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A GENERALIZED ARCHIMEDEAN PROPERTY

Abstract

We introduce and discuss a condition generalizing one of the Archimedean properties characterizing parabolas.

Archimedes was familiar with the following property of parabolas:

> If for any two points A, B on a parabola we denote by S the area of the region between the parabola and the secant AB, and by T the maximum of the area of the triangle ABC, where C is a point on the parabola between A and B, then
>
> $$S = \frac{4}{3}T. \tag{A}$$

Throughout this paper f denotes a continuous real valued function defined on $\mathbb{R}$. We denote by $G(f)$ the graph of f. For $I = [a, b]$ an arbitrary closed nontrivial interval, let $L_I(f) = L_{ab}(f)$ be the linear function that interpolates f at the endpoints of I,

$$L_I(f)(x) = L_{ab}(f)(x) = \frac{(b-x)f(a) + (x-a)f(b)}{b-a},$$

let $f_I = f_{ab} = f - L_I(f)$, and let $G_I(f) = G_{ab}(f)$ denote the graph of f_I over I. (A) can be reformulated as follows:

For every interval $I = [a, b]$,

$$|S_{ab}(f)| = \frac{4}{3}T_{ab}(f), \tag{A'}$$

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where $S_{ab}(f) = S_{ab} = \int_a^b f_{ab}(x)\,dx$ *and* $T_{ab}(f) = T_{ab} = \frac{1}{2}(b-a)\max_{x\in[a,b]}|f_{ab}(x)|$. Motivated by this equality, we introduce the following more general property:

For every interval $I = [a,b]$ *and a fixed real* p, $|S_{ab}(f)| = pT_{ab}(f)$. (A_p)

Observe that (A_p) is of interest only if $p > 0$ and that (A') is exactly $(A_{4/3})$.

The following theorem includes as a special case the converse of Archimedes' result:

Theorem. *Let* $p > 0$ *and let* f *satisfy* (A_p).
(i) If $p < 1$, *then* $G(f)$ *is a line.*
(ii) If $p = 1$, *then* $G(f)$ *is either a line or a pair of half lines.*
(iii) If $p > 1$, *then either* $G(f)$ *is a line, or* $p = 4/3$ *and* $G(f)$ *is a parabola.*

The fact that $(A_{4/3})$ implies that $G(f)$ is a parabola was stated in [1] under excessive regularity hypotheses on f. Moreover, as first noticed by J.L. Garcin, the proof given there contains a flaw. We wish to emphasize that the proof of the theorem as stated above does not rely upon arguments (or a modification thereof) used in [1].

The proof of parts i) and ii) of our theorem depends on the following:

Lemma 1. *If* f *satisfies* (A_p) *for some* $p > 0$, *then for every interval* $I = [a,b]$, *either* $f_I(x) = 0$ *for all* $x \in I$ *or* $f_I(x) \neq 0$ *for all* $x \in I^\circ = (a,b)$. *In particular,* f *is either convex or concave.*

PROOF. Suppose that $f_{ab} \not\equiv 0$ and $f_{ab}(x_0) = 0$ for some $x_0 \in (a,b)$. Then

$$pT_{ab} = |S_{ab}| = |S_{ax_0} + S_{x_0b}| \leq |S_{ax_0}| + |S_{x_0b}| = p(T_{ax_0} + T_{x_0b}) \leq pT_{ab}.$$

It follows that S_{ax_0} and S_{x_0b} are of the same sign and that

$$\max_{x\in(a,x_0)}|f_{ax_0}(x)| = \max_{x\in(x_0,b)}|f_{x_0b}(x)| = \max_{x\in(a,b)}|f_{ab}(x)| > 0.$$

More generally, assuming that f_{ab} has finitely many zeros at $a, x_1, x_2, \ldots, x_k, b$, where $a < x_1 < x_2 < \cdots < x_k < b, k \geq 1$, a similar argument shows that $S_{x_jx_{j+1}}$ are of the same sign and that

$$\max_{x\in(x_j,x_{j+1})}|f_{x_jx_{j+1}}(x)| = \max_{x\in(a,b)}|f_{ab}(x)| > 0.$$

