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ON MEASURABILITY PROPERTIES CONNECTED WITH THE SUPERPOSITION OPERATOR

Abstract

We consider the question of measurability of functions obtained by using the superposition operator which is induced by a given function of two variables. Some related measurability properties of functions of two variables are also discussed.

Let $\Phi : \mathbb{R}^2 \to \mathbb{R}$ be a function of two variables and let F be a class of functions acting from $\mathbb{R}$ into $\mathbb{R}$. For any $f \in F$, we denote by Φ_f the function acting from $\mathbb{R}$ into $\mathbb{R}$ and defined by

$$\Phi_f(x) = \Phi(x, f(x)) \quad (x \in \mathbb{R}).$$

In some sense, Φ plays the role of a superposition operator whose domain coincides with the given class F of functions. A general problem arising in this context is to describe those conditions on Φ under which various nice properties of functions from F are preserved by Φ. For example, suppose that $F = F_L$ is the class of all real-valued Lebesgue measurable functions on $\mathbb{R}$. Then it is natural to try to characterize those Φ which preserve F_L (i.e., preserve the Lebesgue measurability). It is well known that, in general, the Lebesgue measurability of Φ (regarded as a function of two variables) does not guarantee the Lebesgue measurability of Φ_f for $f \in F_L$. Also, it is widely known that if Φ satisfies the so-called Carathéodory conditions, then it preserves the class F_L.

Example 1. For any Lebesgue measurable function $f : \mathbb{R} \to \mathbb{R}$, there exist a function $\Phi : \mathbb{R}^2 \to \mathbb{R}$ satisfying the Carathéodory conditions and a continuous function $g : \mathbb{R} \to \mathbb{R}$, such that $f(x) = g'(x) = \Phi(x, g(x))$ for almost all $x \in \mathbb{R}$.

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Indeed, according to the classical Luzin's theorem (see [1]), there exists a continuous function g such that $g'(x) = f(x)$ for almost all $x \in \mathbb{R}$. Let us define

$$\Phi(x, y) = f(x) + y - g(x) \quad (x \in \mathbb{R},\ y \in \mathbb{R}).$$

Then Φ is measurable with respect to x and linear with respect to y. (Hence, Φ satisfies the Lipschitz condition with respect to the same y.) Obviously, we also have $f(x) = g'(x) = \Phi(x, g(x))$. We thus conclude that any real-valued Lebesgue measurable function can be simultaneously regarded as the derivative (almost everywhere) of a continuous function and as the image of the same continuous function, under an appropriate superposition operator satisfying the Carathéodory conditions.

Several works were devoted to constructions of a non-Lebesgue measurable function Φ which, however, preserves the class F_L (see, e.g., [2], [3], [4], [5]). All those constructions were based on some additional set-theoretical axioms. In this connection, a problem was posed whether it is possible to construct analogous Φ within the **ZFC** theory (cf. [4]). Recently, Roslanowski and Shelah announced that the existence of such a function Φ cannot be established in **ZFC**. (See [6]. A similar question for the Baire property was considered by Ciesielski and Shelah in [7].) In the present paper some related topics will be discussed concerning measurability and sup-measurability properties of functions.

Let E be a set, $\mathcal{S}$ be a σ-algebra of subsets of E and let $\mathcal{I}$ be a σ-ideal of subsets of E, such that $\mathcal{I} \subset \mathcal{S}$. We say that $(E, \mathcal{S}, \mathcal{I})$ is a measurable space with a family of negligible sets (cf. [8]). It will be assumed throughout the paper that $\mathcal{I}$ contains all one-element subsets of E and that the pair $(\mathcal{S}, \mathcal{I})$ satisfies the countable chain condition; i.e., every disjoint family of sets from $\mathcal{S} \setminus \mathcal{I}$ is at most countable.

Let F be a class of real-valued $\mathcal{S}$-measurable functions on E. Suppose also that a function $\Phi : E \times \mathbb{R} \to \mathbb{R}$ is given. In the sequel, this Φ will be treated as a superposition operator (i.e., by using Φ, from any function $f \in F$ we obtain the function Φ_f).

We shall say that Φ is sup-measurable with respect to F if, for each $f \in F$, the corresponding function Φ_f is $\mathcal{S}$-measurable.

