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CRITERIA FOR THE BOUNDEDNESS AND COMPACTNESS OF GENERALIZED ONE-SIDED POTENTIALS

Abstract

Necessary and sufficient conditions are found for the positive Borel measure ν, which provide the boundedness (compactness) of the generalized Riemann–Liouville operator from one Lebesgue space into another Lebesgue space with measure ν. The appropriate problem for the generalized Weyl operator is solved as well.

1 Introduction

In this paper, necessary and sufficient conditions are found, which ensure the boundedness (compactness) of the generalized Riemann-Liouville operator

$$T_\alpha f(x,t) = \int_0^x (x-y+t)^{\alpha-1} f(y)\, dy, \quad x,t \in \mathbb{R}_+,$$

from $L^p(\mathbb{R}_+)$ into $L^q_\nu(\widetilde{\mathbb{R}}^2_+)$, where $0 < p, q < \infty$, $p > 1$, $\alpha > 1/p$, $\mathbb{R}_+ \equiv [0,\infty)$ and ν is a positive σ-finite Borel measure on $\widetilde{\mathbb{R}}^2_+ \equiv \mathbb{R}_+ \times \mathbb{R}_+$ (for $q < p$ it will be assumed that ν is absolutely continuous; i.e., $d\nu(x,t) = v(x,t)\, dx\, dt$, where v is a Lebesgue-measurable almost everywhere positive function on $\widetilde{\mathbb{R}}^2_+$).

An analogous problem for the classical Riemann-Liouville operator

$$R_\alpha f(x) = \int_0^x (x-y)^{\alpha-1} f(y) dy$$

was solved in [17], [18]. Necessary and sufficient conditions for the boundedness of R_α from $L^p_w(\mathbb{R}_+)$ into $L^q_v(\mathbb{R}_+)$ were found for $1 < p < q < \infty$ and $0 < \alpha < 1$

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in [9] (see also [10], Chapter 3). A similar problem was solved for $1 < p \le q < \infty$ and $\alpha > 1$ in [15], [25] and for $1 < q < p < \infty$ and $\alpha > 1$ in [25]. For the compactness of the operator R_α when $1 < p, q < \infty$ and $\alpha > 1$ see [26].

The boundedness problem for the generalized Riesz potential

$$I_\alpha f(x,t) = \int_{\mathbb{R}^n} (|x-y|+t)^{\alpha-n} f(y)\,dy, \quad 0 < \alpha < n,$$

from $L^p(\mathbb{R}^n)$ into $L^q_\nu(\mathbb{R}^n \times \mathbb{R}_+)$ $(1 < p < q < \infty)$ was solved in [1] (Theorem C) (see [8] for more general case).

A complete description of weight pairs (v,w) ensuring the validity of weak (p,q) $(1 < p < q < \infty)$ type inequality for I_α was given in [7] (see also [10], Chapter 3). For the related Hörmander type maximal functions see [10], Chapter 4.

The different (Sawyer type) necessary and sufficient conditions for the validity of two-weight strong (p,q) type inequality for I_α and corresponding Hörmander type fractional maximal functions were established in [23].

Similar operators arise in boundary value problems in PDE, particularly in Polyharmonic Differential Equations. Some applications of operator I_α in weighted estimates for gradients were presented in [27], p. 923.

In this paper, criteria of the boundedness (compactness) from $L^p_\nu(\widetilde{\mathbb{R}}^2_+)$ into $L^q(\mathbb{R}_+)$ are also established for the operator

$$\widetilde{T}_\alpha g(y) = \int_{[y,\infty)\times\mathbb{R}_+} g(x,t)(x-y+t)^{\alpha-1}\,d\nu(x,t).$$

Finally, the upper and lower estimates of the distance of the operator T_α from a space of compact operators are derived in the non-compact case.

Some results of the present paper were announced in [20].

2 Preliminaries

Let ν be a positive σ-finite Borel measure on $\widetilde{\mathbb{R}}^2_+$. For $(0 < q < \infty)$ denote by $L^q_\nu(\widetilde{\mathbb{R}}^2_+)$ the class of all ν-measurable functions $g : \widetilde{\mathbb{R}}^2_+ \to \mathbb{R}^1$ for which

$$\|g\|_{L^q_\nu(\widetilde{\mathbb{R}}^2_+)} \equiv \Big(\int_{\widetilde{\mathbb{R}}^2_+} |g(x,t)|^q\,d\nu(x,t) \Big)^{1/q} < \infty.$$

If ν is absolutely continuous (i.e., $d\nu(x,t) = v(x,t)\,dx\,dt$), then instead of $L^q_\nu(\widetilde{\mathbb{R}}^2_+)$, we will use the notation $L^q_v(\widetilde{\mathbb{R}}^2_+)$, and if $v \equiv 1$, then $L^q_v(\widetilde{\mathbb{R}}^2_+)$ will be denoted by $L^q(\widetilde{\mathbb{R}}^2_+)$.

Let

$$Hf(x) = \int_0^x f(y)\,dy$$

for a measurable function $f : \mathbb{R}_+ \to \mathbb{R}^1$.

Necessary and sufficient conditions for the boundedness of the operator H from $L^p_w(\mathbb{R}_+)$ into $L^q_v(\mathbb{R}_+)$ were found in [3], [12] (see also [16], §1.3) for $1 < p \le q < \infty$, and in [16], §1.3, for $1 \le q < p < \infty$. (For the compactness of H see [5], [22].)

In what follows we will use the notation $U_r \equiv [r, \infty) \times \mathbb{R}_+$, where $r > 0$. It is obvious that $[r, R) \times \mathbb{R}_+ = U_r \setminus U_R$ for $0 < r < R < \infty$.

To prove our main results, we need the following lemma.

Lemma 1. *Let $1 < p \le q < \infty$ and μ be a positive Borel measure on $\widetilde{\mathbb{R}}^2_+$. Then the operator H is bounded from $L^p(\mathbb{R}_+)$ into $L^q_\mu(\widetilde{\mathbb{R}}^2_+)$ if and only if*

$$A \equiv \sup_{r>0} (\mu(U_r))^{1/q} r^{1/p'} < \infty, \ \ p' = p/(p-1).$$

Moreover, $A \le \|H\| \le 4A$.

Proof. *Sufficiency.* Let $f \ge 0$, $f \in L^p(\mathbb{R}_+)$ and $I(t) \equiv \int_0^t f$. Assume that $\int_0^\infty f \in (2^m, 2^{m+1}]$ for some $m \in \mathbb{Z}$. Then there exist x_k ($k \le m$) such that $I(x_k) = 2^k$. It is obvious that $2^k = \int_{x_k}^{x_{k+1}} f$ for $k \le m-1$. The sequence $\{x_k\}$ increases. Moreover, if $\alpha = \lim_{k \to -\infty} x_k$, then $\mathbb{R}_+ = [0, \alpha) \cup (\cup_{k \le m} [x_k, x_{k+1}))$, where $x_{k+1} = \infty$. When $\int_0^\infty f = \infty$, we have $\mathbb{R}_+ = [0, \alpha] \cup (\cup_{k \in \mathbb{Z}} [x_k, x_{k+1}))$ (i.e., $m = \infty$). If $y \in [0, \alpha]$, then $I(y) = 0$, and if $y \in [x_k, x_{k+1})$, then $I(y) \le 2^{k+1}$. We have

$$\begin{aligned}
\|Hf\|^p_{L^q_\mu(\widetilde{\mathbb{R}}^2_+)} &\le \sum_k \|\chi_{U_{x_k} \setminus U_{x_{k+1}}} Hf\|^p_{L^q_\mu(\widetilde{\mathbb{R}}^2_+)} \\
&\le \sum_k 2^{(k+1)p} \|\chi_{U_{x_k} \setminus U_{x_{k+1}}}\|^p_{L^q_\mu(\widetilde{\mathbb{R}}^2_+)} \\
&= 4^p \sum_k \Big(\int_{x_{k-1}}^{x_k} f(y) dy \Big)^p \big(\mu(U_{x_k} \setminus U_{x_{k+1}})\big)^{p/q} \\
&\le 4^p \Big(\int_{x_{k-1}}^{x_k} (f(y))^p dy \Big) (x_k - x_{k-1})^{p-1} \big(\mu(U_{x_k} \setminus U_{x_{k+1}})\big)^{p/q} \\
&\le 4^p A^p \|f\|^p_{L^p(\mathbb{R}_+)}.
\end{aligned}$$