In particular, there may be only finitely many zeros of f_{ab} in (a,b) and f_{ab} is of constant sign, say $f_{ab} \geq 0$ on $[a,b]$. Let $m = \max_{x\in(a,b)} f_{ab}(x) > 0$ and $\{x \in (a,b) : f_{ab}(x) = 0\} = \{x_1, x_2, \ldots, x_k\} \neq \emptyset$, where $x_1 < x_2 < \cdots < x_k$. For an $\epsilon > 0$ sufficiently small ($\epsilon < m$), the line $y = \epsilon$ intersects the graph of f_{ab} at points $y_0, y_1, \ldots, y_{k+1}$, such that $a < y_0 < x_1 < y_1 < x_2 < \cdots < y_{k+1} < b$ and $f_{y_0 y_{k+1}}$ changes sign (it assumes the values $-\epsilon$ and $m-\epsilon$), which is impossible.

To see that f must be either convex or concave, note that if this was not the case, we could find $I = [a, b]$ such that f_{ab} is not identically zero and changes sign in (a, b) - a contradiction. $\square$

Proof of Theorem (i). From Lemma 1, f is either convex or concave. Hence, $T_{ab} \leq |S_{ab}| = pT_{ab}$, implying that $T_{ab} = 0$ for all intervals $I = [a, b]$ and that $G(f)$ is a line. $\square$

Proof of Theorem (ii). We wish to show that $G(f)$ is either a line or a pair of half lines. If f satisfies (A_1) and $S_{ab} \neq 0$ for all $I = [a, b]$, then by continuity, S_{ab} is of constant sign, and f is either strictly convex or strictly concave, in which case $|S_{ab}| > T_{ab}$ - a contradiction. Thus, the graph of f must contain a segment (of a line). We can then consider a maximal segment $\mathcal{L}$ contained in $G(f)$. Suppose that $\mathcal{L}$ lies over the interval (a, b), where $-\infty \leq a < b \leq \infty$. Suppose that $\mathcal{L}$ (and thus $G(f)$) is not a line. Then either $a > -\infty$ or $b < \infty$. For the sake of the argument, suppose that $b < \infty$ and let $a' \in (a, b)$. For every $c > b, f_{a'c} \not\equiv 0$, or else, the segment $\mathcal{L}$ could be extended to one containing the points $(b, f(b)), (c, f(c))$, contradicting the maximality of $\mathcal{L}$. We now show that for every $c > b$, $G_{bc}(f)$ is a line segment; i.e., $f_{bc} \equiv 0$ on (b, c). Otherwise, we would have either $f_{bc} < 0$ on (b, c), or $f_{bc} > 0$ on (b, c). In each of the two cases one can easily check that $|S_{a'c}| > T_{a'c}$, in contradiction with the fact that f satisfies (A_1). It readily follows that $G_{b\infty}(f)$ is a half line. When $a > -\infty$ we proceed analogously to show that $G_{-\infty a}$ is a half line. $\square$

The proof of part iii) of the theorem depends on the following five lemmas.

Lemma 2. *If f satisfies (A_p) for some $p > 1$, and for some interval $I, f_I \equiv 0$ in I, i.e., $G(f)$ contains a line segment, then $G(f)$ is a line. In particular, if $G(f)$ is not a line, then f is strictly convex or strictly concave.*

