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ON DINGHAS-TYPE DERIVATIVES AND CONVEX FUNCTIONS OF HIGHER ORDER†

Abstract

In this paper higher-order convexity properties of real functions are characterized in terms of a Dinghas-type derivative. The main tool used is a mean value inequality for Dinghas-type derivatives.

1 Introduction

A real-valued function $f : I \to \mathbb{R}$ defined on an interval $I \subseteq \mathbb{R}$ is called *Jensen-convex* (c.f. [14]) if it satisfies the functional inequality

$$f\Big(\frac{x+y}{2}\Big) \le \frac{f(x)+f(y)}{2} \text{ for } x, y \in I. \tag{1}$$

Obviously, any convex function is Jensen-convex; however there are nonconvex but Jensen-convex functions. (For a Hamel basis construction of nonconvex but Jensen-convex functions, we refer to [8] and [9, Chapter V].) It is easy to see that $f : I \to \mathbb{R}$ is Jensen-convex if and only if

$$\Delta_h^2 f(x) \ge 0 \text{ for } x \in I,\ h \ge 0 \ \text{ such that } \ x + 2h \in I,$$

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where the *difference operator* Δ_h^n is defined by the following recursion:

$$\Delta_h^1 f(x) = f(x+h) - f(x) \text{ for } x \in I,\ h \in \mathbb{R} \text{ such that } x+h \in I$$
$$\Delta_h^{n+1} f(x) = \Delta_h^1 \Delta_h^n f(x) \qquad \text{for } x \in I,\ h \in \mathbb{R} \text{ such that } x+(n+1)h \in I.$$

The notion of higher-order Jensen-convexity is due to T. Popoviciu (see [12], [13]): A function $f : I \to \mathbb{R}$ is called *Jensen-convex of order* $(n-1)$ (where n is a positive integer), if

$$\Delta_h^n f(x) \geq 0 \text{ for } x \in I,\ h \geq 0 \text{ such that } x+nh \in I. \tag{2}$$

For properties of functions satisfying the above inequality, see e.g. [13], [2], [1], [9, Chapter XV], [14, VIII.83], and the references therein. Generalizations of Jensen-convexity of order $(n-1)$ to higher-dimensional domains were investigated by R. Ger [6], [7].

Clearly, first-order Jensen-convexity is equivalent to Jensen-convexity. The substitution $y = x + nh$ in (2) and a simple calculation yields that f satisfies (2) if and only if

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f\left(\frac{(n-k)x+ky}{n}\right) \geq 0 \text{ for } x, y \in I,\ x \leq y. \tag{3}$$

In the particular case $n = 2$, (3) reduces to (1).

Multiplying the left hand side of (3) by a suitable normalizing factor and taking the $\liminf$ as x and y tend to a fixed point $\xi \in I$ from the left and right, respectively, we can define the so-called n^{th}*-order lower Dinghas interval derivative of* f *at* ξ by

$$\underline{D}^n f(\xi) = \liminf_{\substack{(x,y)\to(\xi,\xi)\\ x\leq\xi\leq y}} \left(\frac{n}{y-x}\right)^n \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f\left(\frac{(n-k)x+ky}{n}\right). \tag{4}$$

If the limit exists, then we speak about Dinghas' interval derivative. This notion was introduced by A. Dinghas [4] as a generalization of the classical derivative. One can obtain that in the n-times differentiable setting, it coincides with the n^{th} derivative of f at ξ. Concerning connections among this interval derivative, other generalized derivatives and the derivative in the classical sense, we refer to the dissertations G. Friedel [5] and P. Volkmann [16].

By putting $y = x + nh$, the derivative $\underline{D}^n f(\xi)$ can be expressed in the following way

$$\underline{D}^n f(\xi) = \liminf_{\substack{(x,h)\to(\xi,0)\\ x\leq\xi\leq x+nh}} \frac{\Delta_h^n f(x)}{h^n}.$$

It is well-known that a function $f : I \to \mathbb{R}$ which is continuous on I and n-times differentiable in the interior of I is Jensen-convex of order $(n-1)$ on I if and only if its n^{th} derivative is nonnegative in the interior of I. The analogous problem formulated by C. E. Weil during the 16^{th} Summer School on Real Functions Theory (Liptovský Ján, Slovakia, 2000), is that whether $(n-1)^{\text{st}}$-order Jensen-convexity can be characterized by the nonnegativity of the corresponding lower Dinghas interval derivative. The necessity of $\underline{D}^n f \geq 0$ is obvious. The proof of the sufficiency will be based on a Goursat-type method due to A. Dinghas [4] and also used by A. Simon and P. Volkmann [15] in the characterization of polynomial functions with the Dinghas derivative.