Starting with the class F, it is reasonable to consider other classes F' of $\mathcal{S}$-measurable real-valued functions, containing F and such that each operator Φ sup-measurable with respect to F remains also sup-measurable with respect to F'. In this case, we shall say that F' extends F with preserving the sup-measurability property. It is also reasonable to try to characterize maximal extensions of F which preserve this property.

It will be demonstrated that, in some natural situations, it is possible to describe such maximal extensions in terms of $(\mathcal{S}, \mathcal{I})$ and F (cf. Theorem 1 below).

Fix a class F of $\mathcal{S}$-measurable real-valued functions. We shall say that $f \in F^*$ if there exist a countable disjoint covering $\{E_n : n \in \omega\} \subset \mathcal{S}$ of E and a countable family $\{f_n : n \in \omega\} \subset F$, such that $E_0 \in \mathcal{I}$ and $f|E_n = f_n|E_n$ for all natural numbers $n > 0$.

Clearly, we have the inclusion $F \subset F^*$. In some cases, this inclusion is reduced to equality. For instance, if F is the family of all $\mathcal{S}$-measurable functions, then $F^* = F$.

Example 2. Let F denote the class of all constant real-valued functions on E. Then it is easy to see that the class F^* coincides with those real-valued functions on E which are $\mathcal{I}$-equivalent to step-functions. (We recall that a step-function on E is any real-valued $\mathcal{S}$-measurable function whose range is at most countable.)

In the sequel, we need a simple auxiliary statement.

Lemma 1. *Let $h : E \to \mathbb{R}$ be an $\mathcal{S}$-measurable function and let F be some family of $\mathcal{S}$-measurable real-valued functions. Then h does not belong to F^* if and only if there exists a set $A \in \mathcal{S} \setminus \mathcal{I}$ possessing the following property: for any subset B of A belonging to $\mathcal{S} \setminus \mathcal{I}$ and for any function $f \in F$, the relation $f|B \neq h|B$ is fulfilled.*

The proof can easily be obtained by the method of transfinite induction, taking into account the countable chain condition for the pair $(\mathcal{S}, \mathcal{I})$.

The next lemma is almost trivial.

Lemma 2. *Let Φ be a sup-measurable operator with respect to F. Then Φ is also sup-measurable with respect to F^*.*

Notice that, for the validity of Lemma 2, the countable chain condition is not necessary.

Let F be a family of $\mathcal{S}$-measurable real-valued functions. We shall say that a family G of real-valued functions is fundamental for F if every function f from F is $\mathcal{I}$-equivalent to some function g from G.

It can easily be shown that the next statement is valid.

Lemma 3. *An operator Φ is sup-measurable with respect to a class F if and only if it is sup-measurable with respect to some class G fundamental for F.*

Example 3. The class of all Borel functions (acting from $\mathbb{R}$ into $\mathbb{R}$) is fundamental for the class of all Lebesgue measurable functions (acting from $\mathbb{R}$ into $\mathbb{R}$). The same class of Borel functions is also fundamental for the class of all those functions which act from $\mathbb{R}$ into $\mathbb{R}$ and possess the Baire property.

We recall that a family $\mathcal{B} \subset \mathcal{S} \setminus \mathcal{I}$ is a pseudo-base for the σ-algebra $\mathcal{S}$ if every set $X \in \mathcal{S} \setminus \mathcal{I}$ contains at least one member of $\mathcal{B}$.

Since the pair $(\mathcal{S}, \mathcal{I})$ satisfies the countable chain condition, every set $X \in \mathcal{S} \setminus \mathcal{I}$ contains a subset Y such that $X \setminus Y \in \mathcal{I}$ and Y is representable as the union of a countable family of members of a pseudo-base $\mathcal{B}$. This circumstance implies the validity of the next auxiliary proposition.

Lemma 4. *Let $\mathcal{B}$ be a pseudo-base for a space $(E, \mathcal{S}, \mathcal{I})$ with* $\mathrm{card}(\mathcal{B}) \geq 2$. *Then there exists a family G of $\mathcal{S}$-measurable real-valued functions, fundamental for the family of all $\mathcal{S}$-measurable real-valued functions and satisfying the relation* $\mathrm{card}(G) \leq (\mathrm{card}(\mathcal{B}))^{\omega}$.

