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DEFINABILITY IN FUNCTION SPACES*

Abstract

We study $C(X) \cap L^p(X)$, the set of all continuous functions in $L^p(X)$, as a subspace of $L^p(X)$, and show that it is $\mathbf{\Pi}^0_3$-complete when X is Polish locally compact, while it is $\mathbf{\Pi}^1_1$-complete when X is Polish not σ-compact. We also show that the subspace of Riemann integrable functions and, for every $k = 1, 2, \ldots, \infty$, $C^k(\mathbb{R})$ are $\mathbf{\Pi}^0_3$-complete in $L^p(\mathbb{R})$. In contrast the subspace of all everywhere differentiable functions is $\mathbf{\Pi}^1_1$-complete in $L^p(\mathbb{R})$. If X is locally compact, we consider $C(X)$ endowed with the compact-open topology and establish the complexity of some of its subspaces, including $C_0(X)$, $C_{00}(X)$ and $UC(X, d)$.

1 Introduction

The main theme of this paper is the study, within the framework of descriptive set theory, of various spaces of continuous or integrable real-valued functions. In particular we are interested in how a given function space "sits" inside a larger function space (whose topology in general does not extend the natural topology of the first space). An instance of this is, for X a compact metrizable space, the space of continuous functions $C(X)$, which "sits" inside the space $L^1(X, \mu)$ of μ-integrable functions (for μ a "reasonable" measure on X).

To make precise the notion of "how a space sits inside another" we use the Wadge hierarchy, which classifies the complexity of subsets of Polish (i.e. separable completely metrizable) spaces. If $A \subseteq X$ and $B \subseteq Y$ are subsets

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of two Polish spaces, then A *is Wadge reducible to* B (write $A \leq_{\mathrm{W}} B$) if there exists a continuous function (called a *reduction of* A *to* B) $f : X \to Y$ such that $f^{-1}(B) = A$. The preordering $\leq_{\mathrm{W}}$ obviously induces an equivalence relation on the class of all subsets of Polish spaces, whose equivalence classes are called the *Wadge degrees*, and a partial ordering on the Wadge degrees, which is called the *Wadge hierarchy*. The Wadge degree of a set is a measure of its complexity (the higher the position in the Wadge hierarchy, the more complex the set). For background information on the Wadge hierarchy we refer the reader to [6].

We will study the Wadge degree of a function space viewed as a subset of another function space. The following terminology is standard and will be used in computing the Wadge degrees. Let $\mathbf{\Gamma}$ denote any of the classes (of subsets of Polish spaces) $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$ (where α is a nonzero countable ordinal), $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$ (where n is finite nonzero). In this case write $\check{\mathbf{\Gamma}}$ for the class where $\mathbf{\Pi}$ and $\mathbf{\Sigma}$ are interchanged. We say that a subset A of an arbitrary Polish space is $\mathbf{\Gamma}$*-hard* if for every subset B in $\mathbf{\Gamma}$ of a zero-dimensional Polish space we have $B \leq_{\mathrm{W}} A$ (intuitively this means that the Wadge degree of A is at least as large as the Wadge degree of a set in $\mathbf{\Gamma}$ can be). If a $\mathbf{\Gamma}$-hard set is also in $\mathbf{\Gamma}$ then we say that it is $\mathbf{\Gamma}$*-complete*. (It is obvious that the $\mathbf{\Gamma}$-complete subsets of zero-dimensional Polish spaces form a Wadge degree.) A set is *true* $\mathbf{\Gamma}$ if it is in $\mathbf{\Gamma}$ but not in $\check{\mathbf{\Gamma}}$.

It is immediate that every $\mathbf{\Gamma}$-complete set is true $\mathbf{\Gamma}$. When $\mathbf{\Gamma}$ is $\mathbf{\Sigma}^0_\alpha$ or $\mathbf{\Pi}^0_\alpha$ the converse holds ([6, exercise 24.20]), but when $\mathbf{\Gamma}$ is $\mathbf{\Sigma}^1_n$ or $\mathbf{\Pi}^1_n$ it is consistent with ZFC that there exist true $\mathbf{\Gamma}$ sets which are not $\mathbf{\Gamma}$-complete.

The prototype of our results is the following fact (see [6, subsection 23.D]): inside $C(\mathbb{T})$, the space of all continuous real valued functions on the unit circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with the sup norm, the set $C^k(\mathbb{T})$ ($k = 1, 2, \ldots, \infty$), is $\mathbf{\Pi}^0_3$-complete, while the set $A(\mathbb{T})$ of all analytic functions is $\mathbf{\Sigma}^0_2$-complete. Other results in the same vein can be found among the examples of [6] and in the survey paper [3].

The main technique for showing that a set B is $\mathbf{\Gamma}$-hard is to show that $A \leq_{\mathrm{W}} B$ for some set A already known to be $\mathbf{\Gamma}$-hard. Here is a short list of complete sets (see [6] for proofs) which will be used to establish the hardness of various spaces.

Examples 1.1.

$$Q_2 = \left\{ \alpha \in 2^{\mathbb{N}} \mid \exists M \, \forall n > M \; \alpha(n) = 0 \right\} \quad \text{is } \mathbf{\Sigma}^0_2\text{-complete}$$

$$c_0 = \{ (x_n) \in I^{\mathbb{N}} \mid \lim x_n = 0 \} \qquad \text{is } \mathbf{\Pi}^0_3\text{-complete}$$
$$c = \{ (x_n) \in I^{\mathbb{N}} \mid \lim x_n \text{ exists} \} \qquad \text{is } \mathbf{\Pi}^0_3\text{-complete}$$

where $I = [0,1]$ and $2^{\mathbb{N}}$ is the Cantor space.

Throughout the paper we follow (as above) the notation and terminology of [6] and refer to this textbook for all standard facts on descriptive set theory. In particular if μ is a measure on a set X we use the following notation for measure quantifiers: $\forall^*_\mu x \in X \; P(x)$ holds if and only if $\mu(\{ x \in X \mid \neg P(x) \}) = 0$, $\exists^*_\mu x \in X \; P(x)$ holds if and only if $\mu(\{ x \in X \mid P(x) \}) > 0$. By a Borel measure we mean a measure defined on the σ-algebra of the Borel subsets of a second countable topological space. When X is a metric space, $x \in X$ and $r > 0$, we write $B(x;r)$ for the open ball of center x and radius r.

If X is a topological space we denote by $C(X)$ the set of continuous functions from X to $\mathbb{R}$. We will also use the following notions from topology (we use $\mathsf{K}(X)$ to denote the hyperspace of the compact subsets of X).

Definition 1.2. A topological space X is *σ-compact* if we can write $X = \bigcup_{n\in\mathbb{N}} K_n$ with $K_n \in \mathsf{K}(X)$; obviously we can always assume that $K_n \subseteq K_{n+1}$. X is *hemicompact* if moreover the K_n's are cofinal in $\mathsf{K}(X)$, i.e. for every $K \in \mathsf{K}(X)$ there exists n such that $K \subseteq K_n$.

A second countable Hausdorff (in particular: separable metrizable) space is hemicompact if and only if it is locally compact (see [4, exercise 3.4.E.c]). If X is locally compact Polish we fix a decomposition $X = \bigcup_n K_n$ with the K_n's cofinal in $\mathsf{K}(X)$. Furthermore, if X is not compact, we may assume that for every n the interior of $K_{n+1} \setminus K_n$ is nonempty.

We now explain the organization of the paper. In section 2 we work in a purely measure-theoretic setting and consider $L^q(X,\mu)$ and the space of all simple functions as subsets of $L^p(X,\mu)$: they are both $\mathbf{\Sigma}^0_2$-complete under the appropriate hypothesis. In section 3 we consider a "reasonable" measure μ on the Polish space X and study the space of all continuous functions which belong to $L^p(X,\mu)$. If X is locally compact this space turns out to be Borel (and precisely $\mathbf{\Pi}^0_3$-complete), while if X is not σ-compact this space is $\mathbf{\Pi}^1_1$-complete and hence not Borel. Section 4 specializes to $X = \mathbb{R}$ and Lebesgue measure: we consider spaces of differentiable and Riemann integrable functions. In section 5 we fix a locally compact Polish space X, endow $C(X)$ with the compact-open topology (which makes it Polish) and study various subspaces of $C(X)$ (often these subspaces are themselves Polish spaces, albeit with different topologies).

2 Some Subspaces of $L^p(X,\mu)$

If $1 \leq p < \infty$ and μ is a σ-finite measure on a countably generated σ-algebra of subsets of a set X we consider the separable Banach (and hence Polish) space $L^p(X,\mu)$ with the p-norm.

The following general lemma (which is related to [6, exercise 17.29]) is interesting in its own right and will be useful also in sections 3 and 4.

Lemma 2.1. *Let X be a separable metrizable space, μ a σ-finite Borel measure on X. Let A be a closed subset of $(X \times \mathbb{R})^k \times X^h$. Then the set B of all $(f, y_1, \dots, y_h) \in L^p(X,\mu) \times X^h$ such that*

$$\forall^*_\mu x_1 \dots x_k \in X \ (x_1, f(x_1), \dots, x_k, f(x_k), y_1, \dots, y_h) \in A$$

is also closed.

Moreover, if $A \subseteq \mathbb{R}^k$, μ can be any σ-finite measure on a countably generated σ-algebra of subsets of X (which does not need to be topological), and the same conclusion holds.

Proof. First of all notice that the definition of B is meaningful:

$$\text{if } \mu(\{\, x \mid f(x) \neq g(x) \,\}) = 0, \text{ then } (f, y_1, \dots, y_h) \in B \iff (g, y_1, \dots, y_h) \in B.$$

For the sake of notational simplicity, we assume $k = h = 1$. Let d be a compatible metric on X and denote by D the metric on $X \times \mathbb{R} \times X$ which is the product of two copies of d and of the Euclidean metric on $\mathbb{R}$.