In this paper, we introduce a more general convexity notion called T-convexity. The main results of the paper show that this general convexity can be characterized in terms of the corresponding lower Dinghas-type interval derivative. As a consequence, we obtain a local characterization of higher-order Jensen-convexity, t-Wright-convexity, etc. Finally, we formulate two open problems concerning t-Jensen-convexity.

2 T-Convex Functions

Let $T = (t_1, \dots, t_n)$ where $t_1, \dots, t_n$ are fixed positive numbers. If $f : I \to \mathbb{R}$, then define the operator $\mathbf{\Delta}_h^T$ by

$$\mathbf{\Delta}_h^T f(x) := \Delta_{t_1 h} \cdots \Delta_{t_n h} f(x) \text{ for } x \in I, h \in \mathbb{R} \text{ such that } x+(t_1+\cdots+t_n)h \in I.$$

We say that $f : I \to \mathbb{R}$ *is T-(Wright-)convex* if $\mathbf{\Delta}_h^T f(x) \geq 0$ for $x \in I,\ h \geq 0$ such that $x + (t_1 + \cdots + t_n)h \in I$. Clearly, T-convexity and cT-convexity are equivalent for $c > 0$. In the case $t_1 = \cdots = t_n = 1$ the notion of T-convexity is obviously the same as Jensen-convexity of order $(n-1)$. Another interesting particular case is the $(t, 1-t)$-convexity, where $0 < t < 1$ is fixed. By definition, f is $(t, 1-t)$-convex if

$$\begin{gathered} f(x+th) + f(x+(1-t)h) \leq f(x) + f(x+h) \\ \text{for } x \in I,\ h \geq 0 \text{ such that } x+h \in I. \end{gathered}$$

which is equivalent to

$$f((1-t)x+ty) + f(tx+(1-t)y) \leq f(x) + f(y) \text{ for } x, y \in I.$$

Functions satisfying the above inequality are called *t-Wright-convex* (see [17] for the origin of this notion). Thus T-convexity can be considered as a generalization of t-Wright-convexity to the higher-order setting. For the connection between t-Wright-convexity and Jensen-convexity, Gy. Maksa, K. Nikodem, and Zs. Páles obtained results in [11].

The *lower T-Dinghas interval derivative of* $f : I \to \mathbb{R}$ is defined by

$$\underline{\mathbf{D}}^T f(\xi) := \liminf_{\substack{(x,h)\to(\xi,0)\\ x\le\xi\le x+(t_1+\cdots+t_n)h}} \frac{\mathbf{\Delta}_h^T f(x)}{(t_1h)\cdots(t_nh)} \text{ for } \xi \in I. \tag{5}$$

In the n-times differentiable setting, one can see that $\underline{\mathbf{D}}^T f(\xi) = f^{(n)}(\xi)$; that is, $\underline{\mathbf{D}}^T$ can be considered as a generalized derivative.

The operator Δ_h^n admits the following well-known decomposition in terms of the operator $\Delta_{h/2}^n$ of half step size.

$$\Delta_h^n f(x) = \sum_{k=0}^{n} \binom{n}{k} \Delta_{h/2}^n f(x + (k/2)h).$$

A similar decomposition is valid for $\mathbf{\Delta}_h^T$ by the following result.

Lemma. *Let $T = (t_1, \dots, t_n)$ where $t_1, \dots, t_n > 0$. Then there exist positive integers $c_0, c_1, \dots, c_m$ with $c_0 + c_1 + \cdots + c_m = 2^n$ and*

$$0 = s_0 < s_1 < \cdots < s_m = \frac{t_1 + \cdots + t_n}{2} \tag{6}$$

such that, for all functions $f : I \to \mathbb{R}$,

$$\begin{gathered} \mathbf{\Delta}_h^T f(x) = \sum_{i=0}^{m} c_i \mathbf{\Delta}_{h/2}^T f(x + s_i h) \\ \text{for } x \in I,\ h \ge 0 \text{ with } x + (t_1 + \cdots + t_n)h \in I. \end{gathered} \tag{7}$$

Proof. Introduce the *translation operator* τ_h for functions $f : I \to \mathbb{R}$ by

$$\tau_h f(x) := f(x + h) \text{ for } x \in I,\ h \in \mathbb{R} \text{ such that } h + x \in I.$$

Then, obviously, $\Delta_h = \tau_h - \tau_0 = (\tau_{h/2} - \tau_0)(\tau_{h/2} + \tau_0) = \Delta_{h/2}(\tau_{h/2} + \tau_0)$. Therefore,

$$\begin{aligned} \mathbf{\Delta}_h^T &= \Delta_{t_1h} \cdots \Delta_{t_nh} \\ &= \big[\Delta_{t_1h/2}(\tau_{t_1h/2} + \tau_0)\big] \cdots \big[\Delta_{t_nh/2}(\tau_{t_nh/2} + \tau_0)\big] \\ &= \Delta_{t_1h/2} \cdots \Delta_{t_nh/2}\big[(\tau_{t_1h/2} + \tau_0) \cdots (\tau_{t_nh/2} + \tau_0)\big] \\ &= \mathbf{\Delta}_{h/2}^T\big[(\tau_{t_1h/2} + \tau_0) \cdots (\tau_{t_nh/2} + \tau_0)\big]. \end{aligned}$$