Necessity. Let $r > 0$ and $f_r(x) = \chi_{[0,r)}(x)$. Then $\|f_r\|_{L^p(\mathbb{R}_+)} = r^{1/p}$. On the other hand,

$$\|Hf\|_{L^q_\mu(\widetilde{\mathbb{R}}^2_+)} \geq \|\chi_{U_r} Hf_r\|_{L^q_\mu(\widetilde{\mathbb{R}}^2_+)} \geq \big(\mu(U_r)\big)^{1/q} r.$$

Hence the boundedness of H implies that $A < \infty$. □

Lemma 2. *Let $0 < q < p < \infty$, $p > 1$ and let v be an almost everywhere positive measurable function on $\widetilde{\mathbb{R}}^2_+$. Then the operator H is bounded from $L^p(\mathbb{R}_+)$ into $L^q_v(\widetilde{\mathbb{R}}^2_+)$ if and only if*

$$A_1 \equiv \bigg(\int_0^\infty \Big(\int_{U_x} v(y,t)\, dy\, dt \Big)^{\frac{p}{p-q}} x^{\frac{(q-1)p}{p-q}}\, dx \bigg)^{\frac{p-q}{pq}} < \infty.$$

Moreover, $\lambda_1 A_1 \leq \|H\| \leq \lambda_2 A_1$, where $\lambda_1 = \Big(\frac{p-q}{p-1}\Big)^{1/q'} q^{1/q}$ and $\lambda_2 = (p')^{1/q'} q^{1/q}$ for $q > 1$, $\lambda_1 = \lambda_2 = 1$ for $q = 1$, $\lambda_1 = (q/p')^{\frac{p-q}{pq}} (p')^{1/p'} q^{1/p} \frac{p-q}{p}$ and $\lambda_2 = \Big(\frac{p}{p-q}\Big)^{\frac{p-q}{pq}} p^{1/p} (p')^{1/p'}$ for $0 < q < 1$.

Proof. Applying Lemma 1.3.2 from [16] for $1 \leq q < p < \infty$ and using the arguments from [24] for $0 < q < 1 < p < \infty$ we find that the condition $A_1 < \infty$ is equivalent to the boundedness of H from $L^p(\mathbb{R}_+)$ into $L^q_{\widetilde{v}}(\mathbb{R}_+)$, where

$$\widetilde{v}(y) = \int_0^\infty v(y,t)\, dt.$$

But

$$\|Hf\|_{L^q_{\widetilde{v}}(\mathbb{R}_+)} = \|Hf\|_{L^q_v(\widetilde{\mathbb{R}}^2_+)}.$$

Therefore the condition $A_1 < \infty$ is equivalent to the boundedness of H from $L^p(\mathbb{R}_+)$ into $L^q_v(\widetilde{\mathbb{R}}^2_+)$. The constants λ_1 and λ_2 are from [16] (Section 1.3.2) for $q \geq 1$, and from [24] (see Theorem 2.4 and Remark) for $0 < q < 1$. □

We need the following theorem which can be obtained from Lemma 2 in [11], Chapter XI (see also [13], Chapter 3).

Theorem A. *Let $1 < p, q < \infty$, ν be a positive σ-finite separable measure on $\widetilde{\mathbb{R}}^2_+$ (i.e., $L^q_\nu(\widetilde{\mathbb{R}}^2_+)$ is separable). If*

$$\|\ \|k(z,\cdot)\|_{L^{p'}(\mathbb{R}_+)}\|_{L^q_\nu(\widetilde{\mathbb{R}}^2_+)} < \infty, \quad k \geq 0,$$

then the operator $Kf(z) = \int_0^\infty k(z,y) f(y)\, dy$, $z \in \widetilde{\mathbb{R}}^2_+$, is compact from $L^p(\mathbb{R}_+)$ into $L^q_\nu(\widetilde{\mathbb{R}}^2_+)$.

3 Boundedness

In this section, criteria of the boundedness of the operators T_α and $\widetilde{T}_\alpha$ are established.

Theorem 1. *Let $1 < p \le q < \infty$, $\alpha > 1/p$, ν be a positive σ-finite measure on $\widetilde{\mathbb{R}}^2_+$. Then the following conditions are equivalent:*

(i) *T_α is bounded from $L^p(\mathbb{R}_+)$ into $L^q_\nu(\widetilde{\mathbb{R}}^2_+)$;*

(ii) $B \equiv \sup\limits_{r>0}\Big(\int_{U_r} (x+t)^{(\alpha-1)q} d\nu(x,t)\Big)^{\frac{1}{q}} r^{\frac{1}{p'}} < \infty$;

(iii) $B_1 \equiv \sup\limits_{k\in Z}\Big(\int_{U_{2^k}\setminus U_{2^{k+1}}} (x+t)^{(\alpha-1)q} x^{q/p'}\, d\nu(x,t)\Big)^{\frac{1}{q}} < \infty.$

Moreover, there exist positive constants b_1, b_2, b_3 and b_4 depending only on p, q and α such that

$$b_1 B \le \|T_\alpha\| \le b_2 B, \;\; b_3 B_1 \le \|T_\alpha\| \le b_4 B_1.$$

PROOF. First we will show that (ii) implies (i). Let $f \ge 0$. If $\alpha \ge 1$, then using Lemma 1 we obtain

$$\begin{aligned}\|T_\alpha f\|_{L^q_\nu} &\le 2^{\alpha-1}\bigg(\int_{\widetilde{\mathbb{R}}^2_+} (x+t)^{(\alpha-1)q}\Big(\int_0^x f(y)\,dy\Big)^q d\nu(x,t)\bigg)^{1/q}\\ &\le 2^{\alpha+1}\|f\|_{L^p(\mathbb{R}_+)}.\end{aligned}$$

Now let $1/p < \alpha < 1$. We have

$$\begin{aligned}\|T_\alpha f\|_{L^q_\nu(\widetilde{\mathbb{R}}^2_+)} \le &\bigg(\int_{\widetilde{\mathbb{R}}^2_+}\Big(\int_0^{x/2} f(y)(x-y+t)^{\alpha-1}\,dy\Big)^q\, d\nu(x,t)\bigg)^{1/q}\\ &+\bigg(\int_{\widetilde{\mathbb{R}}^2_+}\Big(\int_{x/2}^{x} f(y)(x-y+t)^{\alpha-1}\,dy\Big)^q\, d\nu(x,t)\bigg)^{1/q}\\ \equiv &S_1 + S_2.\end{aligned}$$