Suppose towards a contradiction that the sequence (f_n, y_n) converges to (f, y) within $L^p(X,\mu) \times X$, $(f_n, y_n) \in B$ for every n, and $(f, y) \notin B$. Since for μ-almost all x, $D((x, f_n(x), y_n), A) = 0$, then

$$D((x, f(x), y), A) \leq D((x, f(x), y), (x, f_n(x), y)) + D((x, f_n(x), y), (x, f_n(x), y_n))$$

and therefore for μ-almost all x

$$|f(x) - f_n(x)| \geq D((x, f(x), y), A) - d(y, y_n).$$

Since $\{\, x \in X \mid (x, f(x), y) \notin A \,\}$ has positive measure and A is closed, for some $\varepsilon > 0$ we have $\mu(Y_\varepsilon) > 0$, where $Y_\varepsilon = \{\, x \mid D((x, f(x), y), A) > \varepsilon \,\}$. For every n such that $d(y, y_n) < \varepsilon/2$ we have

$$\begin{aligned} \|f - f_n\|_p^p &\geq \int_{Y_\varepsilon} |f(x) - f_n(x)|^p \, d\mu(x) \\ &> \int_{Y_\varepsilon} \left(\varepsilon - \frac{\varepsilon}{2}\right)^p d\mu(x) = \mu(Y_\varepsilon) \left(\frac{\varepsilon}{2}\right)^p > 0. \end{aligned}$$

This contradicts $f_n \to f$ in $L^p(X, \mu)$.

The second part of the theorem follows easily, since in that case no topological properties of X are used. $\square$

An interesting subspace of $L^p(X, \mu)$ is the set $\mathrm{SIM}(X, \mu)$ of all simple functions, which is often used to define integration with respect to μ. Recall that $f \in \mathrm{SIM}(X, \mu)$ if its range is finite and the inverse image of each of its nonzero elements is a measurable set of finite measure. Since a member of $L^p(X, \mu)$ is actually a class of measurable functions which are μ-almost everywhere equal, when we say that $f \in L^p(X, \mu)$ is simple we mean that there is a simple function which is μ-a.e. equal to f. Therefore we can view $\mathrm{SIM}(X, \mu)$ as a subset of $L^p(X, \mu)$.

Theorem 2.2. *Let μ be a σ-finite measure on a countably generated σ-algebra of subsets of an infinite set X. Assume furthermore that for every $\varepsilon > 0$ there exists $A \subseteq X$ such that $0 < \mu(A) \leq \varepsilon$. Then $\mathrm{SIM}(X, \mu)$ is $\mathbf{\Sigma}^0_2$-complete in $L^p(X, \mu)$.*

PROOF. If $f \in L^p(X, \mu)$, then $f \in \mathrm{SIM}(X, \mu)$ if and only if

$$\exists K \in \mathsf{K}(\mathbb{R})(K \text{ is finite } \& \ \forall^*_\mu x \in X \ f(x) \in K).$$

The collection of finite sets is $\mathbf{\Sigma}^0_2$ in $\mathsf{K}(\mathbb{R})$ ([6, exercise 4.30]). Therefore

$$F = \left\{ (K, f) \in \mathsf{K}(\mathbb{R}) \times L^p(X, \mu) \mid K \text{ is finite } \& \ \forall^*_\mu x \in X \ f(x) \in K \right\}$$

is $\mathbf{\Sigma}^0_2$ by lemma 2.1. Since $\mathbb{R}$ is locally compact, it is immediate (using the fact that $\mathsf{K}(X)$ is compact when X is compact) that $\mathsf{K}(\mathbb{R})$ is σ-compact. That $\mathrm{SIM}(X, \mu)$ is $\mathbf{\Sigma}^0_2$ follows at once from the following folklore result, a proof of which can be found, for example, in [2, lemma 1.3].

Lemma 2.3. *If $F \subseteq K \times Y$ is $\mathbf{\Sigma}^0_2$, with K σ-compact and Y Polish, then $\{ y \in Y \mid \exists k \in K \ (k, y) \in F \}$ is $\mathbf{\Sigma}^0_2$.*

By the result mentioned in the introduction, in order to show that $\mathrm{SIM}(X, \mu)$ is $\mathbf{\Sigma}^0_2$-complete it is enough to prove it is not $\mathbf{\Pi}^0_2$. By Baire's category theorem it is enough to prove that $\mathrm{SIM}(X, \mu)$ and $L^p(X, \mu) \setminus \mathrm{SIM}(X, \mu)$ are dense in $L^p(X, \mu)$. The density of $\mathrm{SIM}(X, \mu)$ is immediate, and using the existence of sets of arbitrarily small measure, any $f \in \mathrm{SIM}(X, \mu)$ can be approximated by a $g \notin \mathrm{SIM}(X, \mu)$ with countable range. Therefore $L^p(X, \mu) \setminus \mathrm{SIM}(X, \mu)$ is dense in $L^p(X, \mu)$ and this concludes the proof. $\square$

If μ is finite and $p < q$, then $L^q(X, \mu) \subseteq L^p(X, \mu)$. More in general we can study $L^q(X, \mu) \cap L^p(X, \mu)$ as a subspace of $L^p(X, \mu)$.

Theorem 2.4. *If μ is a σ-finite measure on a countably generated σ-algebra of subsets of a set X, then $L^q(X,\mu) \cap L^p(X,\mu)$ is $\mathbf{\Sigma}^0_2$ in $L^p(X,\mu)$. Moreover if $L^p(X,\mu) \nsubseteq L^q(X,\mu)$, then $L^q(X,\mu) \cap L^p(X,\mu)$ is $\mathbf{\Sigma}^0_2$-complete in $L^p(X,\mu)$.*

Proof. Let $X = \bigcup X_k$ where $\mu(X_k)$ is finite and $X_k \subseteq X_{k+1}$. If $f \in L^p(X,\mu)$, then $f \in L^q(X,\mu)$ if and only if

$$\exists M \ \forall n \ \forall k \ \int_{X_k} \min(|f(x)|^q, n) \, d\mu \leq M.$$

For every n and k the map $L^p(X,\mu) \to \mathbb{R}$, $f \mapsto \int_{X_k} \min(|f(x)|^q, n)\, d\mu$, (which is actually defined for every μ-measurable function f) is continuous and hence the first part of the theorem follows immediately.

Since simple functions are dense in $L^p(X,\mu)$ and belong to $L^q(X,\mu)$, we have that $L^q(X,\mu) \cap L^p(X,\mu)$ is dense in $L^p(X,\mu)$. The hypothesis of the second part of the theorem implies that $L^p(X,\mu) \setminus L^q(X,\mu)$ is also dense in $L^p(X,\mu)$: to see this it suffices to notice that if $f \in L^p(X,\mu) \setminus L^q(X,\mu)$ and $g \in L^q(X,\mu) \cap L^p(X,\mu)$, then for every $\varepsilon \neq 0$, $g + \varepsilon f$ belongs to $L^p(X,\mu) \setminus L^q(X,\mu)$. It now suffices to argue as in the proof of Theorem 2.2, using the Baire category theorem. □

Notice that for the hypothesis of the second part of theorem 2.4 to hold, it suffices either that there exist sets of arbitrarily small positive measure and $p < q$, or that μ is infinite and $p > q$.

3 $C(X)$ as a Subspace of $L^p(X,\mu)$

Suppose X is a compact space and μ a finite Borel measure on X non-vanishing on open sets, i.e. for every nonempty open $U \subseteq X$, $\mu(U) > 0$. Endow $C(X)$ with the sup norm and consider again $L^p(X,\mu)$ with the p-norm ($1 \leq p < \infty$). Let $j : C(X) \to L^p(X,\mu)$ be defined by letting $j(f)$ to be the element of $L^p(X,\mu)$ (which is an equivalence class of Borel functions) determined by f. Then j is injective and continuous.

If X is also metrizable (and hence Polish) $j(C(X))$ is Borel in $L^p(X,\mu)$ (because one-to-one continuous images of Polish spaces are Borel, see e.g. [6, theorem 15.1]). We identify $C(X)$ with its image via j in $L^p(X,\mu)$ and write, with abuse of notation, $C(X) \subseteq L^p(X,\mu)$.

If X is not compact we do not have this inclusion, but it is still worth studying $C(X) \cap L^p(X,\mu)$ as a subspace of $L^p(X,\mu)$.

Lemma 3.1. *Suppose X is Polish and μ is a σ-finite Borel measure on X non-vanishing on open sets. Suppose also there exists $x_0 \in X$ such that $\mu(\{x_0\}) = 0$*

and for some open neighborhood U of x_0, $0 < \mu(U) < \infty$. Then $C(X) \cap L^p(X,\mu)$ is $\mathbf{\Pi}^0_3$-hard in $L^p(X,\mu)$.

Proof. Fix a complete compatible metric d for X; it follows from the hypothesis that $0 < \mu(B(x_0;r_0)) < \infty$ for some $r_0 > 0$. This implies that $\lim_{r\to 0}\mu(B(x_0;r)) = 0$ and we can choose $r_0 > r_1 > \cdots$ such that the sequence $(\mu(B(x_0;r_n)))_n$ is strictly decreasing and converging to 0: we may assume that for every n there exists y_n such that $d(y_n,x_0) = r_n$. Moreover, since μ is non-vanishing on open sets, $\lim_n r_n = 0$.