Proof. Suppose, by way of contradiction, that the graph of f contains a segment (or half line) which does not extend linearly to the right, say, beyond the point $(b, f(b))$. Choosing a suitable $a < b$ and replacing f by f_{ab} we can then assume that $f(x) = 0$ for $x \in [a, b]$ and that for some sequence $b_n \searrow b, f(b_n) \to 0$. Furthermore, by suitably modifying b_n, we may assume that $|f(b_n)| = \max_{x \in [a, b_n]} |f(x)|$. For the sake of the argument, we may let $f(b_n) > 0$. We now consider the condition (A_p) over the interval $[a, b_n]$. We observe that $(f_I)_{I'} = f_{I'}$ if $I \subset I'$ and that the triangle with vertices $(a, 0)$, $(b, 0)$, $(b_n, f(b_n))$, which has area $\frac{1}{2}(b-a)f(b_n)$, appears among triangles with vertices $(a, 0)$, $(x, f(x))$, $(b_n, f(b_n))$, $a < x < b_n$, competing in finding the value T_{ab_n}. Since $|S_{ab_n}| = \left|\frac{b_n-a}{2}f(b_n) - \int_b^{b_n} f(x)\,dx\right|$ and $\int_b^{b_n} |f(x)|\,dx \le (b_n-b)f(b_n)$,

$$p = \frac{|S_{ab_n}|}{T_{ab_n}} \le \frac{b_n-a}{b-a} + \frac{2}{(b-a)f(b_n)}\int_b^{b_n} |f(x)|\,dx \to 1, \text{ as } n \to \infty.$$

This contradicts the assumption that $p > 1$. □

Lemma 3. *If f satisfies (A_p) for some $p > 1$, then f is continuously differentiable.*

PROOF. If $G(f)$ is a line, then there is nothing to prove. Otherwise, because of Lemma 2, we can suppose that f is strictly convex, for the sake of the argument. Then for every x the one sided derivatives $D_-f(x)$ and $D_+f(x)$ exist. Also, for $x < y$, $D_-f(x) \le D_+f(x) < D_-f(y) \le D_+f(y)$. Suppose that there exists a $t \in \mathbb{R}$ such that $D_-f(t) < D_+f(t)$. Let $m = \frac{D_-f(t)+D_+f(t)}{2}$, and notice that $f(x) - f(t) - m(x-t)$ satisfies (A_p). Hence, we may assume that $t = 0$, $f(x) > f(0)$ for $x \neq 0$, and $D_-f(0) = -D_+f(0) = -\beta < 0$. It follows that f is strictly decreasing on $(-\infty, 0)$ and strictly increasing on $(0,\infty)$. Hence for every $a < 0$ there is a unique $b = b(a) > 0$ such that $f(a) = f(b(a))$ and, when $a \nearrow 0$, $b(a) \searrow 0$ and $f(a) \searrow 0$. In this setting, we have $T_{ab} = \frac{1}{2}f(a)(b-a)$ and $f(a) = -\beta a + o(a) = \beta b + o(b)$, which gives

$$T_{ab} = \frac{1}{2}f(a)(-a) + \frac{1}{2}f(a)b = \frac{\beta a^2}{2} + \frac{\beta b^2}{2} + o(a^2) + o(b^2).$$

Similarly, $f(x) = -\beta x + o(x)$ for $x < 0$, and $f(x) = \beta x + o(x)$ for $x > 0$, which implies that

$$S_{ab} = \frac{\beta a^2}{2} + \frac{\beta b^2}{2} + o(a^2) + o(b^2).$$

As $a \to 0$, the ratio $|S_{ab}|/T_{ab}$ converges to 1 rather than to p. It readily follows that f' is continuous: f' is increasing, so if at some point $f'(x+) > f'(x-)$ then we could also have at that point $D_+f(x) > D_-f(x)$, which by the considerations above would yield a contradiction. □

To fix the ideas, we assume from now on that f is strictly convex; i.e., $S_{ab} < 0$ for all intervals $I = [a,b]$. We observe next that the assumptions on f imply that $p < 2$. To see this, note that $|S_{ab}|$ is less than the area of the parallelogram bounded by the secant determined by the two points $(a, f(a)), (b, f(b))$, the tangent to the graph parallel to this secant, and the vertical lines $x = a, x = b$. The area of this parallelogram is $2T_{ab}$. Hence, $2T_{ab} > |S_{ab}| = pT_{ab}$, and $p < 2$. Henceforth we assume $1 < p < 2$.