If $y < x/2$, then $(x-y+t)^{\alpha-1} \le 2^{1-\alpha}(x+t)^{\alpha-1}$. By Lemma 1 we obtain

$$S_1 \le 2^{1-\alpha}\bigg(\int_{\widetilde{\mathbb{R}}^2_+} (Hf(x))^q (x+t)^{(\alpha-1)q}\, d\nu(x,t)\bigg)^{1/q} \le 2^{3-\alpha} B\|f\|_{L^p(\mathbb{R}_+)}.$$

Using the Hölder's inequality, we find that

$$S_2^q \leq \int_{\widetilde{\mathbb{R}}^2_+} \Big(\int_{x/2}^{x} (f(y))^p \, dy \Big)^{q/p} (\varphi(x,t))^{q/p'} \, d\nu(x,t),$$

where

$$\varphi(x,t) \equiv \int_{x/2}^{x} (x-y+t)^{(\alpha-1)p'} \, dy.$$

Moreover, $\varphi(x,t) \leq c_1 (x+t)^{(\alpha-1)p'} x$, where $c_1 = 2^{(1-\alpha)p'-1} 3((\alpha-1)p'+1)^{-1}$. Indeed, if $t \leq x$ then

$$\varphi(x,t) \leq ((\alpha-1)p'+1)^{-1} (x/2+t)^{(\alpha-1)p'+1} \leq c_2 (x+t)^{(\alpha-1)p'} x,$$

where $c_2 = 2^{(1-\alpha)p'-1} 3((\alpha-1)p'+1)^{-1}$. Let $t > x$. Then

$$\varphi(x,t) \leq t^{(\alpha-1)p'} x/2 \leq 2^{(1-\alpha)p'-1} (x+t)^{(\alpha-1)p'} x.$$

Using the Minkowski's inequality we obtain

$$\begin{aligned} S_2^q &\leq c_1^{q/p'} \int_{\widetilde{\mathbb{R}}^2_+} \Big(\int_{x/2}^{x} (f(y))^p \, dy \Big)^{q/p} (x+t)^{(\alpha-1)q} x^{q/p'} \, d\nu(x,t) \\ &\leq c_1^{q/p'} \bigg(\int_0^\infty (f(y))^p \bigg(\int_{U_y \setminus U_{2y}} (x+t)^{(\alpha-1)q} x^{q/p'} \, d\nu(x,t) \bigg)^{p/q} dy \bigg)^{q/p} \\ &\leq 2^{q/p'} c_1^{q/p'} \bigg(\int_0^\infty (f(y))^p \bigg(\int_{U_y} (x+t)^{(\alpha-1)q} \, d\nu(x,t) \bigg)^{p/q} y^{p/p'} \, dy \bigg)^{q/p} \\ &\leq (2c_1)^{q/p'} B^q \|f\|^q_{L^p(\mathbb{R}_+)}. \end{aligned}$$

Now we will show that (i) $\Rightarrow$ (iii). Let $k \in \mathbb{Z}$ and $f_k(x) = \chi_{[0,2^{k-1})}(x)$. Then $\|f_k\|_{L^p(\mathbb{R}_+)} = 2^{(k-1)/p}$. On the other hand,

$$\|T_\alpha f_k\|_{L^q_\nu(\widetilde{\mathbb{R}}^2_+)} \geq c_3 \bigg(\int_{U_{2^k} \setminus U_{2^{k+1}}} (x+t)^{(\alpha-1)q} 2^{(k-1)q} \, d\nu(x,t) \bigg)^{1/q}.$$

Therefore $c_4 B_1 \leq \|T_\alpha\| < \infty$, where $c_4 = 3^{\alpha-1} 2^{-2/p'+1-\alpha}$ if $1/p < \alpha < 1$ and $c_4 = 2^{1-\alpha-2/p'}$ if $\alpha \geq 1$.

Analogously we can show that $c_5 B \leq \|T_\alpha\|$, where $c_5 = 3^{\alpha-1} 2^{1/p-\alpha}$ if $1/p < \alpha < 1$ and $c_5 = 2^{1/p-\alpha}$ for $\alpha \geq 1$.

Let now $r > 0$. Then $r \in [2^m, 2^{m+1})$ for some $m \in \mathbb{Z}$. Therefore

$$\begin{aligned}\Big(\int_{U_r}(x+t)^{(\alpha-1)q}\,d\nu(x,t)\Big)r^{q/p'} &\le 2^{(m+1)q/p'}\int_{U_{2^m}}(x+t)^{(\alpha-1)q}\,d\nu(x,t)\\ &= 2^{q/p'}2^{mq/p'}\sum_{k=m}^{+\infty}\int_{U_{2^k}\setminus U_{2^{k+1}}}(x+t)^{(\alpha-1)q}\,d\nu(x,t)\\ &\le 2^{q/p'}B_1^q 2^{mq/p'}\sum_{k=m}^{+\infty}2^{-kq/p'} = 2^{q/p'}(1-2^{-q/p'})^{-1}B_1^q.\end{aligned}$$

Thus (iii) implies (ii). So that finally (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii). $\square$

Remark 1. For the constants b_1, b_2, b_3 and b_4 from Theorem 1 we have: $b_1 = 3^{\alpha-1}2^{1/p-\alpha}$, $b_2 = 2^{3-\alpha} + 3^{1/p'}2^{1-\alpha}((\alpha-1)p'+1)^{-1/p'}$, $b_3 = 3^{\alpha-1}2^{-2/p'+1-\alpha}$ in the case, where $1/p < \alpha < 1$ and $b_1 = 2^{1/p-\alpha}$, $b_2 = 2^{\alpha+1}$, $b_3 = 2^{-2/p'+1-\alpha}$ if $\alpha \ge 1$. $b_4 = 2^{1/p'}(1-2^{-q/p'})^{-1/q}b_2$.

Let us now consider the case $q < p$.

Theorem 2. *Let $0 < q < p < \infty$, $p > 1$ and $\alpha > 1/p$. Assume that v is an almost everywhere positive Lebesgue-measurable function on $\widetilde{\mathbb{R}}^2_+$. Then the operator T_α is bounded from $L^p(\mathbb{R}_+)$ into $L^q_v(\widetilde{\mathbb{R}}^2_+)$ if and only if*

$$D \equiv \left(\int_0^\infty\Big(\int_{U_x}(y+t)^{(\alpha-1)q}v(y,t)\,dy\,dt\Big)^{\frac{p}{p-q}}x^{\frac{(q-1)p}{p-q}}\,dx\right)^{\frac{p-q}{pq}} < \infty.$$