We show that c (see 1.1) is Wadge reducible to $C(X) \cap L^p(X,\mu)$. For $(z_n) \in I^{\mathbb{N}}$ let $z_{-1} = 0$ and define $f_{(z_n)} : X \to \mathbb{R}$ to be equal to 0 on x_0 and outside the ball $B(x_0;r_0)$, equal to z_n on the surface of $B(x_0;r_{n+1})$, and extended by linearity on the other points:

$$f_{(z_n)}(x) = \begin{cases} 0 & \text{if } d(x,x_0) \geq r_0 \text{ or } x = x_0; \\ tz_n + (1-t)z_{n-1} & \text{if } r_{n+1} \leq d(x,x_0) \leq r_n \text{ and} \\ & \quad d(x,x_0) = r_n - t(r_n - r_{n+1}). \end{cases}$$

Then $f_{(z_n)}$ is bounded and continuous at every $x \neq x_0$; hence $f_{(z_n)} \in L^p(X,\mu)$. Moreover, since $\lim \mu(B(x_0;r_n)) = 0$ and $|f_{(z_n)}(x)| \leq 1$ for every x, the map $I^{\mathbb{N}} \to L^p(X,\mu)$, $(z_n) \mapsto f_{(z_n)}$, is continuous.

Obviously $(z_n) \in c$ if and only if $\lim_{x\to x_0} f_{(z_n)}(x)$ exists. Therefore, if $(z_n) \in c$, then $f_{(z_n)} \in C(X) \cap L^p(X,\mu)$, since the function obtained from $f_{(z_n)}$ by assigning value $\lim_{x\to x_0} f_{(z_n)}(x)$ to x_0 is μ-a.e. equal to $f_{(z_n)}$ and continuous.

On the other hand, if $(z_n) \notin c$ let $\varepsilon = \limsup z_n - \liminf z_n > 0$. For every n let W_n be an open neighborhood of y_n such that $|f_{(z_n)}(w) - z_n| < \varepsilon/3$ for every $w \in W_n$. Let g be μ-a.e. equal to $f_{(z_n)}$. Since μ is non-vanishing on open sets for every n, there exists $w_n \in W_n$ such that $g(w_n) = f_{(z_n)}(w_n)$. Therefore $\limsup g(w_n) - \liminf g(w_n) \geq \varepsilon/3$. Since $\lim w_n = x_0$, g cannot be continuous at x_0 and $f_{(z_n)} \notin C(X) \cap L^p(X,\mu)$. □

Lemma 3.2. *Let X be a compact Polish space and μ a σ-finite Borel measure on X non-vanishing on open sets. Then $C(X)$ is $\mathbf{\Pi}^0_3$ in $L^p(X,\mu)$.*

Proof. Let d be a compatible metric on X. For $f \in L^p(X,\mu)$ we claim that $f \in C(X)$ if and only if

$$\forall\varepsilon > 0\, \exists\delta > 0\, \forall^*_\mu x,y \in X(d(x,y) < \delta \implies |f(x) - f(y)| \leq \varepsilon). \tag{1}$$

Using Lemma 2.1 this equivalence implies that $C(X)$ is $\mathbf{\Pi}^0_3$.

One direction of the equivalence is almost immediate. If f is equal μ-a.e. to a continuous function g, then g is uniformly continuous and hence for every ε there exists δ such that $d(x,y) < \delta$ implies $|g(x) - g(y)| \leq \varepsilon$. This implies that

$$\forall_\mu^* x, y (d(x,y) < \delta \implies |f(x) - f(y)| \leq \varepsilon).$$

For the other direction suppose $f \in L^p(X,\mu)$ satisfies (1). Let δ_n be the δ corresponding to $\varepsilon = 2^{-n}$. Then

$$Z_n = \left\{ x \in X \mid \exists_\mu^* y \in X (d(x,y) < \delta_n \ \& \ |f(x) - f(y)| > 2^{-n}) \right\}$$

has measure 0. Therefore $T = X \setminus \bigcup_n Z_n$ is such that $\mu(X \setminus T) = 0$ and hence, since μ is non-vanishing on open sets, T is dense in X. It suffices to show that f is uniformly continuous on T (in this case for every $x \in X \setminus T$ the limit of $f(y)$ for $y \in T$ and $y \to x$ exists, and these limits provide the definition of a function continuous on X which is equal to f on T).

To complete the proof we show that if $x_1, x_2 \in T$ are such that $d(x_1, x_2) < \delta_{n+1}$, then $|f(x_1) - f(x_2)| \leq 2^{-n}$. As $U = B(x_1; \delta_{n+1}) \cap B(x_2; \delta_{n+1}) \neq \emptyset$ is open we have $\mu(U) > 0$. As $x_i \notin Z_{n+1}$

$$\forall_\mu^* y \in U \ |f(x_i) - f(y)| \leq 2^{-n-1}.$$

Thus there exists $y \in U$ such that $|f(x_i) - f(y)| \leq 2^{-n-1}$ for $i = 1, 2$. Hence

$$|f(x_1) - f(x_2)| \leq |f(x_1) - f(y)| + |f(y) - f(x_2)| \leq 2^{-n},$$

as claimed. □

Lemma 3.2 has been shown independently by H. Becker with a somewhat different proof (personal communication).

Theorem 3.3. *Let X be a compact Polish space, μ a finite Borel measure on X non-vanishing on open sets and suppose there exists $x_0 \in X$ such that $\mu(\{x_0\}) = 0$. Then $C(X)$ is $\mathbf{\Pi}_3^0$-complete in $L^p(X,\mu)$.*

PROOF. Immediate from Lemmas 3.1 and 3.2. □

The assumptions on the finiteness of the measure and on the compactness of the space can be relaxed.

Theorem 3.4. *Let X be locally compact Polish, μ a σ-finite Borel measure on X non-vanishing on open sets which is finite on compact sets. Suppose there exists $x_0 \in X$ such that $\mu(\{x_0\}) = 0$. Then $C(X) \cap L^p(X,\mu)$ is $\mathbf{\Pi}_3^0$-complete in $L^p(X,\mu)$.*

PROOF. The hypotheses of Lemma 3.1 hold and hence $C(X) \cap L^p(X, \mu)$ is $\mathbf{\Pi}^0_3$-hard.

To show that $C(X) \cap L^p(X, \mu)$ is $\mathbf{\Pi}^0_3$ let $X = \bigcup_n K_n$ where each K_n is compact and every compact subset of X is contained in some K_n (see Definition 1.2). If $f \in L^p(X, \mu)$, the cofinality of the K_n's within $\mathsf{K}(X)$ implies that $f \in C(X)$ if and only if $f \restriction K_n$ is continuous for every n, if and only if $f \restriction \mathrm{Int}(K_n)$ is continuous for every n. (Here $\mathrm{Int}(A)$ denotes the interior of the set A.) Then for $f \in L^p(X, \mu)$ we have $f \in C(X)$ if and only if for all n and every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\forall^*_\mu x, y \in X (x, y \in \mathrm{Int}(K_n) \;\&\; d(x, y) < \delta \implies |f(x) - f(y)| \leq \varepsilon).$$

The proof of this equivalence is analogous to the proof of the similar equivalence in lemma 3.2. Lemma 2.1 implies that $C(X) \cap L^p(X, \mu)$ is $\mathbf{\Pi}^0_3$. □

Corollary 3.5. *$C(I)$ is $\mathbf{\Pi}^0_3$-complete in $L^p(I)$, and $C(\mathbb{R}) \cap L^p(\mathbb{R})$ is $\mathbf{\Pi}^0_3$-complete in $L^p(\mathbb{R})$. (Both L^p's are taken with respect to Lebesgue measure.)*

We can also turn the problem around and consider $C(X) \cap L^p(X, \mu)$ as a subspace of $C(X)$ with the compact-open topology.

Lemma 3.6. *Let X be Polish locally compact and non-compact, and let μ be a σ-finite Borel measure on X, non-vanishing on open sets and finite on compact sets. Then $C(X) \cap L^p(X, \mu)$ is $\mathbf{\Sigma}^0_2$-complete in $C(X)$ with the compact-open topology.*

PROOF. Let $X = \bigcup_n K_n$, with $K_n \in \mathsf{K}(X)$, $K_n \subseteq K_{n+1}$, the K_n's cofinal in $\mathsf{K}(X)$, and $\mathrm{Int}(K_{n+1} \setminus K_n) \neq \emptyset$. Recall that the compact open topology on $C(X)$ is generated by the metric

$$d(f, g) = \sum_n 2^{-n} \frac{\sup \{ |f(x) - g(x)| \mid x \in K_n \}}{\sup \{ |f(x) - g(x)| \mid x \in K_n \} + 1}.$$

For $f \in C(X)$ we have

$$f \in L^p(X, \mu) \iff \exists M \, \forall n \int_{K_n} |f|^p \, d\mu \leq M,$$

and since for every compact K the map $C(X) \to \mathbb{R}$, $f \mapsto \int_K |f|^p \, d\mu$, is continuous, then $L^p(X, \mu) \cap C(X)$ is $\mathbf{\Sigma}^0_2$.

Given any $f \in C(X)$ and any n we can find $g, h \in C(X)$ agreeing with f on K_n and such that $g \in L^p(X, \mu)$, and $h \notin L^p(X, \mu)$. Hence $C(X) \cap L^p(X, \mu)$ and $C(X) \setminus L^p(X, \mu)$ are both dense in $C(X)$. Arguing as in the proof of Theorem 2.2 $C(X) \cap L^p(X, \mu)$ is $\mathbf{\Sigma}^0_2$-complete. □

To show that some topological hypothesis on X is necessary in Theorem 3.4 we will show that if X is Polish not σ-compact (which is only slightly stronger than being not locally compact), then $C(X) \cap L^p(X,\mu)$ is $\mathbf{\Pi}^1_1$-complete in $L^p(X,\mu)$.

