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THE ABSOLUTE CONTINUITY OF THE DISTRIBUTION OF RANDON SUMS WITH DIGITS $\{0, 1, \ldots, m-1\}$

Abstract

Let $m \geq 2$ be a natural number. Let ν_λ^m be the distribution of the random sum $\sum_{n=0}^{\infty} \theta_n \lambda^n$, where θ_n are i.i.d. and for every n the random variable θ_n takes value in the set $\{0, \ldots, m-1\}$ with equal probabilities. As a generalization of Solomyak Theorem we prove that for Lebesgue a.e. $\lambda \in (1/m, 1)$ the measure ν_λ^m is absolute continuous w.r.t. the Lebesgue measure. (For smaller λ, the measure ν_λ^m is supported by a Cantor-set, so if $\lambda < 1/m$ then ν_λ^m is singular.)

1 Introduction.

After some results of Erdős in 1930's [3, 4] there have been a continuous interest in the last 60 years about the absolute continuity of some infinite Bernoulli convolutions. The major achievement was made by B. Solomyak in 1995. Answering an almost 60 years open problem he proved [9] the following theorem:

Theorem 1 (Solomyak [9]). *Let ν_λ be the distribution of the random series: $Y_\lambda = \sum_{n=0}^{\infty} \theta_n \lambda^n$, where θ_n are independent random variables taking values in $\{-1, 1\}$ with $1/2, 1/2$ probabilities. Then ν_λ is absolute continuous w.r.t. the Lebesgue measure for almost all $\lambda \in [1/2, 1]$.*

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Obviously this is equivalent to the case when θ_n takes values from $\{0,1\}$ with $1/2, 1/2$ probabilities. See [6, 10] for nice survey papers on this and related results about Bernoulli convolutions. Here we consider the more general situation: Let

$$Y_\lambda^m = \sum_{n=0}^{\infty} \theta_n \lambda^n,$$

where θ_n are i.i.d. and

$$Prob(\theta_n = 0) = Prob(\theta_n = 1) = \ldots = Prob(\theta_n = m-1) = \frac{1}{m}.$$

Further we write ν_λ^m for the distribution of Y_λ^m. Our aim is to prove

Theorem 2 (Main theorem). *For all natural number $m > 1$ and for Lebesgue almost all $\lambda \in \left(\frac{1}{m}, 1\right)$ the measure ν_λ^m is absolute continuous w.r.t. the Lebesgue measure with L^2-density.*

Remark 1. *The case when $m = 2$ is equivalent to Solomyak Theorem. So we may assume in the rest of the paper that $m \geq 3$.*

The measures ν_λ^m are natural self-similar measures and they occur in various settings. They have been studied by Borwein and Girgensohn in connection with functional equations and they occurred in some examples related to quasicrystals [2, p.33.]. Moreover, they arise in fractal geometry, in the investigation of self-affine sets with m pieces, similarly to the work of Przyticki and Urbanski [11] where self-affine sets with 2 pieces were considered.

In the proof we use a theory about infinite Bernoulli convolution developed in the papers of Solomyak and Peres. Roughly speaking, this theory says that if a so called transversality condition holds then the Bernoulli convolution measure is absolute continuous with L^2 density for almost all parameters. Our contribution is merely checking transversality, but this is not completely trivial.

The paper is organized as follows: For the convenience of the reader, in the next section we recall some notation and two important theorems from Peres and Solomyak papers [9, Theorem 1.2] and [7, 8] and [10, Theorem 4.3] . We get our result as an application of these theorems. To apply [9, Theorem 1.2] and [10, Theorem 4.3] we have to verify their assumption, which is in both cases a so called transversality condition. Actually this is what we do in the last section of the paper.

1.1 Some Known Results about ν_λ^m.

As B. Solomyak pointed out [12], as a straight forward generalization of [5], one can see that: if $1/\lambda$ is an algebric number, such that in the minimal polynomial the constant term is $\pm m$ and all roots of the minimal polynomial are outside of the unit disk, then ν_λ^m is absolute continuous w.r.t. Lebesgue measure with density in at least L^∞. Using this for $x^p - m$ one obtains that [1, Theorem 3] for $\lambda = m^{-1/p}, p \in \mathbb{N}, p \geq 1$ the measure ν_λ^m is absolute continuous w.r.t. Lebesgue measure with density in at least L^∞. Also in [1, Theorem 4] it was proved that if λ is the reciprocal of an irrational Pisot number then the measure ν_λ^m is singular.

2 Solomyak and Peres Theory on Bernoulli Convolution.

Since we do not use this theory in its full generality, we will quote only the special cases of the theorems of Solomyak [9, Theorem1.2] and Solomyak, Peres theorem which appeared first in [7] and [8], and in a form what is very close to the way as we use it here, in [10, Theorem 4.3]. Then we prove a lemma which is a slight generalization of [10, Lemma 5.2].