Moreover, there exist positive constants d_1 and d_2 depending only on p, q and α such that

$$d_1 D \le \|T_\alpha\| \le d_2 D.$$

Proof. Let $f \ge 0$ and let $\alpha \ge 1$. Then using Lemma 2 we obtain

$$\begin{aligned}\|T_\alpha f\|_{L^q_v} &\le 2^{\alpha-1}\left(\int_{\widetilde{\mathbb{R}}^2_+}(x+t)^{(\alpha-1)q}\Big(\int_0^x f(y)\,dy\Big)^q v(x,t)\,dx\,dt\right)^{1/q}\\ &\le \lambda_2 2^{\alpha-1}D\|f\|_{L^p(\mathbb{R}_+)},\end{aligned}$$

where λ_2 is from Lemma 2. Now let $1/p < \alpha < 1$. Then as in the proof of

Theorem 1, we have

$$
\begin{aligned}
\|T_\alpha f\|_{L^q_v(\widetilde{\mathbb{R}}^2_+)} \leq & c_1\bigg(\int_{\widetilde{\mathbb{R}}^2_+}\Big(\int_0^{x/2} f(y)(x-y+t)^{\alpha-1}\,dy\Big)^q v(x,t)\,dx\,dt\bigg)^{1/q}\\
& + c_1\bigg(\int_{\widetilde{\mathbb{R}}^2_+}\Big(\int_{x/2}^{x} f(y)(x-y+t)^{\alpha-1}\,dy\Big)^q v(x,t)\,dx\,dt\bigg)^{1/q}\\
\equiv & I_1+I_2,
\end{aligned}
$$

where $c_1=1$ if $q\geq 1$ and $c_1=2^{1/q-1}$ if $0<q<1$. By virtue of Lemma 2, for I_1 we obtain

$$
\begin{aligned}
I_1 &\leq 2^{1-\alpha}c_1\bigg(\int_{\widetilde{\mathbb{R}}^2_+}(Hf(x))^q(x+t)^{(\alpha-1)q}v(x,t)\,dx\,dt\bigg)^{1/q}\\
&\leq c_1\lambda_2 2^{1-\alpha}D\|f\|_{L^p(\mathbb{R}_+)}.
\end{aligned}
$$

Applying the Hölder's inequality twice, we find

$$
\begin{aligned}
I_2^q \leq & c_2\int_{\widetilde{\mathbb{R}}^2_+}\Big(\int_{x/2}^{x}(f(y))^p\,dy\Big)^{q/p}(x+t)^{(\alpha-1)q}x^{q/p'}v(x,t)\,dx\,dt\\
\leq & c_2\sum_{k\in\mathbb{Z}}\Big(\int_{2^{k-1}}^{2^{k+1}}(f(y))^p\,dy\Big)^{q/p}\Big(\int_{U_{2^k}\setminus U_{2^{k+1}}}(x+t)^{(\alpha-1)q}x^{q/p'}v(x,t)\,dx\,dt\Big)\\
\leq & c_2\Big(\sum_{k\in\mathbb{Z}}\int_{2^{k-1}}^{2^{k+1}}(f(y))^p\,dy\Big)^{q/p}\\
& \times\bigg(\sum_{k\in\mathbb{Z}}\Big(\int_{U_{2^k}\setminus U_{2^{k+1}}}(x+t)^{(\alpha-1)q}x^{q/p'}v(x,t)\,dx\,dt\Big)^{\frac{p}{p-q}}\bigg)^{\frac{p-q}{p}}\\
\leq & 2^{q/p}c_2\|f\|^q_{L^p(\mathbb{R}_+)}\widetilde{B}_1,
\end{aligned}
$$

where $c_2=c_1^q(3\cdot 2^{(1-\alpha)p'-1}((\alpha-1)p'+1)^{-1})^{q/p'}$ and

$$
\begin{aligned}
\widetilde{B}_1 &\equiv \bigg(\sum_{k\in\mathbb{Z}}\Big(\int_{U_{2^k}\setminus U_{2^{k+1}}}(x+t)^{(\alpha-1)q}x^{q/p'}v(x,t)\,dx\,dt\Big)^{\frac{p}{p-q}}\bigg)^{\frac{p-q}{p}}\\
&\equiv \Big(\sum_{k\in\mathbb{Z}}\widetilde{B}_{1,k}\Big)^{\frac{p-q}{p}}.
\end{aligned}
$$

For $\widetilde{B}_{1,k}$ we have

$$\begin{aligned}\widetilde{B}_{1,k} &\le 2^{\frac{(k+1)q(p-1)}{p-q}}\Big(\int_{U_{2^k}\setminus U_{2^{k+1}}}(x+t)^{(\alpha-1)q}v(x,t)\,dx\,dt\Big)^{\frac{p}{p-q}}\\ &\le c_3\int_{2^{k-1}}^{2^k} y^{\frac{p(q-1)}{p-q}}\Big(\int_{U_y}(x+t)^{(\alpha-1)q}v(x,t)\,dx\,dt\Big)^{\frac{p}{p-q}}\,dy,\end{aligned}$$

where $c_3=4^{\frac{(p-1)q}{p-q}}\frac{q(p-1)}{p-q}\Big(2^{\frac{(p-1)q}{p-q}}-1\Big)^{-1}$. Therefore $\widetilde{B}_1\le(c_3)^{\frac{p-q}{p}}D^q$. Finally, we obtain $I_2\le c_4D\|f\|_{L^p(\mathbb{R}_+)}$, where $c_4=2^{1/p}(c_2)^{1/q}(c_3)^{\frac{p-q}{pq}}$.

Now let us prove the necessity. Let T_α be bounded from $L^p(\mathbb{R}_+)$ into $L^q_v(\widetilde{\mathbb{R}}^2_+)$. Then for each $x\in(0,\infty)$ we have

$$\int_{U_x}v(y,t)(y+t)^{(\alpha-1)q}\,dy\,dt<\infty.$$

Let $n\in\mathbb{Z}$ and

$$f_n(x)=\Big(\int_x^\infty\overline{v}_n(y)\,dy\Big)^{\frac{1}{p-q}}x^{\frac{q-1}{p-q}},$$

where

$$\overline{v}_n(x)=\Big(\int_0^\infty v(x,t)(x+t)^{(\alpha-1)q}\,dt\Big)\chi_{(1/n,n)}(x).$$

The boundedness of T_α implies that $f_n(x)<\infty$ for each $x\in\mathbb{R}_+$. Applying integration by parts, we obtain

$$\begin{aligned}\|f_n\|_{L^p(\mathbb{R}_+)} &=\Big(\int_0^\infty\Big(\int_x^\infty\overline{v}_n(y)\,dy\Big)^{\frac{p}{p-q}}x^{\frac{p(q-1)}{p-q}}\,dx\Big)^{1/p}\\ &=\Big(\frac{p'}{q}\int_0^\infty\Big(\int_x^\infty\overline{v}_n(y)\,dy\Big)^{\frac{q}{p-q}}\overline{v}_n(x)x^{\frac{q(p-1)}{p-q}}\,dx\Big)^{1/p}<\infty.\end{aligned}$$

On the other hand,

$$
\begin{aligned}
\|T_\alpha\|_{L^q_v(\widetilde{\mathbb{R}}^2_+)} \geq \Big(\int_{\widetilde{\mathbb{R}}^2_+} \Big(\int_0^{x/2} f_n(y)(x-y+t)^{\alpha-1}\,dy\Big)^q v(x,t)\,dx\,dt\Big)^{1/q} \\
\geq & \Big(\int_{\widetilde{\mathbb{R}}^2_+} \Big(\int_x^\infty \overline{v}_n(y)\,dy\Big)^{\frac{q}{p-q}} \Big(\int_0^{x/2} (x-y+t)^{\alpha-1} y^{\frac{q-1}{p-q}}\,dy\Big)^q v(x,t)\,dx\,dt\Big)^{1/q} \\
\geq & c_5\Big(\int_{\widetilde{\mathbb{R}}^2_+} v(x,t)\Big(\int_x^\infty \overline{v}_n(y)\,dy\Big)^{\frac{q}{p-q}} (x+t)^{(\alpha-1)q} x^{\frac{q(p-1)}{p-q}}\,dx\,dt\Big)^{1/q} \\
= & c_5\Big(\int_0^\infty \Big(\int_0^\infty v(x,t)(x+t)^{(\alpha-1)q}\,dt\Big)\Big(\int_x^\infty \overline{v}_n(y)\,dy\Big)^{\frac{q}{p-q}} x^{\frac{(p-1)q}{p-q}}\,dx\Big)^{1/q} \\
\geq & c_5\Big(\int_0^\infty \overline{v}_n(x)\Big(\int_x^\infty \overline{v}_n(y)\,dy\Big)^{\frac{q}{p-q}} x^{\frac{(p-1)q}{p-q}}\,dx\Big)^{1/q} \\
= & c_6\Big(\int_0^\infty \Big(\int_x^\infty \overline{v}_n(y)\,dy\Big)^{\frac{p}{p-q}} x^{\frac{(q-1)p}{p-q}}\,dx\Big)^{1/q},
\end{aligned}
$$

