable functions and L^∞ will denote the space of μ essentially bounded functions. As done in ([1], p. 95), the locally convex space L^1_E will denote the

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space of functions $f : X \to E$ which are μ-integrable by semi-norms. $P(\mu, E)$ denote the locally convex space of Pettis integrable functions; its topology is generated by semi-norms: $\|f\|_p = \sup\{\int |h \circ f|_p : h \in V_p^\circ\}$. $P_1(\mu, E)$ denotes the subspace of $P(\mu, E)$, with induced topology, consisting of those elements of $P(\mu, E)$ which are measurable by semi-norms.

In [4] an interesting result is proven about some special operator $T : L^1_E \to E$ when E is a Banach space. In this paper we extend this result to the some cases when E is a locally convex space and give a different proof.

2 Main results

Theorem 1. *([4], Theorem 2.5) Let $(X, \mathcal{A}, \mu)$ be a finite measure space, E a locally convex space and $T : L^1_E \to E$ a continuous linear operator. Then there is a $g \in L^\infty$ such that $T(f) = \int g f d\mu \Leftrightarrow$ for any $f \in L^1_E$ and $h \in E'$ with $h \circ f = 0$, we have $< h, T(f) >= 0$.*

Proof. ($\Leftarrow$) Fix an $x \in E$ and an $f \in L^1$. We first prove that $T(f \otimes x) = cx$ for some $c \in K$. If $T(f \otimes x) = 0$ there is nothing to prove. Assume $T(f \otimes x) = y$ where x and y are linearly independent. By Hahn-Banach theorem there is a $h \in E'$ with $h(x) = 0,\ h(y) = 1$. Now $< f \otimes x, h >= 0$ and so $< T(f \otimes x), h >= 0$ a contradiction. Now we prove that for any $y, z \in E,\ y \neq 0, z \neq 0$, if $T(f \otimes y) = py$ and $T(f \otimes z) = qz$, then $p = q$. If y and z are linearly dependent, then there is nothing to prove. So assume that y, z are linearly independent. This means $y, y - z$ are linearly independent. By Hahn-Banach theorem there is a $h \in E'$ with $h(y) = 1,\ h(y - z) = 0$. Now $< f \otimes (y - z), h >= 0$ and so $< T(f \otimes (y - z), h >= 0$. But $< T(f \otimes (y - z), h >= p - q \neq 0$, a contradiction.

Fix an $x \in E,\ x \neq 0$ and put $T((f \otimes x) = \nu(f)x$. It is easily verified that $\nu : L^1 \to K$ is continuous and so there is a $g \in L^\infty$ such that $\nu = g\mu$. Thus $T(.) = \int .\, g\, d\mu$ on $L^1 \otimes E$. Since $L^1 \otimes E$ is dense in L^1_E ([1], Theorem 3.1, p. 95), it is a simple verification that $T(.) = \int\ .\, g\, d\mu$ on L^1_E.

($\Rightarrow$) It is a trivial verification.

Now we consider the locally convex space $P(\mu, E)$ of Pettis integrable functions. Here we assume that the $(X, \mathcal{A}, \mu)$ is perfect–this will insure that simple functions are dense in $P(\mu, E)$ ([7]).

Theorem 2. *([4]) Let $(X, \mathcal{A}, \mu)$ be a perfect finite measure space, E a locally convex space, $P(\mu, E)$ the locally convex space of Pettis integrable functions, and $T : P(X, E) \to E$ a continuous linear operator. Then there is a $g \in L^\infty$ such that $T(f) = \int g f d\mu \Leftrightarrow$ for any $f \in P(X, E)$ and $h \in E'$ with $h \circ f = 0$, we have $< h, T(f) >= 0$.*

Proof. ($\Leftarrow$) Fix an $x \in E$ and an $f \in L^1$. We first prove that $T(f \otimes x) = cx$ for some $c \in K$. If $T(f \otimes x) = 0$ there is nothing to prove. Assume $T(f \otimes x) = y$ where x and y are linearly independent. By Hahn-Banach theorem there is a $h \in E'$ with $h(x) = 0,\ h(y) = 1$. Now $< f \otimes x, h >= 0$ and so $< T(f \otimes x), h >= 0$ a contradiction. Now we prove that for any $y, z \in E,\ y \neq 0, z \neq 0$, if $T(f \otimes y) = py$ and $T(f \otimes z) = qz$, then $p = q$. If y and z are linearly dependent, then there is nothing to prove. So assume that y, z are linearly independent. This means $y, y - z$ are linearly independent. By Hahn-Banach theorem there is a $h \in E'$ with $h(y) = 1,\ h(y - z) = 0$. Now $< f \otimes (y - z), h >= 0$ and so $< T(f \otimes (y - z), h >= 0$. But $< T(f \otimes (y - z), h >= p - q \neq 0$, a contradiction.

Fix an $x \in E,\ x \neq 0$ and put $T((f \otimes x) = \nu(f)x$. It is easily verified that $\nu : L^1 \to K$ is continuous and so there is a $g \in L^\infty$ such that $\nu = g\mu$. Thus $T(.) = \int .\, g\, d\mu$ on $L^1 \otimes E$. Since $L^1 \otimes E$ is dense in L^1_E ([1], Theorem 3.1, p. 95), it is a simple verification that $T(.) = \int\ .\, g\, d\mu$ on L^1_E.

($\Rightarrow$) It is a trivial verification.

In the locally convex space $P_1(X, E)$, simple functions are dense in $P_1(X, E)$ ([2]), and so proceeding as in Theorem 2, we get:

Theorem 3. *Let $(X, \mathcal{A}, \mu)$ be a finite measure space, E a locally convex space, $P_1(\mu, E)$ the locally convex space of Pettis integrable functions which are measurable by semi-norms, and $T : P_1(X, E) \to E$ a continuous linear operator. Then there is a $g \in L^\infty$ such that $T(f) = \int gf d\mu \Leftrightarrow$ for any $f \in P_1(X, E)$ and $h \in E'$ with $h \circ f = 0$, we have $< h, T(f) >= 0$.*

If F is another locally convex space whose topology is generated by increasing family of seminorms $\{q : q \in Q\}$ and $T : L^1_E \to F$ a continuous mapping, then we get, in a canonical way, a unique measure $\nu : \mathcal{A} \to L(E, F)$ which is countably additive when the topology of pointwise convergence on E, with the original topology of F, is considered on $L(E, F)$; also it is of finite semi-variation in the sense that, for every $q \in Q$, $\sup\{| \sum(\nu(A_i)(x_i)|_q\} < \infty$ as $\{A_i\}$ varies as a finite disjoint collection of elements from $\mathcal{A}$ along with $\{x_i\} \subset E$ having $|x_i|_p \leq 1\ \forall i$. Conversely given such a measure we get a unique continuous $T : L^1_E \to F$. The Theorem 1 can now be stated in terms of this measure:

Theorem 4. *([4], Theorem 3.16) Let $(X, \mathcal{A}, \mu)$ be a finite measure space, E a locally convex space and $T : L^1_E \to E$ a continuous linear operator with $\nu : \mathcal{A} \to L(E, E)$ the associated measure. Then there is a finite measure λ absolutely continuous with respect to μ such that $(\nu(A))(x) = (\lambda(A))x$ for every $A \in \mathcal{A}$ and every $x \in E \Leftrightarrow$ For any $f \in L^1_E$ and $h \in E'$ with $h \circ f = 0$, we have $< h, T(f) >= 0$.*

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