Lemma 5. *Suppose that the following conditions are satisfied for a space $(E, \mathcal{S}, \mathcal{I})$:*

1) there exists a pseudo-base $\mathcal{B}$ containing at least two members and such that $(\mathrm{card}(\mathcal{B}))^{\omega} \leq \mathrm{card}(E)$;

2) for any set $X \in \mathcal{S} \setminus \mathcal{I}$ and for any family $\{X_\theta : \theta \in \Theta\} \subset \mathcal{I}$ with $\mathrm{card}(\Theta) < \mathrm{card}(E)$, *we have* $X \setminus \cup\{X_\theta : \theta \in \Theta\} \neq \emptyset$;

3) each subset of E with cardinality strictly less than card(E) belongs to $\mathcal{I}$;

Let F be a family of $\mathcal{S}$-measurable real-valued functions and let h be any $\mathcal{S}$-measurable real-valued function not belonging to F^*. Then there exists a superposition operator

$$\Phi : E \times \mathbb{R} \to \mathbb{R}$$

such that Φ is sup-measurable with respect to F, but is not sup-measurable with respect to the one-element class $\{h\}$.

PROOF. Lemma 4 and condition 1) readily imply that there exists a family G of $\mathcal{S}$-measurable real-valued functions, fundamental for F and such that $\mathrm{card}(G) \leq \mathrm{card}(E)$. We may also assume, without loss of generality, that every function from G is $\mathcal{I}$-equivalent to some function from F.

Let α denote the least ordinal number whose cardinality is equal to $\mathrm{card}(E)$ and let $\{g_\xi : \xi < \alpha\}$ be an enumeration of all functions from G. Taking into account the relation $h \notin F^*$ and applying Lemma 1, we can find a set $A \in \mathcal{S} \setminus \mathcal{I}$ such that h differs from any $f \in F$ on each $\mathcal{S}$-measurable subset of

A not belonging to the ideal $\mathcal{I}$. Obviously, the same is true for all functions from G; i.e., for any $g \in G$ the function h differs from g on each $\mathcal{S}$-measurable subset of A not belonging to $\mathcal{I}$. We may assume in the sequel (without loss of generality) that $A = E$.

Let $\{B_\xi : \xi < \alpha\}$ be an enumeration of all members from the pseudo-base $\mathcal{B}$. Applying the method of transfinite recursion, let us construct two injective disjoint α-sequences $\{x_\xi : \xi < \alpha\}$ and $\{x'_\xi : \xi < \alpha\}$ of points of the space E. Suppose that, for an ordinal $\xi < \alpha$, the partial families $\{x_\zeta : \zeta < \xi\}$ and $\{x'_\zeta : \zeta < \xi\}$ have already been constructed. For any ordinal $\zeta < \xi$, let

$$A_\zeta = \{z \in E : g_\zeta(z) = h(z)\}.$$

Then it is clear that $A_\zeta \in \mathcal{I}$. Consider the set

$$P_\xi = B_\xi \setminus ((\cup\{A_\zeta : \zeta < \xi\}) \cup \{x_\zeta : \zeta < \xi\} \cup \{x'_\zeta : \zeta < \xi\}).$$

In view of the conditions 2) and 3), we have $\mathrm{card}(P_\xi) = \mathrm{card}(E)$. Therefore, we can choose two distinct points x and x' from P_ξ. Finally, we put $x_\xi = x$ and $x'_\xi = x'$.

Proceeding in this manner, we will be able to construct the required α-sequences of points. Now, we define a function $\Phi : E \times \mathbb{R} \to \mathbb{R}$ as follows:

$$\begin{aligned} \Phi(x_\xi, h(x_\xi)) &= 1 \text{ for each ordinal } \xi < \alpha, \\ \Phi(x'_\xi, h(x'_\xi)) &= -1 \text{ for each ordinal } \xi < \alpha, \\ \Phi(x, t) &= 0 \text{ for all other pairs } (x, t) \in E \times \mathbb{R}. \end{aligned}$$

Let us show that Φ is sup-measurable with respect to F and, at the same time, the function Φ_h is not $\mathcal{S}$-measurable. Indeed, take any $f \in F$ and find a function $g \in G$ which is $\mathcal{I}$-equivalent to f. Clearly, g coincides with some g_η where $\eta < \alpha$. Further, introduce the set

$$Z = \{z \in E : \Phi(z, g(z)) \neq 0\}.$$

For each $z \in Z$, the relation

$$\Phi(z, g(z)) = 1 \quad \vee \quad \Phi(z, g(z)) = -1$$

must be valid. This implies that either $z = x_\xi$ and $g(z) = h(z)$, or $z = x'_\xi$ and $g(z) = h(z)$. It follows directly from our construction that, in both cases above, $\xi \leq \eta$. Consequently, the cardinality of Z must be strictly less than $\mathrm{card}(E)$. Hence, in view of condition 3), the function Φ_g must be $\mathcal{S}$-measurable, and the same is true for Φ_f.