Let x_0 be arbitrary. Note that if we replace $f(x)$ by $f(x + x_0) - f(x_0) - f'(x_0)x$ then the hypotheses on f are preserved. Hence, we may assume that $f(x_0) = f'(x_0) = 0$ and that $x_0 = 0$. Since f' is strictly increasing, f has a minimum at 0, and f is decreasing for $x < 0$ and increasing for $x > 0$. Hence, for every $x > 0$ there is a unique $\varphi(x)$ such that $\varphi(x) < 0, \varphi'(x) < 0$, and $f(\varphi(x)) = f(x)$.

Lemma 4. *There is a constant $\lambda > 0$ such that $f(x) = \lambda(x - \varphi(x))^q$, where $q = \frac{p}{2-p}$.*

Proof. Note that $q > 1$ since $1 < p < 2$, and $q = 2$ if $p = \frac{4}{3}$. For $x > 0$, the hypothesis on f implies that

$$(x - \varphi(x))f(x) - \int_{\varphi(x)}^{x} f(t)\,dt = \frac{p}{2} f(x)(x - \varphi(x))$$

or

$$\frac{2-p}{2} f(x)(x - \varphi(x)) = \int_{\varphi(x)}^{x} f(t)\,dt.$$

If we differentiate the last equality and use the fact that $f(x) = f(\varphi(x))$, we obtain

$$\frac{2-p}{2}(1 - \varphi'(x))f(x) + (x - \varphi(x))f'(x) = (1 - \varphi'(x))f(x)$$

or

$$\frac{f'(x)}{f(x)} = q\frac{1 - \varphi'(x)}{x - \varphi(x)} \quad \text{for } x \neq 0,$$

which implies that $f(x) = \lambda(x - \varphi(x))^q$, where λ is a positive constant. $\square$

Remark. We have $f'(x) = q\lambda(x - \varphi(x))^{q-1}(1 - \varphi'(x))$. For $x > 0, f'(x) > q\lambda x^{q-1}$, since $\lambda > 0$. In particular,

$$\lim_{x\to\infty} f'(x) = \infty,$$

a condition invariant under addition of an affine function.

Lemma 5. *Let (x_0, y_0) be the point of intersection of the tangent lines to $y = f(x)$ at $(a, f(a))$ and $(\varphi(a), f(\varphi(a)))$ respectively. Then the area of the triangle with vertices at $(a, f(a)), (\varphi(a), f(\varphi(a))), (x_0, y_0)$ is q times the area of the triangle with vertices at $(a, f(a)), (\varphi(a), f(\varphi(a))), (0, 0)$.*

PROOF. The equations of the tangent lines to $y = f(x)$ at $(a, f(a))$ and $(\varphi(a), f(a)) = (\varphi(a), f(\varphi(a)))$ are

$$y - f(a) = f'(a)(x-a) \quad \text{and} \quad y - f(a) = f'(\varphi(a))(x - \varphi(a)).$$

Using the fact that $f'(\varphi(a))\varphi'(a) = f'(a)$, we find that the point of intersection (x_0, y_0) has its coordinates given by

$$x_0 = \frac{a\varphi'(a) - \varphi(a)}{\varphi'(a) - 1}, \quad y_0 = f(a) + f'(a)\frac{a - \varphi(a)}{\varphi'(a) - 1}.$$

But $f(x) = \lambda(x - \varphi(x))^q$ implies that $f'(x) = q\lambda(x - \varphi(x))^{q-1}(1 - \varphi'(x))$, hence

$$y_0 = f(a) + q\lambda\frac{(a - \varphi(a))^q(1 - \varphi'(a))}{\varphi'(a) - 1} = (1-q)f(a).$$

In particular, the area of the triangle with vertices $(a, f(a)), (\varphi(a), f(\varphi(a)))$, (x_0, y_0) is q times the area of the triangle with vertices $(a, f(a)), (\varphi(a), f(\varphi(a)))$, $(0,0)$, a property of f observed by G. Muraz when $p = \frac{4}{3}$ $(q = 2)$. □

Consider next the area of the region between the graph of f and the secant at two arbitrary points. We may assume, by adding as before an affine function, that the points are $(0,0), (x, f(x))$ and that $f(0) = f'(0) = 0$. In this case, the following lemma holds.