First we obtain the following quite general upper bound for the complexity of $C(X) \cap L^p(X,\mu)$ within $L^p(X,\mu)$:

Lemma 3.7. *Let X be a Polish space and μ a σ-finite Borel measure on X. Then $C(X) \cap L^p(X,\mu)$ is $\mathbf{\Pi}^1_1$ in $L^p(X,\mu)$.*

PROOF. Let X' be the set of all elements of X such that μ assigns positive measure to any open neighborhood of x. Notice that X' is closed in X and $\mu(X \setminus X') = 0$. The map $L^p(X,\mu) \to L^p(X',\mu)$, $f \mapsto f \restriction X'$, is a homeomorphism and its restriction to $C(X) \cap L^p(X,\mu)$ is onto $C(X') \cap L^p(X',\mu)$ by the Tietze extension theorem. Hence it suffices to show that $C(X') \cap L^p(X',\mu)$ is $\mathbf{\Pi}^1_1$ in $L^p(X',\mu)$. Therefore we may assume that μ is non-vanishing on open sets.

Let d be a complete metric on X and let $f \in L^p(X,\mu)$. We claim that $f \in C(X) \cap L^p(X,\mu)$ if and only if

$$\forall x \in X\ \exists! r \in \mathbb{R}\ \forall \varepsilon > 0\ \exists \delta > 0\ \forall^*_\mu y \in X (d(x,y) < \delta \implies |f(y) - r| \leq \varepsilon). \quad (2)$$

Using Lemma 2.1 and the well known fact that $\exists!$ in front of a Borel matrix behaves as a universal quantifier (see e.g. [6, theorem 18.11]), the claim suffices to prove the lemma.

If $f \in C(X) \cap L^p(X,\mu)$, then f is equal μ-a.e. to a unique continuous function g. For any $x \in X$ let $r = g(x)$ and, given ε let δ be such that $\forall y \in X(d(x,y) < \delta \implies |g(y) - r| \leq \varepsilon)$. Then $\forall^*_\mu y(d(x,y) < \delta \implies |f(y) - r| \leq \varepsilon)$. Since μ is non-vanishing on open sets, no other r could work for x and hence (2) is satisfied.

Now suppose $f \in L^p(X,\mu)$ satisfies (2) and define $g : X \to \mathbb{R}$ by letting $g(x)$ be the unique r given by (2). We claim that $f = g$ μ-a.e. and that g is continuous, so that $f \in C(X) \cap L^p(X,\mu)$.

Suppose $f \neq g$ on a set of positive measure. Then for some $\varepsilon > 0$ we have that for a set E with $\mu(E) > 0$, $|f(x) - g(x)| \geq 2\varepsilon$ for every $x \in E$. Since f can be assumed to be Borel measurable, there exists a Borel set $E' \subseteq E$ with $\mu(E') > 0$ such that $f \restriction E'$ is continuous ([6, theorem 17.12]). For every $x \in E'$ there exists $\delta_x > 0$ such that for every $y \in E'$ with $d(x,y) < \delta_x$ we have $|f(y) - f(x)| < \varepsilon$. If δ_x is small enough, using (2) and the definition of g, we have

$$\forall^*_\mu y(d(x,y) < \delta_x \implies |f(y) - g(x)| \leq \varepsilon).$$

Using the fact that E' is a Lindelöf space it follows that $\mu(E' \cap B(x;\delta_x)) > 0$ for some $x \in E'$. Fix such an x and let

$$F = E' \cap B(x;\delta_x) \cap \{ y \mid |f(y) - g(x)| \leq \varepsilon \},$$

so that $\mu(F) > 0$. Then if $y \in F$ we have $d(x,y) < \delta_x$, $y \in E'$ and $|f(y) - g(x)| \leq \varepsilon$. Therefore

$$|f(x) - g(x)| \leq |f(x) - f(y)| + |f(y) - g(x)| < 2\varepsilon,$$

against $x \in E' \subseteq E$ which implies $|f(x) - g(x)| \geq 2\varepsilon$. Therefore $f = g$ μ-a.e.

Now suppose g is not continuous at some $x \in X$. There exist $\varepsilon > 0$ and a sequence (x_n) converging to x such that $|g(x_n) - g(x)| > 2\varepsilon$ for every n. By (2) there exists $\delta > 0$ such that

$$\forall^*_\mu y(d(x,y) < \delta \implies |f(y) - g(x)| \leq \varepsilon).$$

Let n be such that $d(x_n, x) < \delta/2$ and (again by (2) and the definition of g) let $\delta_n < \delta/2$ be such that

$$\forall^*_\mu y(d(x_n,y) < \delta_n \implies |f(y) - g(x_n)| \leq \varepsilon).$$

Notice that $d(x_n, y) < \delta_n$ implies $d(x,y) < \delta$ and let

$$D_n = \{ y \mid d(x_n,y) < \delta_n \ \& \ |f(y) - g(x_n)| \leq \varepsilon \ \& \ |f(y) - g(x)| \leq \varepsilon \},$$

so that $\mu(D_n) > 0$. If $y \in D_n$, then $|f(y) - g(x_n)| \leq \varepsilon$ and $|f(y) - g(x)| \leq \varepsilon$ which imply $|g(x_n) - g(x)| \leq 2\varepsilon$, a contradiction. □

In the proof of the next theorem we will have to deal with the Baire space $\mathbb{N}^\mathbb{N}$ and use the following notation. $\mathbb{N}^{<\mathbb{N}}$ is the set of all finite sequences of natural numbers; if $\alpha \in \mathbb{N}^\mathbb{N}$ is an infinite sequence of natural numbers, then $\alpha \restriction n \in \mathbb{N}^{<\mathbb{N}}$ is the finite initial segment of α with length n.

Theorem 3.8. *Let X be a Polish space which is not σ-compact and μ a finite Borel measure on X which is non-vanishing on open sets. Then $C(X) \cap L^p(X,\mu)$ is $\mathbf{\Pi}^1_1$-complete in $L^p(X,\mu)$.*

PROOF. By Lemma 3.7 $C(X) \cap L^p(X,\mu)$ is $\mathbf{\Pi}^1_1$ in $L^p(X,\mu)$. By Hurewicz theorem ([6, theorem 7.10]) there exists a set $F \subseteq X$ which is closed in X and homeomorphic to $\mathbb{N}^\mathbb{N}$. We identify F with $\mathbb{N}^\mathbb{N}$. Furthermore we may assume that F is nowhere dense in X. If this were not the case we can substitute F with its closed and nowhere dense subset $\{ \alpha \in F \mid \forall n \, \alpha(2n) = 0 \}$, which is still homeomorphic to $\mathbb{N}^\mathbb{N}$.

Fix a complete metric d on X. It is the metric we use when we speak of the diameter of a subset of X. Also fix a bijection $\# : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ and, for every $s \in \mathbb{N}^{<\mathbb{N}}$, denote by s^* a fixed element of $\mathbb{N}^{\mathbb{N}}$ which extends s (e.g. s concatenated with infinitely many 0's). For every $s \in \mathbb{N}^{<\mathbb{N}}$ we define an open set W_s such that

i) W_s has at least two elements;

ii) $\operatorname{diam}(W_s) < 2^{-\#(s)}$;

iii) the sets W_s are pairwise disjoint;

iv) $d(s^*, W_s) < 2^{-\#(s)}$ (here s^* is viewed as an element of F);

v) the closure of W_s is disjoint from F.

To see that this construction is possible we proceed by induction on $\#(s)$. Assume we have defined W_t for all $t \in \mathbb{N}^{<\mathbb{N}}$ with $\#(t) < \#(s)$. Since s^* is not in the closure of $\bigcup_{\#(t)<\#(s)} W_t$, there is an open neighborhood U_s of s^* with diameter less than $2^{-\#(s)}$ which is disjoint from $\bigcup_{\#(t)<\#(s)} W_t$. Since F is nowhere dense, $U_s \setminus F$ contains at least two points and we can define W_s as the union of two open neighborhoods of these points which are contained in U_s and have positive distance from F. It is immediate to check that W_s satisfies *i)*-*v)*.

Notice that for every $\varepsilon > 0$ there is a finite number of s such that W_s has measure greater than ε. This follows from the finiteness of μ and the fact that the W_s's are pairwise disjoint.

Let (s_n) be a sequence of distinct elements of $\mathbb{N}^{<\mathbb{N}}$; then $\lim \#(s_n) = +\infty$ and hence, if $x_n \in W_{s_n}$ for every n and $\lim x_n = x$, then $x \in F$. In fact $d(s_n^*, x_n) < 2^{-\#(s_n)} \to 0$, so $x = \lim s_n^* \in F$.

For every $s \in \mathbb{N}^{<\mathbb{N}}$ define a continuous function $f_s : X \to [0,1]$ such that $f_s = 0$ outside W_s, and f_s takes the values 0 and 1 on elements of W_s. Such a function exists by *i)* and the Tietze extension theorem. Suppose now that Y is a Polish space and $A \subseteq Y$ is $\mathbf{\Sigma}^1_1$. We will define a continuous function $Y \to L^p(X,\mu)$, $y \mapsto g_y$, such that $y \notin A$ if and only if $g_y \in C(X) \cap L^p(X,\mu)$. This shows $Y \setminus A \leq_{\mathrm{W}} C(X) \cap L^p(X,\mu)$ and completes the proof. For every $y \in Y$, g_y is of the form $\sum_{s\in\mathbb{N}^{<\mathbb{N}}} h_s(y) f_s$ where each $h_s : Y \to [0,1]$ is continuous. Notice that this implies that the range of g_y is contained in $[0,1]$. To define h_s fix a compatible metric d' on Y and let $\varphi : \mathbb{N}^{\mathbb{N}} \to A$ be a continuous onto map (φ exists because A is $\mathbf{\Sigma}^1_1$). Let

$$\delta_s(y) = \inf \{\, d'(\varphi(\alpha), y) \mid \alpha \text{ extends } s \,\},$$

and define $h_s(y) = \dfrac{1}{1+\#(s)\delta_s(y)}$. It is easy to check that $g_y \in L^p(X,\mu)$ and that $y \mapsto g_y$ is continuous. (This uses the fact noted above that only finitely many W_s's have measure greater than a given $\varepsilon > 0$.) Moreover g_y is a continuous function on $X \setminus F$ for every $y \in Y$.