Now, it is easy to see that there exist positive integers $c_0, c_1, \dots, c_m$ with $c_0 + c_1 + \dots + c_m = 2^n$ and $s_0, s_1, \dots, s_m$ such that (6) and

$$(\tau_{t_1 h/2} + \tau_0) \cdots (\tau_{t_n h/2} + \tau_0) = \sum_{i=0}^{m} c_i \tau_{s_i h}$$

hold. Thus $\mathbf{\Delta}_h^T = \mathbf{\Delta}_{h/2}^T \Big[\sum_{i=0}^m c_i \tau_{s_i h}\Big] = \sum_{i=0}^m c_i \mathbf{\Delta}_{h/2}^T \tau_{s_i h}$, which yields (7) immediately. □

3 Main Results

Our first main result offers a mean value theorem for the operator $\mathbf{\Delta}_h^T$ in terms of the corresponding Dinghas-type derivative.

Theorem. (Mean Value Inequality) Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$, $T = (t_1, \dots, t_n)$ where $t_1, \dots, t_n > 0$, and let $x \in I$, $h > 0$ with $x + (t_1 + \dots + t_n)h \in I$. Then there exists a point $\xi \in [x, x + (t_1 + \dots + t_n)h]$ such that

$$\mathbf{\Delta}_h^T f(x) \geq (t_1 h) \cdots (t_n h) \underline{\mathbf{D}}^T f(\xi). \tag{8}$$

PROOF. Let $x \in I$, and $h \geq 0$ such that $x + (t_1 + \dots + t_n)h \in I$. Let $A := \mathbf{\Delta}_h^T f(x)$, $x_0 := x$ and $y_0 := x + (t_1 + \dots + t_n)h$. Using induction, we are going to construct sequences (x_k) and (y_k) such that, for all $k \geq 0$,

$$x_k \leq x_{k+1}, \quad y_{k+1} \leq y_k, \tag{9}$$

$$y_k - x_k = \frac{t_1 + \dots + t_n}{2^k} h, \tag{10}$$

and

$$\mathbf{\Delta}_{h/2^k}^T f(x_k) \leq \frac{A}{2^{kn}}. \tag{11}$$

Clearly, x_0 and y_0 satisfy (10) and (11). Assume that we have constructed $x_0 \leq x_1 \leq \dots \leq x_k$ and $y_0 \geq y_1 \geq \dots \geq y_k$ such that (10) and (11) hold.

Applying the Lemma of the previous section and (11), we have the existence of positive integers $c_0, c_1, \dots, c_m$ with $c_0 + c_1 + \dots + c_m = 2^n$ and $s_0, s_1, \dots, s_m$ satisfying (6) such that (7) is valid. Then

$$\sum_{i=0}^{m} c_i \mathbf{\Delta}_{h/2^{k+1}}^T f\Big(x_k + s_i \frac{h}{2^k}\Big) = \mathbf{\Delta}_{h/2^k}^T f(x_k) \leq \frac{A}{2^{kn}}.$$

The sum of the coefficients on the left hand side being 2^n, there exists an integer $0 \le j \le m$ such that

$$2^n \mathbf{\Delta}^T_{h/2^{k+1}} f\Big(x_k + s_j \frac{h}{2^k}\Big) \le \frac{A}{2^{kn}}. \tag{12}$$

Writing $x_{k+1} := x_k + s_j \dfrac{h}{2^k}$ and $y_{k+1} := x_k + s_j \dfrac{h}{2^k} + \dfrac{t_1 + \cdots + t_n}{2^{k+1}} h$, we can see that (12) reduces to (11) with $k+1$ instead of k, (10) for $k+1$ follows from the above definition of x_{k+1} and y_{k+1}. The inequality $x_k \le x_{k+1}$ is obvious by $s_j \ge 0$. On the other hand, (7) and (10) yield that

$$\begin{aligned} y_{k+1} &\le x_k + s_m \frac{h}{2^k} + \frac{t_1 + \cdots + t_n}{2^{k+1}} h \\ &= x_k + \frac{t_1 + \cdots + t_n}{2} \cdot \frac{h}{2^k} + \frac{t_1 + \cdots + t_n}{2^{k+1}} h \\ &= x_k + \frac{t_1 + \cdots + t_n}{2^k} h = y_k. \end{aligned}$$

Therefore, we also have $y_{k+1} \le y_k$ and we have proved the existence of the sequences (x_k) and (y_k) satisfying (9), (10), and (11).