Lemma 6. *For $x > 0, f(x) = cx^r$, where $r = \frac{2-p}{p-1}$ (c is some positive constant).*

PROOF. Note that $r > 0$ since $1 < p < 2$, and $r = 2$ when $p = \frac{4}{3}$. Let $(c, f(c))$ be the point where the tangent line to the graph of f is parallel to the secant determined by the points $(0,0), (x, f(x))$, and let S denote the area of the region between the graph of f and this secant and T the area of the triangle determined by the points $(0,0), (c, f(c)), (x, f(x))$. From Lemma 5, we have $\tilde{T} = qT$, where $\tilde{T}$ is the triangle with vertices at the points $(0,0), (\xi, 0), (x, f(x))$, and $(\xi, 0)$ is the point of intersection of the $x-$axis with the tangent to the graph of f at $(x, f(x))$. Note that $\xi = x - \frac{f(x)}{f'(x)}$. The condition $(A_p), S = pT = \frac{p}{q}\tilde{T}$, can be rewritten as

$$\frac{1}{2}xf(x) - \int_0^x f(t)\,dt = \frac{p}{q}\left(x - \frac{f(x)}{f'(x)}\right)\frac{f(x)}{2},$$

which is equivalent to

$$\frac{p-1}{2}xf(x) - \int_0^x f(t)\,dt = \left(\frac{p}{2} - 1\right)\frac{f(x)^2}{f'(x)}. \tag{A'_p}$$

Note that f being continuously differentiable implies, solving for f' in (A'_p), that f has derivatives of any order wherever $f(x) \neq 0$. Differentiate both sides of the equality (A'_p) to obtain:

$$\frac{xf'(x) - f(x)}{f(x)^2} = r\frac{f''(x)}{f'(x)^2}; \quad \text{i.e.,} \quad \left(\frac{x}{f(x)}\right)' = -r\frac{f''(x)}{f'(x)^2}.$$

If we integrate the last equality we get

$$\frac{x}{f(x)} = \frac{r}{f'(x)} + C.$$

Let now $x \to \infty$ in the last equality. Using the remark following Lemma 4 and l'Hôpital's rule, we conclude that $C = 0$, $xf'(x) - rf(x) = 0$ and finally that $f(x) = cx^r, c > 0$. □

Proof of Theorem (iii). We have reduced the proof of Part iii) to the case when $1 < p < 2$, f is strictly convex and there is a unique function φ such that $\varphi(x) < 0$, $\varphi'(x) < 0$ and $f(\varphi(x)) = f(x)$ for all $x > 0$. Using Lemmas 4 and 6 we get

$$x - \varphi = (\frac{c}{\lambda})^{\frac{1}{q}} x^{\frac{r}{q}}$$

and

$$\varphi'(x) = 1 - (\frac{c}{\lambda})^{\frac{1}{q}} \frac{r}{q} x^{\frac{r}{q}-1}.$$

If $\frac{r}{q} > 1$, then the equation for φ' implies that there is an $\varepsilon > 0$ such that $\varphi'(x) > 0$ for $0 < x < \varepsilon$. If $\frac{r}{q} < 1$ the equation for φ' implies that $\varphi'(x) \to 1$ as $x \to \infty$. Both of these cases contradict the fact that $\varphi'(x) < 0$ for all $x > 0$. Therefore $r = q$, from which it follows, that $p = \frac{4}{3}$ and the proof is complete. □