If g_y fails to be continuous at some $\alpha \in F$, then (since $g_y(\alpha) = 0$), there exists a sequence (x_n) with $\lim x_n = \alpha$ and $g_y(x_n) \geq \eta > 0$. This implies that for every n there exists $s_n \in \mathbb{N}^{<\mathbb{N}}$ such that $x_n \in W_{s_n}$ and $h_{s_n}(y) \geq \eta$. Clearly $\lim s_n^* = \alpha$ and by *v)* we may assume that the s_n's are all distinct. Since $\lim \#(s_n) = +\infty$, we have $\lim \delta_{s_n}(y) = 0$. From $\lim \delta_{s_n}(y) = d'(\varphi(\alpha), y)$ it follows that $\varphi(\alpha) = y$, i.e. $y \in A$. Therefore if $y \notin A$, then $g_y \in C(X) \cap L^p(X,\mu)$.

If $y \in A$, let g be a function which is μ-a.e. equal to g_y and fix $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $\varphi(\alpha) = y$. For every n let $s_n = \alpha \restriction n$, so that $\delta_{s_n}(y) = 0$ and for every $x \in W_{s_n}$ we have $g_y(x) = f_{s_n}(x)$. For every n let $V_n^0, V_n^1 \subseteq W_{s_n}$ be nonempty open sets such that $f_{s_n}(x) < \frac{1}{4}$ for every $x \in V_n^0$ and $f_{s_n}(x) > \frac{3}{4}$ for every $x \in V_n^1$. Since $\mu(V_n^i) > 0$ (because μ is non-vanishing on open sets), there exists $x_n^i \in V_n^i$ such that $g(x_n^i) = g_y(x_n^i) = f_{s_n}(x_n^i)$. $x_n^i \in W_{s_n}$ implies that $\lim x_n^i = \alpha$ for $i = 0, 1$. Since $x_n^i \in V_n^i$, it follows that g is not continuous at α. Therefore if $y \in A$, then $g_y \notin C(X) \cap L^p(X,\mu)$. $\square$

The crucial idea in the preceding proof is due to the anonymous referee. In an earlier draft we could only prove Theorem 3.8 under more specific assumptions, i.e. when $X = \mathbb{N}^{\mathbb{N}}$ or X is an infinite-dimensional separable Banach space.

Theorem 3.8 can be improved by relaxing the hypothesis that the measure is finite. It suffices to assume that μ (beside being non-vanishing on open sets) is σ-finite and that $\lim_{\varepsilon \to 0} \mu(B(x;\varepsilon)) = 0$ for every $x \in X$. The proof is essentially the same, but we require that W_s satisfies the additional condition $\mu(W_s) < 2^{-\#(s)}$. This can be achieved because each point of X has neighborhoods with arbitrarily small positive measure. Therefore also in this case only finitely many W_s's have measure greater than a given $\varepsilon > 0$, and this was the only consequence of the finiteness of μ which was actually used in the proof.

4 Differentiable and Integrable Functions in $L^p(\mathbb{R})$

As usual, let $1 \leq p < \infty$. Throughout this section we will use the Lebesgue measure on I and $\mathbb{R}$, which we denote by m, but actually any Borel measure non-vanishing on open sets which is finite on compact sets will work in the same way. Similarly the results in this section can be generalized to the case of several variables.

We start with an extension of Corollary 3.5. Consider, for $k = 1, 2, \dots, \infty$, $C^k(I)$ as a subset of $L^p(I)$ and $C^k(\mathbb{R}) \cap L^p(\mathbb{R})$ as a subset of $L^p(\mathbb{R})$. In this context $C^k(I)$ (and similarly $C^k(\mathbb{R})$) is the space of $L^p(I)$ functions which are m-a.e. equal to a (necessarily unique) k-times continuously differentiable real valued function defined on I.

Lemma 4.1. *For every $k = 1, 2, \dots, \infty$, $C^k(I)$ is $\mathbf{\Pi}^0_3$-hard in $L^p(I)$ and $C^\infty(\mathbb{R}) \cap L^p(\mathbb{R})$ is $\mathbf{\Pi}^0_3$-hard in $L^p(\mathbb{R})$.*

PROOF. Notice that the proof of theorem 23.14 of [6] shows that $C^\infty(I)$ is $\mathbf{\Pi}^0_3$-hard in $C(I)$ and can be easily converted to a proof that the set $C^\infty(\mathbb{R}) \cap \{ f \in C(\mathbb{R}) \mid \operatorname{Supp}(f) \subseteq I \}$ is $\mathbf{\Pi}^0_3$-hard in $\{ f \in C(\mathbb{R}) \mid \operatorname{Supp}(f) \subseteq I \}$ (as usual $\operatorname{Supp}(f)$ denotes the support of f).

As noticed at the beginning of section 3, $C(I)$ (with the topology of the sup norm) continuously embeds in $L^p(I)$ and therefore, by the same token, $\{ f \in C(\mathbb{R}) \mid \operatorname{Supp}(f) \subseteq I \}$ continuously embeds in $L^p(\mathbb{R})$. Therefore any $\mathbf{\Pi}^0_3$-hard subset of $C(I)$ is also $\mathbf{\Pi}^0_3$-hard in $L^p(I)$ (and the same holds for $L^p(\mathbb{R})$), and the case $k = \infty$ is proved.

If $k < \infty$ the same sort of argument applies but, since in [6, theorem 23.14] this part of the proof is left to the reader, we provide some more details and show that $c_0 \leq_{\mathrm{W}} C^k(I)$. For every n let g_n be a C^∞ function defined on I with $\operatorname{Supp}(g_n) \subseteq (2^{-n-1}, 2^{-n})$ such that $\sup\left\{ |g_n^{(k)}(x)| \mid x \in I \right\} = 1$ and therefore $\forall h < k$ $\sup\left\{ |g_n^{(h)}(x)| \mid x \in I \right\} \leq 2^{-n(k-h)}$. For every $(y_n) \in I^{\mathbb{N}}$ let $f_{(y_n)}(x) = \sum_n y_n g_n(x)$. It is clear that the map $I^{\mathbb{N}} \to L^p(I)$, $(y_n) \mapsto f_{(y_n)}$ is continuous. It is also easy to check that $f_{(y_n)} \in C^{k-1}(I)$, and that $f_{(y_n)}$ is C^k on $I \setminus \{0\}$, for every $(y_n) \in I^{\mathbb{N}}$. It is now clear that

$$\begin{aligned} f_{(y_n)} \in C^k(I) &\iff f^{(k)}_{(y_n)}(0) \text{ exists and } f^{(k)}_{(y_n)} \text{ is continuous at } 0 \\ &\iff \lim_{x \to 0} f^{(k)}_{(y_n)}(x) = 0 \\ &\iff (y_n) \in c_0. \end{aligned}$$

The above proof shows also that $c_0 \leq_{\mathrm{W}} C^k(\mathbb{R}) \cap L^p(\mathbb{R})$: it suffices to let $f_{(y_n)}(x) = 0$ for $x \notin I$. $\square$

Now we take care of the upper bound. To this end we need to express the k-th derivative in terms of the original function. This is accomplished by a calculus exercise resulting in the following lemma.

Lemma 4.2. *If $k \geq 1$ and $f \in C^{k-1}(I)$, then for every $x \in I$ we have*

$$f^{(k)}(x) = \lim_{y_1 \to x} \cdots \lim_{y_k \to x} \frac{\displaystyle\sum_{S \subseteq \{1,\dots,k\}} (-1)^{k-|S|} f\left(\sum_{i \in S} y_i - (|S|-1)x\right)}{\displaystyle\prod_{i=1}^{k} (y_i - x)}, \tag{3}$$

i.e. either both sides exist and are equal or both sides do not exist.

PROOF. The proof is by induction on k. For $k = 1$ the right hand side of (3) is just $\lim_{y_1 \to x} \dfrac{f(y_1) - f(x)}{y_1 - x}$.