Denote by ξ the (unique) element of the intersection $\bigcap_{k=0}^\infty [x_k, y_k]$ and let $h_k := \dfrac{y_k - x_k}{t_1 + \cdots + t_n} = \frac{h}{2^k}$. Then $x_k \le \xi \le y_k = x_k + (t_1 + \cdots + t_n) h_k$ and (11) can be rewritten as $\frac{\mathbf{\Delta}^T_{h_k} f(x_k)}{h_k^n} \le \frac{A}{h^n}$. Therefore, we have

$$\begin{aligned} \underline{\mathbf{D}}^T f(\xi) &= \liminf_{\substack{(x,h)\to(\xi,0) \\ x \le \xi \le x + (t_1 + \cdots + t_n) h}} \frac{\mathbf{\Delta}^T_h f(x)}{(t_1 h) \cdots (t_n h)} \\ &\le \liminf_{k\to\infty} \frac{\mathbf{\Delta}^T_{h_k} f(x_k)}{(t_1 h_k) \cdots (t_n h_k)} \le \frac{A}{(t_1 h) \cdots (t_n h)}. \end{aligned}$$

Thus the proof of (8) is complete. □

If one replaces f by $-f$, then a mean value inequality for the upper Dinghas-type derivative can be deduced which is defined via (5) with "lim sup" instead of "lim inf".

If the theorem is applied to the special case $t_1 = \cdots = t_n = 1$, then we get a mean value theorem for the Δ^n_h operator in terms of the lower Dinghas interval derivative $\underline{D}^n$ defined in (4).

As an immediate consequence of the above theorem, we get the following characterization of T-convexity.

Corollary 1. *Let $T = (t_1, \dots, t_n)$ with $t_1, \dots, t_n > 0$. A function $f : I \to \mathbb{R}$ is T-convex on I if and only if $\underline{\mathbf{D}}^T f(\xi) \geq 0$ for $\xi \in I$.*

Proof. If f is T-convex, then, clearly $\underline{\mathbf{D}}^T f \geq 0$. Conversely, if $\underline{\mathbf{D}}^T f$ is nonnegative on I, then, by our Theorem, $\mathbf{\Delta}_h^T f(x) \geq 0$ for all $x \in I$ and $h \geq 0$ with $x + (t_1 + \cdots + t_n)h \in I$. □

In the special case $t_1 = \cdots = t_n = 1$, the above corollary yields that f is Jensen-convex of order $(n-1)$ if and only if the lower Dinghas interval derivative $\underline{D}^n f$ is nonnegative on I. Thus the problem of C. E. Weil is answered in the affirmative. A similar result can be derived for t-Wright-convexity when we apply our Theorem to the $(t, 1-t)$-convexity setting.

Another obvious but interesting consequence of Corollary 1 is that the T-convexity property is *localizable* in the following sense.

Corollary 2. *A function $f : I \to \mathbb{R}$ is T-convex on I if and only if, for each point $\xi \in I$, there exists a neighborhood U of ξ such that f is T-convex on $I \cap U$.*

Thus, Jensen-convexity of order $(n-1)$, and also t-Wright-convexity are localizable properties of functions. There are convexity properties, however, that may not have this localization property. A function $f : I \to \mathbb{R}$ is called t-Jensen-convex on I (where $0 < t < 1$ is fixed), if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \text{ for } x, y \in I.$$

Clearly, t-Jensen-convexity implies t-Wright-convexity, but the converse is not true in general (see [11]). We can formulate two open problems concerning t-Jensen-convexity.

Problem 1. Let $0 < t < 1$ be fixed. Is t-Jensen-convexity equivalent to the property

$$\underline{\delta}_t^2 f(\xi) = \liminf_{\substack{(x,y)\to(\xi,\xi) \\ x\leq\xi\leq y}} \frac{tf(x) + (1-t)f(y) - f(tx + (1-t)y)}{(y-x)^2} \geq 0 \text{ for } \xi \in I$$

for all functions $f : I \to \mathbb{R}$?

Of course, for $t = 1/2$, the answer is affirmative, because the $(1/2)$-Jensen and the $(1/2)$-Wright convexities are equivalent.

Problem 2. Let $0 < t < 1$ be fixed. Is the t-Jensen-convexity property localizable?

If the first problem has a positive answer, then the second problem can also be answered positively, but the converse may not be true. If t is rational, then the (local) t-Jensen-convexity is equivalent to the (local) Jensen-convexity by the results of N. Kuhn [10] and Z. Daróczy and Zs. Páles [3]. Thus, for rational t, the t-Jensen-convexity property is localizable. However, for irrational t, Jensen-convexity does not imply t-Jensen-convexity. Therefore, in this case, Problem 2 cannot be solved or disproved in such an easy way.

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