with $c_6 = (q/p')^{1/q} 2^{-\frac{p-1}{p-q}} \frac{p-q}{p-1} c_7$, where $c_7 = (\frac{3}{2})^{\alpha-1}$ if $1/p < \alpha < 1$ and $c_7 = (\frac{1}{2})^{\alpha-1}$ if $\alpha \geq 1$. Therefore

$$
c_6\Big(\int_0^\infty \Big(\int_x^\infty \overline{v}_n(y)\,dy\Big)^{\frac{p}{p-q}} x^{\frac{(q-1)p}{p-q}}\,dx\Big)^{\frac{p-q}{pq}} \leq \|T_\alpha\|.
$$

By virtue of Fatou's lemma we finally conclude that $c_6 D \leq \|T_\alpha\| < \infty$. $\square$

Remark 2. It follows from the proof of Theorem 2 that for the constants d_1 and d_2 we have: $d_1 = \Big(\frac{q}{p'}\Big)^{1/q} 2^{\frac{1-p}{p-q}} \frac{p-q}{p-1}\gamma_1(\alpha)$, where $\gamma_1(\alpha) = (3/2)^{\alpha-1}$ if $1/p < \alpha < 1$ and $\gamma_1(\alpha) = (1/2)^{\alpha-1}$ if $\alpha \geq 1$, $d_2 = \lambda_2 2^{\alpha-1}$ for $\alpha \geq 1$, and if $1/p < \alpha < 1$, then

$$
\begin{aligned}
d_2 = & \lambda_2\gamma_2(q) 2^{1-\alpha} + 2^{2/p-\alpha} 3^{1/p'}((\alpha-1)p'+1)^{-1/p'} 4^{1/p'} \\
& \times \Big(\frac{q(p-1)}{p-q}\Big)^{\frac{p-q}{pq}} \Big(2^{\frac{(p-1)q}{p-q}} - 1\Big)^{-\frac{p-q}{pq}} \gamma_2(q),
\end{aligned}
$$

where $\gamma_2(q) = 1$ for $q \geq 1$, $\gamma_2(q) = 2^{1/q-1}$ for $0 < q < 1$.

Using dual arguments, we readily obtain the following theorems:

Theorem 3. *Let $1 < p \leq q < \infty$, $\alpha > (q-1)/q$. Then the following conditions are equivalent:*

(i) $\widetilde{T}_\alpha$ *is bounded from* $L^p_\nu(\widetilde{\mathbb{R}}^2_+)$ *into* $L^q(\mathbb{R}_+)$;

(ii) $\widetilde{B} \equiv \sup\limits_{r>0}\Big(\int_{U_r} (x+t)^{(\alpha-1)p'} d\nu(x,t)\Big)^{\frac{1}{p'}} r^{\frac{1}{q}} < \infty$;

(iii) $\widetilde{B}_1 \equiv \sup\limits_{k\in Z}\Big(\int_{U_{2^k}\setminus U_{2^{k+1}}} (x+t)^{(\alpha-1)p'} x^{p'/q}\, d\nu(x,t)\Big)^{\frac{1}{p'}} < \infty.$

Moreover, there exist positive constants $\widetilde{b}_1$, $\widetilde{b}_2$, $\widetilde{b}_3$ *and* $\widetilde{b}_4$ *depending only on* p, q *and* α *such that*

$$\widetilde{b}_1\widetilde{B} \le \|\widetilde{T}_\alpha\| \le \widetilde{b}_2\widetilde{B}, \quad \widetilde{b}_3\widetilde{B}_1 \le \|\widetilde{T}_\alpha\| \le \widetilde{b}_4\widetilde{B}_1.$$

Theorem 4. *Let* $1 < q < p < \infty$ *and* $\alpha > (q-1)/q$. *Let* ν *be absolutely continuous, i.e.* $d\nu(x,y) = w(x,t)\,dx\,dt$. *Then* $\widetilde{R}_\alpha$ *is bounded from* $L^p_w(\widetilde{\mathbb{R}}^2_+)$ *into* $L^q(\mathbb{R}_+)$ *if and only if*

$$\widetilde{D} \equiv \bigg(\int_0^\infty \Big(\int_{U_x} (y+t)^{(\alpha-1)p'} w(y,t)\, dydt\Big)^{\frac{q(p-1)}{p-q}} x^{\frac{q}{p-q}}\, dx\bigg)^{\frac{p-q}{pq}} < \infty.$$

Moreover, $\widetilde{d}_1\widetilde{D} \le \|\widetilde{T}_\alpha\| \le \widetilde{d}_2\widetilde{D}$, *where the positive constants* $\widetilde{d}_1$ *and* $\widetilde{d}_2$ *depend only on* p, q *and* α.

4 Compactness

In this section, criteria for the compactness of the operators T_α and $\widetilde{T}_\alpha$ are established. First we will prove

Lemma 3. *Let* $1 < p \le q < \infty$, $\alpha > 1/p$ *and let* ν *be separable measure. If*

(i) $B < \infty$;

(ii) $\lim\limits_{a\to 0} B^{(a)} = \lim\limits_{b\to +\infty} B^{(b)} = 0$, *where*

$$B^{(a)} \equiv \sup_{0<r<a}\Big(\int_{U_r\setminus U_a} (x+t)^{(\alpha-1)q}\, d\nu(x,t)\Big)^{1/q} r^{1/p'},$$
$$B^{(b)} \equiv \sup_{r>b}\Big(\int_{U_r} (x+t)^{(\alpha-1)q}\, d\nu(x,t)\Big)^{1/q} r^{1/p'},$$

then T_α *is compact from* $L^p(\mathbb{R}_+)$ *into* $L^q_\nu(\widetilde{\mathbb{R}}^2_+)$.

Proof. Let us represent T_α as

$$\begin{aligned}T_\alpha f =&\chi_{V_a} T_\alpha(\chi_{[0,a)} f) + \chi_{V_b\setminus V_a} T_\alpha(\chi_{(0,b)} f) + \chi_{U_b} T_\alpha(\chi_{(0,b/2]} f)\\ &+ \chi_{U_b} T_\alpha(\chi_{(b/2,\infty)} f) \equiv P_1 f + P_2 f + P_3 f + P_4 f,\end{aligned}$$

where $V_r \equiv [0,r) \times \mathbb{R}_+$. (It is obvious that $[a,b) \times \mathbb{R}_+ = V_b \setminus V_a$.)