On the other hand, the definition of Φ also yields that the function Φ_h cannot be $\mathcal{S}$-measurable. Indeed, for any set $B \in \mathcal{B}$, we have from our construction that $\{-1, 1\} \subset \mathrm{ran}(\Phi_h|B)$. Remembering that $\mathcal{B}$ is a pseudo-base for $(E, \mathcal{S}, \mathcal{I})$, we obtain that the sets $\Phi_h^{-1}(-1)$ and $\Phi_h^{-1}(1)$ are $\mathcal{S}$-thick in E. This fact immediately implies that both these sets are not $\mathcal{S}$-measurable and hence Φ_h is also not $\mathcal{S}$-measurable. $\square$

Taking into account the preceding lemmas, we are able to formulate the following statement.

Theorem 1. *Let a space $(E, \mathcal{S}, \mathcal{I})$ be given, let F be a class of $\mathcal{S}$-measurable real-valued functions and let the assumptions of Lemma 5 be fulfilled. Then the class F^* is the largest extension of F which preserves the sup-measurability property.*

Lemma 5 and hence Theorem 1 were proved under assumptions of a somewhat set-theoretical flavor. These assumptions are known to be consistent for canonical measurable spaces with negligible sets, studied in real analysis (cf. Examples 5 and 6 below). However, we do not know whether the same assumptions are essential for the validity of the result.

Now, let us give several examples illustrating the theorem obtained above. We begin with the following very simple example.

Example 4. Let $E = \mathbb{R}$, let $\mathcal{S}$ be the σ-algebra of all Lebesgue measurable sets in $\mathbb{R}$ and let $\mathcal{I}$ be the σ-ideal of all Lebesgue measure zero subsets of $\mathbb{R}$. Denote by F the class of all real-valued constant functions on $\mathbb{R}$. Obviously, there are many real-valued functions h on $\mathbb{R}$ not belonging to F^*. (For instance, any strictly monotone function can be taken as h.) Thus, we obtain that there exists a superposition operator $\Phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ which is Lebesgue measurable with respect to the first variable but produces non-Lebesgue measurable functions of type Φ_h. Actually, this result needs no additional set-theoretical assumptions.

The next example is less trivial.

Example 5. Again, let $E = \mathbb{R}$, let $\mathcal{S}$ be the σ-algebra of all Lebesgue measurable sets in $\mathbb{R}$ and let $\mathcal{I}$ be the σ-ideal of all Lebesgue measure zero subsets of $\mathbb{R}$. We denote by F the family of all real-valued continuous functions on $\mathbb{R}$ differentiable almost everywhere (with respect to the Lebesgue measure). Let h be a real-valued continuous function on $\mathbb{R}$ such that it is nowhere approximately differentiable. Then $h \notin F^*$. Therefore, under the corresponding set-theoretical assumptions on $(E, \mathcal{S}, \mathcal{I})$, there exists a superposition operator Φ sup-measurable with respect to F, for which the function Φ_h is not Lebesgue measurable (cf. [9]).

Example 6. Let $E = \mathbb{R}$, let $\mathcal{S}$ be the σ-algebra of all those sets in $\mathbb{R}$ which possess the Baire property and let $\mathcal{I}$ be the σ-ideal of all first category subsets of $\mathbb{R}$. We denote by F the family of all real-valued continuous functions f on $\mathbb{R}$ having the property that each nonempty open subinterval of $\mathbb{R}$ contains at least one point at which f is differentiable. Take any real-valued continuous function h on $\mathbb{R}$ which is nowhere differentiable. Then it is not hard to demonstrate that $h \notin F^*$. Therefore, under the corresponding set-theoretical assumptions on $(E, \mathcal{S}, \mathcal{I})$, there exists a superposition operator Φ sup-measurable with respect to F, for which the function Φ_h does not possess the Baire property (cf. [9]).