Now assume the lemma holds for k and let $f \in C^k(I)$; then $f' \in C^{k-1}(I)$ and we can apply the induction hypothesis to f'. We have

$$\begin{aligned} f^{(k+1)}(x) &= (f')^{(k)}(x) \\ &= \lim_{y_1 \to x} \cdots \lim_{y_k \to x} \frac{\displaystyle\sum_{S \subseteq \{1,\dots,k\}} (-1)^{k-|S|} f'\left(\sum_{i \in S} y_i - (|S|-1)x\right)}{\displaystyle\prod_{i=1}^{k} (y_i - x)}. \end{aligned}$$

Fix $S \subseteq \{1, \dots, k\}$

$$\begin{aligned} & f'\left(\sum_{i \in S} y_i - (|S|-1)x\right) \\ &= \lim_{y_{k+1} \to x} \frac{f\left(\sum_{i \in S} y_i - (|S|-1)x + y_{k+1} - x\right) - f\left(\sum_{i \in S} y_i - (|S|-1)x\right)}{y_{k+1} - x} \\ &= \lim_{y_{k+1} \to x} \frac{f\left(\sum_{i \in S'} y_i - (|S'|-1)x\right) - f\left(\sum_{i \in S} y_i - (|S|-1)x\right)}{y_{k+1} - x} \end{aligned}$$

where $S' = S \cup \{k+1\}$. Hence $f^{(k+1)}(x)$ equals

$$\lim_{y_1 \to x} \dots \lim_{y_k \to x} \lim_{y_{k+1} \to x} \frac{\displaystyle\sum_{T \subseteq \{1,\dots,k+1\}} (-1)^{k+1-|T|} f\left(\sum_{i \in T} y_i - (|T|-1)x\right)}{\displaystyle\prod_{i=1}^{k+1} (y_i - x)}. \qquad \square$$

Lemma 4.3. *For every $k = 1, 2, \dots, \infty$, $C^k(I)$ is $\mathbf{\Pi}^0_3$ in $L^p(I)$.*

PROOF. First of all notice that the case $k = \infty$ follows from the other cases because $C^\infty(I) = \bigcap_k C^k(I)$ and $\mathbf{\Pi}^0_3$ is closed under countable intersections. For $k < \infty$ the proof is by induction on k and we start with the case $k = 1$.

Let $f \in L^p(I)$. We claim that $f \in C^1(I)$ if and only if $f \in C(I)$ and for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\forall^*_m x, y, z \left(0 < |x - y|, |x - z| < \delta \implies \left| \frac{f(x) - f(y)}{x - y} - \frac{f(x) - f(z)}{x - z} \right| \leq \varepsilon \right). \tag{4}$$

The result follows from this equivalence, using Corollary 3.5 and Lemma 2.1.

To prove the forward direction of the equivalence suppose f is a Borel function such that (its equivalence class) belongs to $L^p(I)$ and to $C^1(I) \subseteq C(I)$. Let $\varepsilon > 0$ and let g be the unique continuously differentiable function which is a.e. equal to f. Since I is compact, there exists $\delta > 0$ such that for every $x, y \in I$ with $0 < |x - y| < \delta$ we have $\left| g'(x) - \dfrac{g(x) - g(y)}{x - y} \right| \leq \dfrac{\varepsilon}{2}$. Then

$$\forall x, y, z \left(0 < |x - y|, |x - z| < \delta \implies \left| \frac{g(x) - g(y)}{x - y} - \frac{g(x) - g(z)}{x - z} \right| \leq \varepsilon \right)$$

and hence (using $m(\{ x \mid f(x) \neq g(x) \}) = 0$)

$$\forall^*_m x, y, z \left(0 < |x - y|, |x - z| < \delta \implies \left| \frac{f(x) - f(y)}{x - y} - \frac{f(x) - f(z)}{x - z} \right| \leq \varepsilon \right).$$

Now assume that (the equivalence class of) f belongs to $C(I)$ and for all ε there exists δ satisfying (4). Let g be the unique continuous function which is a.e. equal to f. Notice that $f \in C^1(I)$ if and only if g is continuously differentiable. We have that for all ε there exists δ such that

$$\forall^*_m x, y, z \left(0 < |x - y|, |x - z| < \delta \implies \left| \frac{g(x) - g(y)}{x - y} - \frac{g(x) - g(z)}{x - z} \right| \leq \varepsilon \right)$$

which, by the continuity of g, implies

$$\forall x, y, z \left(0 < |x - y|, |x - z| < \delta \implies \left| \frac{g(x) - g(y)}{x - y} - \frac{g(x) - g(z)}{x - z} \right| \leq \varepsilon \right).$$

This implies that $\lim_{y \to x} \dfrac{g(x) - g(y)}{x - y}$ converges uniformly in x to a continuous function $g'(x)$; i.e., that g is continuously differentiable.

If $1 < k < \infty$, we need to generalize appropriately (4). By a proof analogous to the one for $k = 1$, using Lemma 4.2, it turns out that, for $f \in L^p(I)$, $f \in C^k(I)$ if and only if $f \in C^{k-1}(I)$ and for all $\varepsilon > 0$ there exists $\delta > 0$ such that for m-almost all $x, y_1, \dots, y_k, z_1, \dots, z_k \in I$, if $0 < |x - y_1|, \dots, |x - y_k|, |x - z_1|, \dots, |x - z_k| < \delta$, then the absolute value of the difference between

$$\frac{\displaystyle\sum_{S \subseteq \{1,\dots,k\}} (-1)^{k-|S|} f\left(\sum_{i \in S} y_i - (|S| - 1)x\right)}{\displaystyle\prod_{i=1}^{k} (y_i - x)} \quad \text{and}$$

$$\frac{\displaystyle\sum_{S \subseteq \{1,\dots,k\}} (-1)^{k-|S|} f\left(\sum_{i \in S} z_i - (|S| - 1)x\right)}{\displaystyle\prod_{i=1}^{k} (z_i - x)}$$

is less than or equal to ε.

The result follows from this equivalence, using the the induction hypothesis (asserting that $C^{k-1}(I)$ is $\mathbf{\Pi}^0_3$) and Lemma 2.1. □

Theorem 4.4. *For every $k = 1, 2, \dots, \infty$, $C^k(I)$ is $\mathbf{\Pi}^0_3$-complete in $L^p(I)$.*

PROOF. Immediate from Lemmas 4.1 and 4.3. □

Theorem 4.5. *For every $k = 1, 2, \dots, \infty$, $C^k(\mathbb{R}) \cap L^p(\mathbb{R})$ is $\mathbf{\Pi}^0_3$-complete in $L^p(\mathbb{R})$.*

PROOF. The hardness part is contained in Lemma 4.1. For $k < \infty$ to show that $C^k(\mathbb{R}) \cap L^p(\mathbb{R})$ is $\mathbf{\Pi}^0_3$ we can modify the proof of Theorem 4.3 exactly as to prove the analogous part of theorem 3.4 we modified the proof of Lemma 3.2. Then $C^\infty(\mathbb{R}) \cap L^p(\mathbb{R}) = \bigcap_k C^k(\mathbb{R}) \cap L^p(\mathbb{R})$ is also $\mathbf{\Pi}^0_3$. □

Let $\mathrm{DIFF}(I)$ and $\mathrm{DIFF}(\mathbb{R})$ be the set of all continuous functions (on I or $\mathbb{R}$, respectively) which are m-a.e. equal to a (necessarily unique) everywhere differentiable function.

Theorem 4.6. $\mathrm{DIFF}(I)$ *is $\mathbf{\Pi}^1_1$-complete in $L^p(I)$ and* $\mathrm{DIFF}(\mathbb{R}) \cap L^p(\mathbb{R})$ *is $\mathbf{\Pi}^1_1$-complete in $L^p(\mathbb{R})$.*

Proof. The set of everywhere differentiable functions on I is $\mathbf{\Pi}^1_1$-complete in $C(I)$ by a classical theorem of Mazurkiewicz (see e.g. [6, theorem 33.9]). Arguing as in the proof of Lemma 4.1, this implies that $\mathrm{DIFF}(I)$ is $\mathbf{\Pi}^1_1$-hard in $L^p(I)$ and $\mathrm{DIFF}(\mathbb{R}) \cap L^p(\mathbb{R})$ is $\mathbf{\Pi}^1_1$-hard in $L^p(\mathbb{R})$.

To prove that the two sets are also $\mathbf{\Pi}^1_1$ we claim that for $f \in L^p(\mathbb{R})$ we have that $f \in \mathrm{DIFF}(\mathbb{R})$ if and only if for all $x \in \mathbb{R}$ there exists a unique pair $(r_0, r_1) \in \mathbb{R}^2$ such that

$$\forall \varepsilon \, \exists \delta \, \forall^*_m y \in \mathbb{R} \left(|y - x| < \delta \implies |f(y) - r_0| \le \varepsilon \,\&\, \left| \frac{f(y) - r_0}{y - x} - r_1 \right| \le \varepsilon \right).$$

The latter condition is $\mathbf{\Pi}^1_1$ and an argument analogous to the one of the proof of Lemma 3.7 shows that the claim holds, completing the proof. □

Let $\mathrm{RIEM}(I)$ and $\mathrm{RIEM}(\mathbb{R})$ be the set of all functions (on I or $\mathbb{R}$, respectively) which are m-a.e. equal to a (non-unique) Riemann integrable function. Notice that with this definition some of the classical examples of non-Riemann integrable functions, such as the Dirichlet function which takes value 1 on $\mathbb{Q}$ and 0 elsewhere, do belong to $\mathrm{RIEM}(\mathbb{R})$.

Theorem 4.7. $\mathrm{RIEM}(I) \cap L^p(I)$ *is* $\mathbf{\Pi}^0_3$*-complete in* $L^p(I)$ *and* $\mathrm{RIEM}(\mathbb{R}) \cap L^p(\mathbb{R})$ *is* $\mathbf{\Pi}^0_3$*-complete in* $L^p(\mathbb{R})$.

Proof. If $f \in L^p(I)$, then $f \in \mathrm{RIEM}(I)$ if and only if for all $\varepsilon > 0$ there exist three finite sequences of real numbers, $(r_i)_{i=0,\ldots,n+1}$, $(m_i)_{i=0,\ldots,n}$, and $(M_i)_{i=0,\ldots,n}$, such that $0 = r_0 < r_1 < \cdots < r_{n+1} = 1$,

$$\sum_{i=0}^{n} (r_{i+1} - r_i)(M_i - m_i) \le \varepsilon$$

and

$$\forall i \le n \, \forall^*_m x \in I \, (r_i < x < r_{i+1} \implies m_i \le f(x) \le M_i).$$

Using Lemmas 2.1 and 2.3 as in the proof of Theorem 2.2 this shows that $\mathrm{RIEM}(I) \cap L^p(I)$ is $\mathbf{\Pi}^0_3$ in $L^p(I)$.