Throughout this section we always use the following notation: Given $m \in \mathbb{N}, m \geq 2$ (when we say $\mathbb{N}$ we mean that $0 \in \mathbb{N}$) and also given $0 < l \leq n$. Further, given a set $I \subset \mathbb{N}$, with the following property: I consists of l arithmetic progressions with difference equal to n, and the first elements of these arithmetic progressions are in the set $\{0,1,\ldots,n-1\}$. We denote

$$\sigma^k I = \{i - k : i \in I, i \geq k\}.$$

In this paper I will be either $\mathbb{N}$ (most frequently) but in some important cases I will be $\{i \in \mathbb{N} : i \neq 3k+2, k \geq 0\}$. Then $\sigma I = \{i \in \mathbb{N} : i \neq 3k+1, k \geq 0\}$. Finally given a probability vector $\mathbf{p} = (p_0, \ldots, p_{m-1})$.

For a $\lambda \in (0,1)$ we consider the random sum

$$Y_{\lambda,\mathbf{p}}^{m,I} := \sum_{i \in I} \theta_i \lambda^i,$$

where the random variables θ_i are i.i.d. and

$$Prob(\theta_i = 0) = p_0; \ldots; Prob(\theta_i = m-1) = p_{m-1}.$$

To make the notation simpler we omit the probability vector $\mathbf{p}$ from $Y_{\lambda,\mathbf{p}}^{m,I}$ if $\mathbf{p} = (1/m, \ldots, 1/m)$. Similarly, we omit the set I from $Y_{\lambda,\mathbf{p}}^{m,I}$ if $I = \mathbb{N}$. So,

$$Y_{\lambda,\mathbf{p}}^{m} = Y_{\lambda,\mathbf{p}}^{m,\mathbb{N}} \text{ and } Y_\lambda^{m,I} = Y_{\lambda,(\frac{1}{m} \ldots \frac{1}{m})}^{m,I}$$

We write $\nu_{\lambda,\mathbf{p}}^{m,I}$ for the distribution of $Y_{\lambda,\mathbf{p}}^{m,I}$. (We also omit I and $\mathbf{p}$ in the most natural cases like above.) We will frequently use the following family of power series:

$$\mathcal{B}_{m,I} := \left\{ 1 + \sum_{i \in I \setminus \{0\}} a_i x^i : |a_i| \leq m-1 \right\}.$$

As usual if $I = \mathbb{N}$ we write simply $\mathcal{B}_m$ for $\mathcal{B}_{m,\mathbb{N}}$. Let $J \subset (0,1)$ be a closed interval and $\delta > 0$. After [7, 8] and [10] we define:

Definition 1. *We say that the δ-transversality condition holds for $\mathcal{B}_{m,I}$ on J, if for $\forall k \in I, k < n$ and for $\forall g \in \mathcal{B}_{m,\sigma^k I}$ and for $\forall \lambda \in J$:*

$$g(\lambda) < \delta \Rightarrow g'(\lambda) < -\delta. \tag{2.1}$$

Remark 2. *Let f be an arbitrary power series with integer coefficients smaller than or equal to $m-1$ in modulus, and $f(0) \neq 0$. Then it follows from the expression (15) of [9], that: If the δ-transversality condition holds on J for $\mathcal{B}_{m,I}$ then f has no double zero on J.*

The following theorem is a combination of [9, Theorem 1.2], [7], [8, Theorem 1.3] and see also [10, Theorem 4.3].

Theorem 3 (Peres, Solomyak). *Assume that for some $\delta > 0$ the δ-transversality condition holds on a closed interval $J \subset [0,1)$ for $\mathcal{B}_{m,I}$. Then:*

1. *For a.e. $\lambda \in J \cap \left(\sum_{i=0}^{m-1} p_i^2, 1 \right)$, the measure $\nu_\lambda^{m,\mathbf{p}}$ is absolutely continuous with $L^2(\mathbb{R})$ density. (Here we assumed that $I = \mathbb{N}$)*
2. *For a.e. $\lambda \in J \cap \left(m^{-l/n}, 1 \right)$ the measure $\nu_{\lambda,I}^m$ is absolutely continuous with $L^2(\mathbb{R})$ density. (Here we assumed that $\mathbf{p} = (1/m, \ldots, 1/m)$.)*

Remark 3. *When $p_0 = \ldots = p_{m-1}$ then the lower bound $\sum_{i=0}^{m-1} p_i^2$ in Theorem (3) is $1/m$.*

The following is a slight modification of [10, Lemma 5.2]. The difference is that here the summation is taken over I instead of $\mathbb{N}$. Therefore the same statement does not hold always. We define the family of the so called (*) functions for $\mathcal{B}_{m,I}$. The idea of (*) functions was developed by Solomyak [9] and significantly further developed in [7, 8].