For P_2 we have

$$P_2 f(x,t) = \int_0^\infty \overline{k}(x,t,y) f(y)\, dy,$$

where $\overline{k}(x,t,y) = \chi_{V_b\setminus V_a}(x,t)\chi_{(0,x)}(y)(x-y+t)^{\alpha-1}$. Moreover, using the inequality

$$\int_0^x (x-y+t)^{(\alpha-1)p'}\, dy \le b(x+t)^{(\alpha-1)p'} x,$$

where the constant $b > 0$ is independent of x and t, we get

$$\begin{aligned}\|\|\overline{k}(x,t,y)\|_{L^{p'}(\mathbb{R}_+)}\|_{L^q_\nu(\widetilde{\mathbb{R}}^2_+)} =&\Big(\int_{V_b\setminus V_a}\Big(\int_0^x (x-y+t)^{(\alpha-1)p'}\, dy\Big)^{q/p'} d\nu(x,t)\Big)^{1/q}\\ \le& c_1\Big(\int_{V_b\setminus V_a} (x+t)^{(\alpha-1)q} x^{q/p'}\, d\nu(x,t)\Big)^{1/q} < \infty.\end{aligned}$$

For P_3 we obtain $P_3 f(x,t) = \int_0^\infty \widetilde{k}(x,t,y) f(y)\, dy$, where

$$\widetilde{k}(x,t,y) = \chi_{U_b}(x,t)\chi_{(0,b/2]}(y)(x-y+t)^{\alpha-1}.$$

It can be easily verified that $\|\|\widetilde{k}(x,t,y)\|_{L^{p'}(\mathbb{R}_+)}\|_{L^q_\nu(\widetilde{\mathbb{R}}^2_+)} < \infty$. Using Theorem A we conclude that P_2 and P_3 are compact operators.

By Theorem 1 we have

$$\|P_1\| \le b_2 B^{(a)} < \infty \quad \textit{and} \quad \|P_4\| \le b_2 B^{(b/2)} < \infty, \tag{1}$$

where b_2 is from Theorem 1. Hence we obtain

$$\|T_\alpha - P_2 - P_3\| \le \|P_1\| + \|P_4\| \to 0 \tag{2}$$

as $a \to 0$ and $b \to +\infty$. Therefore T_α is compact as a limit of the sequence of compact operators. □

Theorem 5. *Let p, q, α and ν satisfy the conditions of Lemma* 3. *Then the following conditions are equivalent:*

(i) T_α *is compact from* $L^p(\mathbb{R}_+)$ *to* $L^q_\nu(\widetilde{\mathbb{R}}^2_+)$;

(ii) $B<\infty$ *and* $\lim\limits_{a\to 0} B^{(a)} = \lim\limits_{b\to+\infty} B^{(b)} = 0$;

(iii) $B<\infty$ *and* $\lim\limits_{r\to 0} B(r) = \lim\limits_{r\to+\infty} B(r) = 0$, *where*

$$B(r) \equiv \Big(\int_{U_r} (x+t)^{(\alpha-1)q} d\nu(x,t) \Big)^{\frac{1}{q}} r^{\frac{1}{p'}};$$

(iv) $B_1<\infty$ *and* $\lim\limits_{k\to-\infty} B_1(k) = \lim\limits_{k\to+\infty} B_1(k) = 0$, *where*

$$B_1(k) \equiv \Big(\int_{U_{2^k}\setminus U_{2^{k+1}}} (x+t)^{(\alpha-1)q} x^{q/p'}\, d\nu(x,t) \Big)^{\frac{1}{q}}.$$

PROOF. By Lemma 3 we have (ii) $\Rightarrow$ (i). Now let us show that (iii) $\Rightarrow$ (ii). Since

$$B^{(a)} \le \sup_{0<r<a} B(r) \text{ and } B^{(b)} = \sup_{r>b} B(r),$$

we obtain $B^{(a)} \to 0$ as $a\to 0$ and $B^{(b)} \to +\infty$ as $b\to\infty$. Therefore (iii) $\Rightarrow$ (ii). Let now T_α be compact from $L^p(\mathbb{R}_+)$ into $L^q_\nu(\widetilde{\mathbb{R}}^2_+)$. Let $r>0$ and $f_r(x) = \chi_{(0,r/2)}(x) r^{-1/p}$. Now it can be easily verified that f_r weakly converges to 0 if $r\to 0$. On the other hand, $\|T_\alpha f_r\|_{L^q_\nu(\widetilde{\mathbb{R}}^2_+)} \ge c_1 B(r) \to 0$ as $r\to 0$, since $T_\alpha f_r$ strongly converges to 0. Now, if we take

$$g_r(x,t) = \chi_{U_r}(x,t)(x+t)^{(\alpha-1)(q-1)} \Big(\int_{U_r} (y+t)^{(\alpha-1)q}\, d\nu(y,t) \Big)^{-1/q'},$$

then we readily find that g_r weakly converges to 0 as $r\to+\infty$. Since $\widetilde{T}_\alpha$ is compact from $L^{q'}_\nu(\widetilde{\mathbb{R}}^2_+)$ into $L^{p'}(\mathbb{R}_+)$ and $\|\widetilde{T}_\alpha g_r\|_{L^{p'}(\mathbb{R}_+)} \ge c_2 B(r)$, we obtain $\lim\limits_{r\to+\infty} B(r) = 0$. Therefore (i) $\Rightarrow$ (iii).

Now we will prove that (ii) follows from (iv). Using Theorem 1, we establish the fact that $B \le b_1 B_1$. Let $a>0$. Then $a\in[2^m, 2^{m+1})$ for some $m\in\mathbb{Z}$. Therefore $B^{(a)} \le \sup\limits_{0<r<2^m} B_{2^m,r} \equiv B^{(2^m)}$, where

$$B_{2^m,r} \equiv \Big(\int_{U_r\setminus U_{2^m}} (x+t)^{(\alpha-1)q} d\nu(x,t) \Big)^{\frac{1}{q}} r^{\frac{1}{p'}}.$$

If $r\in[0,2^m)$, then $r\in[2^{j-1},2^j)$ for some $j\in\mathbb{Z}$, $j\le m$. Furthermore,

$$B^q_{2^m,r} \le 2^{\frac{jq}{p'}} \sum_{k=j}^{m} \int_{U_{2^{k-1}}\setminus U_{2^k}} (x+t)^{(\alpha-1)q} d\nu(x,t) \le c_3 \Big(\sup_{k\le m} B_1(k-1) \Big)^q.$$

Hence we have $B^{(2^m)} \le c_4 B_1^{(m)}$, where $B_1^{(m)} \equiv \sup\limits_{k\le m} B_1(k-1)$. If $a\to 0$, then $m\to -\infty$ and $B_1^{(m)} \to 0$. Therefore $\lim\limits_{a\to 0} B^{(a)} = 0$.