Example 7. It is not difficult to show that the existence of a Sierpiński subset of the Euclidean plane $\mathbb{R}^2$ implies the existence of a superposition operator $\Phi : \mathbb{R}^2 \to \mathbb{R}$ which is sup-measurable (with respect to the class F_L) but is not Lebesgue measurable as a function of two variables. Indeed, it suffices to take as Φ the characteristic function of a Sierpiński set on the plane. (We shall say that such a Φ determines a Sierpiński superposition operator.) Moreover, it can be observed that the same Φ yields Lebesgue measurable functions Φ_f for all those functions $f : \mathbb{R} \to \mathbb{R}$ whose graphs are sets of Lebesgue measure zero (in $\mathbb{R}^2$). Obviously, there are many non-Lebesgue measurable functions among those f.

An analogous situation holds in terms of category and the Baire property. In this case, the existence of a Luzin subset of the plane is needed for constructing an appropriate example.

In connection with Example 7, the next statement is of some interest (cf. Theorem 1 above).

Theorem 2. *Let $h : \mathbb{R} \to \mathbb{R}$ be a function whose graph has positive outer Lebesgue measure. Then, under the Continuum Hypothesis* (**CH**)*, there exists a Sierpiński superposition operator Φ such that Φ_h is not Lebesgue measurable.*

Proof. Let $\lambda = \lambda_1$ denote the standard one-dimensional Lebesgue measure on $\mathbb{R}$, let $\lambda_2 = \lambda_1 \times \lambda_1$ denote the usual two-dimensional Lebesgue measure on $\mathbb{R}^2$ and let $\Gamma \subset \mathbb{R}^2$ be the graph of h. In view of the assumption of the theorem, Γ is not contained in a set of λ_2-measure zero.

Let $\{X_\xi : \xi < \omega_1\}$ be the family of all Borel sets (in $\mathbb{R}$) of λ-measure zero and let $\{B_\xi : \xi < \omega_1\}$ be the family of all Borel sets (in $\mathbb{R}^2$) of λ_2-measure zero. We shall construct, by applying the method of transfinite recursion, an injective family $\{x_\xi : \xi < \omega_1\}$ of points in $\mathbb{R}$. Suppose that, for an ordinal $\xi < \omega_1$, the partial family of points $\{x_\zeta : \zeta < \xi\}$ has already been defined. Consider the set

$$T_\xi = (\cup\{B_\zeta : \zeta < \xi\}) \cup (\cup\{X_\zeta \times \mathbb{R} : \zeta < \xi\}) \cup (\cup\{\{x_\zeta\} \times \mathbb{R} : \zeta < \xi\}).$$

Clearly, we have $\lambda_2(T_\xi) = 0$. Hence, $\Gamma \setminus T_\xi \neq \emptyset$ and there exists a point $(x, y) \in \Gamma \setminus T_\xi$. We put $x_\xi = x$. Proceeding in this way, we will be able to construct the required family of points $\{x_\xi : \xi < \omega_1\}$. It follows immediately from our construction that:

(i) the set $\{x_\xi : \xi < \omega_1\}$ is a Sierpiński subset of the real line $\mathbb{R}$;

(ii) the set $\{(x_\xi, h(x_\xi)) : \xi < \omega_1\}$ is a Sierpiński subset of the plane $\mathbb{R}^2$.

Let Φ denote the characteristic function of the latter set. Then Φ is a Sierpiński superposition operator. At the same time, considering the function Φ_h, we easily observe that $\Phi_h^{-1}(1) = \{x_\xi : \xi < \omega_1\}$. Thus, Φ_h is not Lebesgue measurable since no Sierpiński subset of $\mathbb{R}$ is λ-measurable (see, e.g., [10]). $\square$

Actually, the argument presented above yields (under **CH**) a more general result. Namely, for any set $\Gamma \subset \mathbb{R}^2$ of positive outer Lebesgue measure, there exists a partial function h acting from $\mathbb{R}$ into $\mathbb{R}$ and having the following properties:

(1) the graph of h is contained in Γ;

(2) the graph of h is a Sierpiński subset of $\mathbb{R}^2$;

(3) the domain of h is a Sierpiński subset of $\mathbb{R}$.

An analogous result is valid (under **CH**) in terms of category, Baire property and Luzin sets.

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