Definition 2. *We say that the function $h(x)$ is a (*) function for $\mathcal{B}_{m,I}$ if there is an $a \in \mathbb{R}$ ($|a| > m-1$ is allowed here) such that*

$$h(x) = 1 - (m-1) \sum_{i<k, i \in I \setminus \{0\}} x^i + a x^k + (m-1) \sum_{i>k, i \in I \setminus \{0\}} x^i,$$

where $k \in I$. Notice that if $h(x)$ is a (*) function then there is at most one sign change in the coefficients of $h'(x)$ an $h''(x)$. So, the functions $h'(x), h''(x)$ have at most one positive zero.

Lemma 1. *Suppose that for every $j \in I, j < n$, there exists a (*) function $h_j(x)$ for $\mathcal{B}_{m,\sigma^j I}$ satisfying:*

$$h_j(x_0) > \delta_j \text{ and } h_j'(x_0) < -\delta_j \tag{2.2}$$

for some $x_0 \in (0,1)$ and $\delta_j > 0$. Then for every $0 < \varepsilon < x_0$ there exists a $\tau > 0$ such that the τ-transversality condition holds on $[\varepsilon, x_0]$ for $\mathcal{B}_{m,I}$.

The proof follows the idea of [10, Lemma 5.2] with the necessary modifications. Before we present the proof we need to prove the following:

Claim 1. *Fix $0 < \varepsilon < x_0$. There exists $\tau_j > 0$ ($j < n, j \in I$) such that*

$$h_j(x) > \tau_j,\ \forall x \in [0, x_0] \text{ and } h_j'(x) < -\tau_j,\ \forall x \in [\varepsilon, x_0]. \tag{2.3}$$

Proof. Choose an arbitrary $j \in I, j < n$. We write $I' := \sigma^j I$. It is obvious that (2.3) holds if $k = 1$ since in this case $h'(x)$ is increasing. From now we assume that $k > 1$. Let $k_0 = \min\{I' \setminus \{\emptyset\}\}$. We distinguish two cases:

First we assume that $k_0 = 1$. In this case the proof is the same as in [10, Lemma 5.2]. We argue by contradiction. Since $h'(0) = -(m-1)$ if there exists $u \in (0, x_0)$ such that $h'(u) > -\delta$ then by the mean value theorem there exist $v_1 \in (0, u)$ and $v_2 \in (u, x_0)$ such that $h''(v_1) > 0$ and $h''(v_2) < 0$. Using that $\lim\limits_{x\to 1} h''(x) = \infty$ we obtain that h'' has two sign changes what is impossible.

Now we assume that $k_0 > 1$. In this case $h'(x) = 0$. First we point out that

$$h'(x) < 0\ \forall x \in (0, x_0). \tag{2.4}$$

First we assume that $k > k_0 > 1$. To see (2.4) we also argue by contradiction. Assume that there exists $w \in (0, x_0)$ such that $h'(w) \geq 0$. Notice that $h'(x) < 0$ for all x small enough. Let $w_1 \in (0, w)$ be chosen such that $h'(w_1) < 0$. Then there exist $z_1 \in (0, w_1)$ and $z_2 \in (w_1, w)$ such that $h''(z_1) < 0$ and $h''(z_2) > 0$.Using that $h'(x_0) < -\delta$ by the mean value theorem there exists $z_3 \in (w, x_0)$ such that $h''(z_3) < 0$. In this way h'' changes signs more then one times. This contradiction proofs that (2.4) holds.

Now we assume that $k = k_0 > 1$. In this case $h'(0) = 0$. To prove (2.4) first observe that $a < 0$. Namely, if $a \geq 0$ then $h'(x)$ is increasing. What is impossible since $h'(0) = 0$ and $h'(x_0) < 0$. Notice that $a < 0$ implies that for every $x > 0$ small enough $h'(x) < 0$ holds. Fix such an x. If there was a $w \in (0, x_0)$ such that $h'(w) \geq 0$ then using that $\lim_{x\to 1} h'(x) = \infty$, we obtain that there exist z_1, z_2, z_3 such that: $z_1 \in (x, w)$, $z_2 \in (w, x_0)$, $z_3 \in (x_0, 1)$ and

$h''(z_1) > 0$, $h''(z_2) < 0$, $h''(z_3) > 0$. That is h'' should change signs at least twice, what is impossible. This proves that (2.4) holds.

Let

$$\tau_j := -\frac{1}{2} \max \{h'(x) : \varepsilon < x < x_0\} .$$

It follows from (2.4) that $\tau_j > 0$. Then $h'(x) < \tau_j$ for all $x \in [\varepsilon, x_0]$. Since $h'(x) < 0$ for all $x \in (0, x_0]$ we obtain that $h(x) > \delta \geq \tau_j$ holds for all $x \in [\varepsilon, x_0]$. □

The rest of the proof Lemma 1 proceeds exactly as that of [10, Lemma 5.2]. For the convenience of the reader we present it here.