Let now $\tau > 0$. Then $\tau \in [2^m, 2^{m+1})$ and we have

$$B^q(\tau) \le c_5 B^q(2^m) = c_5 2^{\frac{mq}{p'}} \sum_{k=m}^{+\infty} \int_{U_{2^k}\setminus U_{2^{k+1}}} (x+t)^{(\alpha-1)q} d\nu(x,t)$$
$$\le c_6 (\sup_{k\ge m} B_1(k))^q.$$

Hence it readily follows that $\lim\limits_{\tau\to+\infty} B(\tau) \le c_7 \lim\limits_{m\to+\infty} \sup\limits_{k\ge m} B_1(k) = 0$ and $\lim\limits_{b\to+\infty} B^{(b)} = 0$. Thus (iv) $\Rightarrow$ (ii). Let now T_α is compact from $L^p(\mathbb{R}_+)$ into $L^q_\nu(\widetilde{\mathbb{R}}^2_+)$, $k\in\mathbb{Z}$ and $f_k(x) = \chi_{[2^{k-2},2^{k-1})}(x) 2^{-k/p}$. Then the sequence f_k weakly converges to 0 as $k\to-\infty$ or $k\to+\infty$. Moreover, it is easy to show that $\|T_\alpha f_k\|_{L^q_\nu(\widetilde{\mathbb{R}}^2_+)} \ge c_8 B_1(k)$. Therefore (iv) is valid. Finally, we obtain (i) $\Leftrightarrow$ (iii), (iv) $\Rightarrow$ (ii) $\Rightarrow$ (i) $\Rightarrow$ (iv). $\square$

Our next theorem is proved in a similar manner. It is also a corollary of the well-known Ando's theorem (see, e.g., [2] and [14], §5).

Theorem 6. *Let p, q, α and v satisfy the condition of Theorem* 2. *Then T_α is compact from $L^p(\mathbb{R}_+)$ into $L^q_v(\widetilde{\mathbb{R}}^2_+)$ if and only if $D<\infty$.*

By dual arguments we obtain the following theorems.

Theorem 7. *Let $1<p\le q<\infty$, $\alpha > \frac{q-1}{q}$. It is assumed that ν is a positive σ-finite measure such that the space $L^p_\nu(\widetilde{\mathbb{R}}^2_+)$ is separable. Then the following conditions are equivalent:*

(i) *$\widetilde{T}_\alpha$ is compact from $L^p_\nu(\widetilde{\mathbb{R}}^2_+)$ into $L^q(\mathbb{R}_+)$;*

(ii) *$\widetilde{B}<\infty$ and $\lim\limits_{a\to 0} \widetilde{B}^{(a)} = \lim\limits_{b\to+\infty} \widetilde{B}^{(b)} = 0$, where*

$$\widetilde{B}^{(a)} \equiv \sup_{0<r<a} \Big(\int_{U_r\setminus U_a} (x+t)^{(\alpha-1)p'}\, d\nu(x,t) \Big)^{1/p'} r^{1/q},$$
$$\widetilde{B}^{(b)} \equiv \sup_{r>b} \widetilde{B}(r) \equiv \sup_{r>b} \Big(\int_{U_r} (x+t)^{(\alpha-1)p'}\, d\nu(x,t) \Big)^{1/p'} r^{1/q};$$

(iii) *$\widetilde{B}<\infty$ and $\lim\limits_{r\to 0} \widetilde{B}(r) = \lim\limits_{r\to+\infty} \widetilde{B}(r) = 0$;*

(iv) $\widetilde{B}_1 < \infty$ *and* $\lim\limits_{k\to-\infty} \widetilde{B}_1(k) = \lim\limits_{k\to+\infty} \widetilde{B}_1(k) = 0$, *where*

$$\widetilde{B}_1(k) \equiv \Big(\int_{U_{2^k} \setminus U_{2^{k+1}}} (x+t)^{(\alpha-1)p'} x^{p'/q} \, d\nu(x,t) \Big)^{\frac{1}{q}}.$$

Theorem 8. *Let $1 < q < p < \infty$ and $\alpha > \frac{q-1}{q}$. Suppose that $d\nu(x,t) = w(x,t)\,dx\,dt$, where w is a measurable a.e. positive function on $\widetilde{\mathbb{R}}^2_+$. Then $\widetilde{T}_\alpha$ is compact from $L^p_w(\widetilde{\mathbb{R}}^2_+)$ into $L^q(\mathbb{R}_+)$ if and only if $\widetilde{D} < \infty$.*

5 Measure of Non-Compactness

In this section, the distance of the operator T_α from a space of compact operators is estimated.

Let X and Y be Banach spaces. Denote by $\mathbb{B}(X,Y)$ a space of bounded operators from X into Y. Let $\mathbb{K}(X,Y)$ be a class of all compact operators from X into Y, $\mathbb{F}_r(X,Y)$ be a space of operators of finite rank.

It is assumed that v is a Lebesgue-measurable almost everywhere positive function on $\widetilde{\mathbb{R}}^2_+$.

We need the following lemmas.

Lemma 4. [[4], Chapter V, Corollary 5.4]. *Let $1 \le q < \infty$ and $P \in \mathbb{B}(X,Y)$, where $Y = L^q(\widetilde{\mathbb{R}}^2_+)$. Then*

$$\operatorname{dist}(P, \mathbb{K}(X,Y)) = \operatorname{dist}(P, \mathbb{F}_r(X,Y)).$$

Our next lemma is proved like Lemma V.5.6 in [4] (see also [21], Lemma 2.2).

Lemma 5. *Let $1 \le q < \infty$ and $Y = L^q(\widetilde{\mathbb{R}}^2_+)$. It is assumed that $P \in \mathbb{F}_r(X,Y)$ and $\epsilon > 0$. Then there exist $T \in \mathbb{F}_r(X,Y)$ and $[\alpha,\beta] \subset (0,\infty)$ such that $\|P - T\| < \epsilon$ and supp$Tf \subset [\alpha,\beta] \times \mathbb{R}_+$ for any $f \in X$.*

Let $T'_\alpha (0 < \alpha < 1)$ be an operator of the form $T'_\alpha f(x,t) = v^{1/q}(x,t) T_\alpha f(x,t)$. We denote

$$\widetilde{I} \equiv \operatorname{dist}(T_\alpha, \mathbb{K}(X, L^q_v(\widetilde{\mathbb{R}}^2_+)), \text{ and } \overline{I} \equiv \operatorname{dist}(T'_\alpha, \mathbb{K}(X, L^q(\widetilde{\mathbb{R}}^2_+)).$$

Lemma 6. *Let $1 \le q < \infty$. Then $\widetilde{I} = \overline{I}$.*

PROOF. Let $E \equiv \{f : \|f\|_X \le 1\}$ and $P \in \mathbb{K}(X, L^q_v(\widetilde{\mathbb{R}}^2_+))$. Then

$$\begin{aligned}\|T_\alpha - P\| &= \sup_E \|(T_\alpha - P)f\|_{L^q_v(\widetilde{\mathbb{R}}^2_+)} \\ &= \sup_E \|T'_\alpha f - v^{1/q}Pf\|_{L^q(\widetilde{\mathbb{R}}^2_+)} = \|T'_\alpha - \overline{P}\|,\end{aligned}$$

where $\overline{P} = v^{1/q}P$. But $\overline{P} \in \mathbb{K}(X, L^q(\mathbb{R}^2_+))$. Therefore $\overline{I} \le \widetilde{I}$. Similarly, we obtain $\widetilde{I} \le \overline{I}$. □

Theorem 9. *Let $1 < p \le q < \infty$, $\alpha > 1/p$ and let $X = L^p(\mathbb{R}_+)$, $Y = L^q_v(\widetilde{\mathbb{R}}^2_+)$. Assume that $B < \infty$ for $d\nu(x,t) = v(x,t)\,dx\,dt$. Then there exist positive constants ϵ_1 and ϵ_2 depending only on p, q and α such that*

$$\epsilon_1 J \le dist(T_\alpha, \mathbb{K}(X,Y)) \le \epsilon_2 J,$$

where $J = \lim_{a\to 0} J^{(a)} + \lim_{d\to+\infty} J^{(d)}$,

$$J^{(a)} \equiv \sup_{0<r<a} \Big(\int_{U_r\setminus U_a} v(x,t)(x+t)^{(\alpha-1)q}\,dx\,dt\Big)^{1/q} r^{1/p'},$$

$$J^{(d)} \equiv \sup_{r>d} \Big(\int_{U_r} v(x,t)(x+t)^{(\alpha-1)q}\,dx\,dt\Big)^{1/q} r^{1/p'}.$$