To obtain the upper bound for $\mathrm{RIEM}(\mathbb{R}) \cap L^p(\mathbb{R})$ let $f \in L^p(\mathbb{R})$; then $f \in \mathrm{RIEM}(\mathbb{R})$ if and only if $f \in L^1(\mathbb{R})$ and $f \in \mathrm{RIEM}([-n, n])$ for every n. Thus Theorem 2.4 and the above result about I (which obviously holds for every bounded closed interval) imply that $\mathrm{RIEM}(\mathbb{R}) \cap L^p(\mathbb{R})$ is $\mathbf{\Pi}^0_3$ in $L^p(\mathbb{R})$.

To show that $\mathrm{RIEM}(I) \cap L^p(I)$ is $\mathbf{\Pi}^0_3$-hard we show that $c_0 \le_{\mathrm{W}} \mathrm{RIEM}(I) \cap L^p(I)$ by defining a map $I^{\mathbb{N}} \to L^p(I)$, $(z_n) \mapsto f_{(z_n)}$. First of all let us fix a Cantor set of positive measure $C \subseteq I$. This can be obtained by modifying a bit the standard construction of the usual Cantor 1/3-set so that we remove smaller portions of the intervals as the construction progresses. Let us denote by A_n the set removed at stage n (so that A_n consists of 2^n disjoint open intervals and $C = I \setminus \bigcup_n A_n$). Let also D be the countable set of all the endpoints of the intervals appearing in some A_n.

For every $(z_n) \in I^{\mathbb{N}}$ define $f_{(z_n)} : I \to \mathbb{R}$ by

$$f_{(z_n)}(x) = \begin{cases} 0 & \text{if } x \in C, \\ z_n & \text{if } x \in A_n. \end{cases}$$

Since C and the A_n's are Borel, it is immediate that $f_{(z_n)} \in L^p(I)$ holds.

To check that $(z_n) \in c_0$ if and only if $f_{(z_n)} \in \mathrm{RIEM}(I)$ we use the classical fact (see e.g. [7, theorem 11.33]) that a bounded real-valued function f defined on I is Riemann integrable if and only the set of points of discontinuity of f has measure 0. Thus we need to check the continuity of $f_{(z_n)}$ on I and, since $m(D) = 0$, we can disregard the points in D. Moreover $f_{(z_n)}$ is continuous at every point of $\bigcup_n A_n$ and we can focus on the points in $C \setminus D$, which is a set of positive measure. Fix $x \in C \setminus D$. Notice that every open neighborhood of x intersects every A_n for n large enough. Conversely, given N we can find an open neighborhood of x which intersects only the A_n's with $n > N$. These facts imply that $f_{(z_n)}$ is continuous at x if and only if $(z_n) \in c_0$. Since this holds for every point of a set of positive measure, $f_{(z_n)}$ is Riemann integrable if and only if $(z_n) \in c_0$. Since each A_n has positive measure, every function which is m-a.e. equal to $f_{(z_n)}$ with $(z_n) \notin c_0$ is also discontinuous at almost every $x \in C \setminus D$ (in fact, at every $x \in C \setminus D$) and hence $(z_n) \in c_0$ if and only if $f_{(z_n)} \in \mathrm{RIEM}(I)$.

The above proof that $\mathrm{RIEM}(I) \cap L^p(I)$ is $\mathbf{\Pi}^0_3$-hard can easily be turned into a proof that $\mathrm{RIEM}(\mathbb{R}) \cap L^p(\mathbb{R})$ is $\mathbf{\Pi}^0_3$-hard (just define $f_{(z_n)}(x) = 0$ when $x \notin I$), completing the proof of the theorem. $\square$

5 $C(X)$ and its Subspaces with the Compact-Open Topology

If X is locally compact Polish, then the compact-open topology makes $C(X)$ a separable Frechet space; i.e., a vector space whose topology is induced by a complete metric. In particular $C(X)$ is a Polish group. For various purposes different subsets (which are usually also vector spaces, and in particular subgroups) of $C(X)$ are of interest. In general these are not Polish (as it will follow from our results, using the fact that a subset of a Polish space is Polish if and only if it is $\mathbf{\Pi}^0_2$) with the induced compact-open topology, but some of them come equipped with their own Polish topology.

Definition 5.1. If X is Polish locally compact, let $C_{00}(X) = \{f \in C(X) \mid f$ has compact support$\}$ equipped with the sup metric and let $C_0(X)$ be the completion of $C_{00}(X)$. $C_0(X)$ is Polish (with respect to the sup metric) and its elements are called functions vanishing at infinity.

The above terminology is justified by the fact that $f \in C_0(X)$ if and only if $f \in C(X)$ and for every $\varepsilon > 0$ the set of points such that $|f(x)| \geq \varepsilon$ has compact closure.

The following lemmas imply also that neither $C_{00}(X)$ nor $C_0(X)$ is Polish with the topology inherited from $C(X)$, unless X is compact (in which case $C_{00}(X) = C_0(X) = C(X)$).

Theorem 5.2. *If X is Polish locally compact and not compact, then $C_{00}(X)$ is $\mathbf{\Sigma}^0_2$-complete as a subset of $C(X)$.*

Proof. Let $(K_n) \subseteq \mathsf{K}(X)$ be a decomposition of X as described in the introduction; we may also assume $K_0 = \emptyset$. For $f \in C(X)$ we have $f \in C_{00}(X)$ if and only if $\exists n\, \forall x \notin K_n\; f(x) = 0$. Since $\{\, f \in C(X) \mid f \restriction (X \setminus K_n) = 0 \,\}$ is clearly closed within $C(X)$, we have that $C_{00}(X)$ is $\mathbf{\Sigma}^0_2$.

To see that $C_{00}(X)$ is $\mathbf{\Sigma}^0_2$-hard we use Q_2 (see 1.1). For every n let y_n be an element of the interior of $K_{n+1} \setminus K_n$. Since the K_n's are cofinal in $\mathsf{K}(X)$, the sequence (y_n) has no cluster points. We define a continuous map $2^{\mathbb{N}} \to C(X)$, $\alpha \mapsto f_\alpha$, showing $Q_2 \leq_{\mathrm{W}} C_{00}(X)$. For every n let $L_n = K_n \cup (X \setminus K_{n+1})$ and $\eta_n = d(y_n, L_n) > 0$. Given $x \in X$ let $n(x)$ be the least n such that $x \in K_{n+1}$ and let

$$f_\alpha(x) = \frac{\min(\eta_{n(x)}, d(x, L_{n(x)}))}{\eta_{n(x)}} \cdot \alpha(n(x)).$$

It is easy to check that f_α is continuous (on the boundary of each K_n takes value 0) and that $\alpha \mapsto F_\alpha$ is continuous. Since $f_\alpha(y_n) = \alpha(n)$, for every n it follows that $f_\alpha \in C_{00}(X)$ if and only if $\alpha \in Q_2$. $\square$

Remark 5.3. A slight variant of the proof of Theorem 5.2 (in which we replace $\alpha(n(x))$ with $\frac{\alpha(n(x))}{n(x)+1}$ in the definition of f_α) shows also that if X is Polish, then $C_{00}(X)$ is $\mathbf{\Sigma}^0_2$-complete as a subset of $C_0(X)$.

Theorem 5.4. *If X is Polish locally compact and not compact, then $C_0(X)$ is $\mathbf{\Pi}^0_3$-complete as a subset of $C(X)$.*

PROOF. Let K_n be as in the proof of Theorem 5.2 and fix a countable dense set $D \subseteq X$. If $f \in C(X)$ we have that $f \in C_0(X)$ is equivalent to

$$\forall \varepsilon > 0\ \exists n\ \forall x \in D (x \notin K_n \implies |f(x)| \leq \varepsilon).$$

This shows that $C_0(X)$ is $\mathbf{\Pi}^0_3$.

To see that it is $\mathbf{\Pi}^0_3$-hard we use c_0 (see 1.1) and follow the notation of the proof of theorem 5.2. We map continuously each $(x_k) \in I^{\mathbb{N}}$ to the function $f_{(x_k)} \in C(X)$ defined by

$$f_{(x_k)}(x) = \frac{\min(\eta_{n(x)}, d(x, L_{n(x)}))}{\eta_{n(x)}} \cdot x_{n(x)}.$$

This shows that $c_0 \leq_{\mathrm{W}} C_0(X)$ and completes the proof. □

We can also consider $C_{00}(X)$ and $C_0(X)$ as subspaces of $L^p(X, \mu)$.

Theorem 5.5. *Let X be Polish locally compact and μ a Borel measure on X non-vanishing on open sets and finite on compact sets. Suppose also there exists $x_0 \in X$ such that $\mu(\{x_0\}) = 0$. Then $C_{00}(X)$ and $C_0(X) \cap L^p(X, \mu)$ are $\mathbf{\Pi}^0_3$-complete in $L^p(X, \mu)$.*

PROOF. For $f \in L^p(X, \mu)$ we have

$$f \in C_{00}(X) \iff f \in C(X)\ \&\ \exists n \int_{X \setminus K_n} |f|^p \, d\mu = 0.$$

By Theorem 3.4 $C(X)$ is $\mathbf{\Pi}^0_3$. Hence $C_{00}(X)$ is the intersection of a $\mathbf{\Pi}^0_3$ and a $\mathbf{\Sigma}^0_2$ set, and is $\mathbf{\Pi}^0_3$. The reduction in the proof of Lemma 3.1 actually shows that $C_{00}(X)$ is $\mathbf{\Pi}^0_3$-hard in $L^p(X, \mu)$.