A modified condition

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. In the work above we considered the following condition (A_p):

$$\left| \int_a^b f_{ab}(x)\, dx \right| = \frac{p}{2}(b-a) \max_{a<x<b} |f_{ab}(x)|, \quad \text{for all } I = [a, b].$$

We now consider a modified condition, call it $(\tilde{A}_p)$, obtained from (A_p) by moving the absolute value sign inside the integral in the left hand side of the equality:

$$\int_a^b |f_{ab}(x)|\, dx = \frac{p}{2}(b-a) \max_{a<x<b} |f_{ab}(x)|, \quad \text{for all } I = [a, b].$$

Let

$$\tilde{S}_{ab} = \int_a^b |f_{ab}(x)|\ dx$$

and

$$T_{ab} = \frac{1}{2}(b-a) \max_{a<x<b} |f_{ab}(x)|.$$

The condition $(\tilde{A}_p)$ can be rewritten as

$$\tilde{S}_{ab} = pT_{ab}, \text{ for all } I = [a, b].$$

Let $p > 0$. We have shown (Lemma 1) that for every interval $I = [a, b]$, f_I is either identically zero or f_I has a constant sign. In particular, (A_p) implies $(\tilde{A}_p)$. It might come as a surprise, but the reverse implication is true as well. We now show that $(\tilde{A}_p)$ implies (A_p).

We first prove, similarly to our previous work, that if f satisfies $(\tilde{A}_p)$ for $p > 1$ and the graph of f contains a line segment, then f is an affine function– this is the case when $(\tilde{A}_p)$ holds with both sides of the equality equal to zero. Suppose that the graph contains a segment which starts at the point $(a, f(a))$ and ends at the point $(b, f(b))$, and f is not affine. This implies that $f_{ab} = 0$ in $[a, b]$ but there exists a sequence $b_n \searrow b$ such that $f_{ab}(b_n) \neq 0$. Then we have

$$\tilde{S}_{ab_n} = \frac{1}{2}(b-a)|f_{ab_n}(b)| + (b_n - b)|f_{ab_n}(s_n)| \leq \left[\frac{1}{2}(b-a) + (b_n - b)\right] |f_{ab_n}(c_n)|,$$

for some $s_n, c_n \in [b, b_n)$ and $|f_{ab_n}|$ attains its maximum at c_n. On the other hand, $T_{ab_n} = \frac{1}{2}(b-a)|f_{ab_n}(c_n)|$ and it follows that

$$p = \lim_{n\to\infty} \frac{\tilde{S}_{ab_n}}{T_{ab_n}} \leq 1,$$

a contradiction. Next we note that as in the proof of Lemma 1, if f is not an affine function and if for some $a < c < b$ we have $f_{ab}(c) = 0$, then

$$T_{ab} = \frac{1}{p}\tilde{S}_{ab} = \frac{1}{p}(\tilde{S}_{ac} + \tilde{S}_{cb}) = T_{ac} + T_{cb},$$

an equality which holds only if the maxima of $|f_{ac}|$ and $|f_{cb}|$ are the same. In particular, f_{ab} can have at most finitely many zeros in (a, b). By suitably restricting the interval, we may assume that there is only one zero of f_{ab}, say at c, in (a, b). Suppose first that f_{ab} changes sign at c, say $f_{ac} > 0$ and $f_{cb} < 0$.

Let $b' \in (c, b)$ be sufficiently close to b so that $f_{ab'}$ changes sign at c' in (a, b'). Then

$$\max_{a<x<c'} f_{ab'}(x) > \max_{a<x<c} f_{ab}(x) = \max_{c<x<b}(-f_{ab}(x)) > \max_{c'<x<b}(-f_{ab'}(x)),$$

a contradiction. If $f_{ab} > 0$ in the intervals (a, c) and (c, b) then, for $b' \in (c, b)$ sufficiently close to b, $f_{ab'}$ changes sign in (a, b') and we are back in the previous case. It follows that, for $a < b$, f_{ab} is of constant sign in (a, b) and $(\tilde{A}_p)$ implies (A_p).

References

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