Proof. By the inequalities (1) and (2) from the proof of Lemma 3, we obtain $\widetilde{I} \equiv \mathrm{dist}(T_\alpha, \mathbb{K}(X,Y)) \le b_2 J$, where b_2 is from Theorem 1. Let $\lambda > \widetilde{I}$. By Lemma 6 we have $\widetilde{I} = \overline{I}$. Using Lemma 4, we find that there exists an operator of finite rank $P : X \to L^q(\widetilde{\mathbb{R}}^2_+)$ such that $\|T'_\alpha - P\| < \lambda$. From Lemma 5 it follows that for $\epsilon = (\lambda - \|T'_\alpha - P\|)/2$ there are $T \in \mathbb{F}_r(X, L^q(\widetilde{\mathbb{R}}^2_+))$ and $[\alpha,\beta] \subset (0,\infty)$ such that $\|P - T\| < \epsilon$ and $\mathrm{supp}\, Tf \subset [\alpha,\beta] \times \mathbb{R}_+$. Therefore for all $f \in X$ we have $\|T'_\alpha f - Tf\|_{L^q(\widetilde{\mathbb{R}}^2_+)} \le \lambda \|f\|_X$. Moreover,

$$\int_{[0,\alpha]\times\mathbb{R}_+} |T'_\alpha f(x,t)|^q\,dx\,dt + \int_{[\beta,\infty)\times\mathbb{R}_+} |T'_\alpha f(x,t)|^q\,dx\,dt \le \lambda^q \|f\|^q_{L^p(\mathbb{R}_+)}. \quad (3)$$

Let now $d > \beta$ and $r \in (d,\infty)$. Assume that $f_r(y) = \chi_{0,r/2)}(y)$. Then $\|f_r\|^q_{L^p(\mathbb{R}_+)} = 2^{-q/p} r^{q/p}$. On the other hand,

$$\begin{aligned}\int_{U_r} |T'_\alpha f_r(x,t)|^q j\,dt &\ge \int_{U_r} \Big(\int_0^{r/2} (x-y+t)^{\alpha-1}\,dy\Big)^q v(x,t)\,dx\,dt \\ &\ge c_1 \Big(\int_{U_r} v(x,t)(x+t)^{(\alpha-1)q}\,dx\,dt\Big) r^q,\end{aligned}$$

where $c_1 = 3^{(\alpha-1)q}2^{-\alpha q}$ if $1/p < \alpha < 1$ and $c_1 = 2^{-\alpha q}$ for $\alpha \geq 1$. Therefore

$$\lambda \geq c_1^{1/q}2^{1/p}\Big(\int_{U_r} v(x,t)(x+t)^{(\alpha-1)q}\,dx\,dt\Big)^{1/q} r^{1/p'}.$$

for all $r > d$. Hence we have $c_2 J^{(d)} \leq \lambda$ for any $d > \beta$ and, finally, we obtain $c_2 \lim\limits_{d\to+\infty} J^{(d)} \leq \lambda$. Since λ is arbitrarily close to $\widetilde{I}$, we conclude that $c_2 \lim\limits_{d\to+\infty} J^{(d)} \leq \widetilde{I}$, where $c_2 = c_1^{1/q}2^{1/p}$.

Let us choose $n \in \mathbb{Z}$ such that $2^n < \alpha$. Assume that $j \in \mathbb{Z}$, $j \leq n-1$ and $f_j(y) = \chi_{(0,2^{j-1})}(y)$. Then we obtain

$$\begin{aligned}\int_{U_{2^j}\setminus U_{2^{j+1}}} |T'_\alpha f(x,t)|^q\,dx\,dt &\geq \int_{U_{2^j}\setminus U_{2^{j+1}}} v(x,y)\bigg(\int_0^{2^{j-1}}(x-y+t)^{\alpha-1}\,dy\bigg)^q, dx\,dt\\ &\geq c_3 \int_{U_{2^j}\setminus U_{2^{j+1}}} v(x,y)(x+t)^{(\alpha-1)q}2^{(j-1)q}\,dx\,dt,\end{aligned}$$

where $c_3 = (3/2)^{(\alpha-1)q}$ in the case, where $1/p < \alpha < 1$ and $c_3 = (1/2)^{(\alpha-1)q}$ for $\alpha \geq 1$. On the other hand, $\|f_j\|_X^q = 2^{(j-1)q/p}$. By (3) we find that

$$c_3^{1/q}4^{-1/p'}\overline{B}_1(j) \leq \lambda$$

for every integer j, $j \leq n-1$, where

$$\overline{B}(j) \equiv \bigg(\int_{U_{2^j}\setminus U_{2^{j+1}}} v(x,t)(x+t)^{(\alpha-1)q}x^{q/p'}\,dx\,dt\bigg)^{1/q}.$$

Consequently $c_3^{1/q}4^{-1/p'}\sup_{j\leq n}\overline{B}_1(j) \leq \lambda$ for every integer n with the condition $2^n < \alpha$. Let $a < 2^n < \alpha$. Then $a \in [2^m, 2^{m+1})$ for some m, $m \leq n-1$. As in the proof of Theorem 5 we have that

$$B^{(a)} \leq B^{(2^m)} \leq 2^{1/p'}(1-2^{-q/p'})^{-1/q}\sup_{j\leq m}\overline{B}_1(j),$$

where

$$B^{(2^m)} \equiv \sup_{0<r<2^m}\bigg(\int_{U_r\setminus U_{2^m}} v(x,t)(x+t)^{(\alpha-1)q}\,dx\,dt\bigg)^{1/q} r^{1/p'}.$$

Therefore $c_4 \lim\limits_{a\to 0} B^{(a)} \leq \lambda$ with $c_4 = 2^{-3/p'}c_3^{1/q}(1-2^{-q/p'})^{1/q}$. Finally we obtain $c_5 J \leq \widetilde{I}$, where $c_5 = 1/2\min\{c_2, c_4\}$. $\square$

An analogous theorem for the classical Riemann-Liouville operator R_α is proved for $\alpha > 1/p$ in [19]. Estimates of the distance of R_α from the class of compact operators in the case of two weights for $\alpha > 1$ are obtained in [6], [21] (for the case $\alpha = 1$ see [5]).

Remark 3. For the constants ϵ_1 and ϵ_2 from Theorem 9 we have: $\epsilon_2 = b_2$, $\epsilon_1 = 1/2 \min\{\beta_1, \beta_2\}$, where $\beta_1 = 2^{1/p}\gamma_3$, $\beta_2 = 2^{-3/p'}(1 - 2^{-q/p'})^{1/q}\gamma_4$ with $\gamma_3 = 3^{\alpha-1}2^{-\alpha}$ for $1/p < \alpha < 1$, $\gamma_3 = 2^{-\alpha}$ for $\alpha \geq 1$ and $\gamma_4 = (3/2)^{\alpha-1}$ for $1/p < \alpha < 1$, $\gamma_4 = (1/2)^{\alpha-1}$ if $\alpha \geq 1$.

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