Fix a countable basis $\{U_m\}$ for the topology of X. We may assume that every U_m is included in some K_n and hence has finite measure. For $f \in L^p(X, \mu)$ we have $f \in C_0(X)$ if and only if

$$f \in C(X)\ \&\ \forall \varepsilon > 0\ \exists n\ \forall m \left(U_m \cap K_n = \emptyset \implies \int_{U_m} |f|^p \, d\mu \leq \varepsilon^p \cdot \mu(U_m) \right).$$

As the map $f \mapsto \int_{U_m} |f|^p \, d\mu$ is continuous, $C_0(X)$ is the intersection of two $\mathbf{\Pi}^0_3$ sets and hence it is $\mathbf{\Pi}^0_3$. The reduction in the proof of Lemma 3.1 shows also that $C_0(X) \cap L^p(X, \mu)$ is $\mathbf{\Pi}^0_3$-hard in $L^p(X, \mu)$. □

A generalization of $C_0(X)$ is given by the following definition.

Definition 5.6. If X is Polish locally compact let $C_c(X)$ be the set of all $f \in C(X)$ such that there exists $L \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists $K \in \mathsf{K}(X)$ such that for every $x \notin K$ we have $|f(x) - L| < \varepsilon$. The elements of $C_c(X)$ are called functions constant at infinity. The sup metric can be used also to make $C_c(X)$ a Polish space (homeomorphic to $C_0(X) \times \mathbb{R}$ via the obvious bijection).

Theorem 5.7. *If X is Polish locally compact and not compact, then $C_c(X)$ is $\mathbf{\Pi}^0_3$-complete as a subset of $C(X)$.*

Proof. Let K_n, y_n and D be as in the proof of Theorem 5.4. To see that $C_c(X)$ is $\mathbf{\Pi}^0_3$ notice that if $f \in C(X)$ we have that $f \in C_c(X)$ is equivalent to

$$\forall \varepsilon > 0\, \exists n \in \mathbb{N}\, \forall x, y \in D(x, y \notin K_n \implies |f(x) - f(y)| \leq \varepsilon).$$

To prove that $C_c(X)$ is $\mathbf{\Pi}^0_3$-hard notice that the reduction used in the proof of theorem 5.4 shows that $c \leq_{\mathrm{W}} C_c(X)$. $\square$

A further extension of $C_c(X)$ is the set $C_b(X)$ of all $f \in C(X)$ such that f is bounded. The sup metric can be used also in $C_b(X)$ but if X is not compact the resulting topology is not separable and hence not Polish. $C_b(X)$ inherits a topology from $C(X)$. The next theorem implies that $C_b(X)$ is not Polish with this topology as well.

Theorem 5.8. *If X is Polish locally compact and not compact, then $C_b(X)$ is $\mathbf{\Sigma}^0_2$-complete as a subset of $C(X)$.*

Proof. Let K_n, y_n and D be as in the proof of Theorem 5.4. For $f \in C(X)$ we have $f \in C_b(X)$ if and only if $\exists M \in \mathbb{N}\, \forall x \in D\, |f(x)| \leq M$. This shows that $C_b(X)$ is $\mathbf{\Sigma}^0_2$.

To see that $C_b(X)$ is $\mathbf{\Sigma}^0_2$-hard we use again Q_2. Using again the notation of the proof of Theorem 5.2 we define a continuous map $\alpha \mapsto f_\alpha$ from $2^{\mathbb{N}}$ to $C(X)$ setting

$$f_\alpha(x) = \frac{\min(\eta_{n(x)}, d(x, L_{n(x)}))}{\eta_{n(x)}} \cdot \sum_{i=0}^{n(x)} \alpha(i),$$

so that $f_\alpha(y_n) = \sum_{i=0}^{n} \alpha(i)$. Clearly this shows $Q_2 \leq_{\mathrm{W}} C_b(X)$ and completes the proof. $\square$

Let (X, d) be a separable metric space. Let

$$UC(X, d) = \{\, f \in C(X) \mid f \text{ is } d\text{-uniformly continuous} \,\}.$$

Metric spaces such that $UC(X, d) = C(X)$ are usually called Atsuji spaces. Notice that this is a metric property: the same metrizable space may be Atsuji with respect to one compatible metric and not Atsuji with respect to another.

Theorem 5.9. *If (X, d) is separable metric which is not Atsuji and X is locally compact, then $UC(X, d)$ is $\mathbf{\Pi}^0_3$-complete in $C(X)$ with the compact-open topology.*

PROOF. Let $D \subseteq X$ be countable dense. For $f \in C(X)$ we have that

$$f \in UC(X, d) \iff \forall \varepsilon\, \exists \delta\, \forall x, y \in D(d(x, y) < \delta \implies |f(x) - f(y)| \leq \varepsilon).$$

This shows that $UC(X, d)$ is $\mathbf{\Pi}^0_3$. Since X is not Atsuji, there exist two disjoint sequences (a_n) and (b_n) of elements of X such that $\lim_{n\to\infty} d(a_n, b_n) = 0$ and neither sequence has cluster points (see e.g. [5]). For every n let

$$\eta_n = \frac{1}{2} \inf \left(\{\, d(a_m, b_n) \mid m \in \mathbb{N} \,\} \cup \{\, d(b_m, b_n) \mid m \neq n \,\}\right).$$

Notice that $\eta_n > 0$ and that $\lim \eta_n = 0$. Let $B_n = B(b_n; \eta_n)$: we have $B_n \cap B_m = \emptyset$ whenever $n \neq m$ and $a_m \notin B_n$ for every m. We will show that $c_0 \leq_{\mathrm{W}} UC(X, d)$ by defining a map $I^{\mathbb{N}} \to C(X)$, $(z_n) \mapsto f_{(z_n)}$. If $x \in X \setminus \bigcup B_n$ we set $f_{(z_n)}(x) = 0$. If $x \in B_n$ we set

$$f_{(z_n)}(x) = \frac{\eta_n - d(x, b_n)}{\eta_n} z_n.$$

It is easy to check that $f_{(z_n)} \in C(X)$ and that the map $(z_n) \mapsto f_{(z_n)}$ is continuous. (For the latter fact notice that every compact subset of X intersects finitely many B_n's.) Notice also that $\sup \{\, f_{(z_n)}(x) \mid x \in B_n \,\} = z_n$.

If $(z_n) \in c_0$ and $\varepsilon > 0$ let $\delta = \varepsilon \min \{\, \eta_n / z_n \mid z_n \geq \varepsilon \,\} > 0$. (The minimum is on a finite set.) We want to show that $d(x, y) < \delta \implies |f_{(z_n)}(x) - f_{(z_n)}(y)| < \varepsilon$. If $d(x, y) < \delta$, then the only possibility for $|f_{(z_n)}(x) - f_{(z_n)}(y)| \geq \varepsilon$ to hold is if at least one of x and y belongs to some B_n with $z_n \geq \varepsilon$, say $x \in B_n$. If $y \in B_m$ with $n \neq m$, then

$$\frac{\eta_n - d(y, b_n)}{\eta_n} z_n < 0 < f_{(z_n)}(y).$$

Hence

$$\begin{aligned} |f_{(z_n)}(x) - f_{(z_n)}(y)| &\leq \left| \frac{\eta_n - d(x, b_n) - \eta_n + d(y, b_n)}{\eta_n} z_n \right| \\ &\leq \frac{z_n}{\eta_n} d(x, y) \\ &< \frac{z_n}{\eta_n} \delta \\ &\leq \frac{z_n}{\eta_n} \varepsilon \frac{\eta_n}{z_n} = \varepsilon. \end{aligned}$$

If $y \in B_n$ or if $y \notin \bigcup_m B_m$, then the first inequality above holds trivially so we are done again. Therefore $f_{(z_n)} \in UC(X,d)$.

If $(z_n) \notin c_0$ for some $\varepsilon > 0$ there exist infinitely many n's such that $z_n > \varepsilon$. For every δ there exists one such n with $d(a_n, b_n) < \delta$. Then $f_{(z_n)}(a_n) = 0$ and $f_{(z_n)}(b_n) = z_n$ show that δ does not work for ε in the definition of uniform continuity. Therefore $f_{(z_n)} \notin UC(X,d)$. $\square$

If X is not locally compact $C(X)$ does not have a natural Polish topology. Still, if $D \subseteq X$ is countable dense it makes sense to view $UC(X,d)$ as a subset of $\mathbb{R}^D$ exactly as we did with the whole $C(X)$ in [1]. Let

$$U\tilde{C}_p(X,d;D) = \{ (y_a)_{a \in D} \mid \exists f \in UC(X,d)\ \forall a \in D\ f(a) = y_a \}.$$

Theorem 5.10. *If (X,d) is separable metric space which is not Atsuji and $D \subseteq X$ is countable dense, then $U\tilde{C}_p(X,d;D)$ is $\mathbf{\Pi}^0_3$-complete in $\mathbb{R}^D$.*

PROOF. The proof is the same of Theorem 5.9. It suffices to restrict $f_{(z_n)}$ to D. $\square$

The above theorem implies (using the analogue of lemma 2.2 of [1]) that $UC(X,d)$ with the Borel structure of the pointwise topology is a standard Borel space for every separable metric space (X,d). (A measurable space $(X,\mathcal{S})$ is standard Borel if there exists a Polish topology on X whose Borel sets are $\mathcal{S}$.) Contrast this with the results obtained in [1], where it is shown that $C(X)$ with the Borel structure of the pointwise topology can fail to be a standard Borel space even when X is a Polish space.

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