Proof. [Proof of Lemma 1] As above, we choose an arbitrary $j \in I, j < n$ and we write $I' = \sigma^j I$ and $h(x) = h_j(x)$. Let $g \in \mathcal{B}_{m,I'}$. Put $f(x) = g(x) - h(x)$. Then

$$f(x) = \sum_{i<l, i \in I' \setminus \{\emptyset\}} c_i x^i - \sum_{i \geq l, i \in I' \setminus \{\emptyset\}} c_i x^i,$$

where either $l = k$ or $l = \max\{i \in I, i < k\}$ and $c_i \geq 0$. Notice that

$$f(x) < 0 \Rightarrow f'(x) < 0. \tag{2.5}$$

Namely, if $f(x) < 0$ then

$$\sum_{i<l, i \in I' \setminus \{\emptyset\}} c_i x^i < \sum_{i \geq l, i \in I' \setminus \{\emptyset\}} c_i x^i$$

so,

$$\sum_{i<l, i \in I' \setminus \{\emptyset\}} i c_i x^{i-1} < \sum_{i \geq l, i \in I' \setminus \{\emptyset\}} i c_i x^{i-1}$$

thus $f'(x) < 0$.

Assume that for an $x \in [\varepsilon, x_0]$ $g(x) < \tau_j$. Then from the Claim above $f(x) < 0$. This implies, as we just saw that $f'(x) < 0$. So, $g'(x) < h'(x) < -\tau_j$. Then $\tau := \min \{\tau_j : j \in I, j < n\} > 0$ satisfies (2.3). □

Remark 4. *It follows from the proof that if $I = \mathbb{N}$ then ε in Lemma 1 can be 0. This is exactly the case considered in [10, Lemma 5.2].*

3 The Fourier Transform of ν_λ^m.

As usual we denote the Fourier-transform of the measure ν_λ^m by $\hat{\nu}_\lambda^m$. By Plancherel-Theorem (see also [10, p.16]) $\hat{\nu}_\lambda^m \in L^2$ if and only if $\nu_\lambda^m \ll \mathcal{L}eb$ with L^2 density. To find the Fourier transform of ν_λ^m observe that

$$\nu_\lambda^m = * \prod_{n=0}^{\infty} \left(\frac{\delta_0}{m} + \frac{\delta_{\lambda^n}}{m} + \ldots + \frac{\delta_{(m-1)\lambda^n}}{m} \right).$$

Thus

$$\begin{aligned} \hat{\nu}_\lambda^m(\xi) &= \prod_{n=0}^{\infty} \frac{1}{m} \left(1 + e^{-i\lambda^n \xi} + \ldots + e^{-i(m-1)\lambda^n \xi} \right) \\ &= \prod_{n=0}^{\infty} \frac{1}{m} \cdot \frac{e^{-im\lambda^n \xi} - 1}{e^{-i\lambda^n \xi} - 1}. \end{aligned} \tag{3.1}$$

We verify some simple properties of $\hat{\nu}_\lambda^m(\xi)$ what we frequently use in the paper:

Claim 2. *Let $z = e^{ix}$ $(x \in \mathbb{R})$. (We will use this with $x = \lambda^n \xi$ most frequently.)*

1. $\left| \frac{z^m - 1}{z - 1} \right| = \left| \frac{\sin(m \cdot x/2)}{\sin(x/2)} \right|$,
2. $\frac{1}{m} \left| \frac{z^m - 1}{z - 1} \right| \leq 1$ *and* $m \left| \frac{z-1}{z^m - 1} \right| - 1 = O(x^2)$ *as* $x \to 0$.
3. *The Fourier-transform is a convergent infinite product:*

$$|\hat{\nu}_\lambda^m(\xi)| = \prod_{n=0}^{\infty} \frac{1}{m} \left| \frac{\sin(m\lambda^n \xi/2)}{\sin(\lambda^n \xi/2)} \right|. \tag{3.2}$$

Proof. (i) $\left| e^{ix} - 1 \right| = 2 \left| \sin(x/2) \right|$. Using this for mx instead of x gives the statement of (i).

(ii) The first part is straightforward. We obtain the second part by a multiple application of L'Hospital rule.

(iii) This is immediate from (i), (ii) and (3.1). □

Corollary 1. *We can rearrange the terms of the infinite product in (3.1). Like in [9] this yields that for any $\lambda \in (0, 1)$:*

$$\hat{\nu}_{\sqrt{\lambda}}^m(\xi) = \hat{\nu}_\lambda^m(\xi) \cdot \hat{\nu}_\lambda^m(\sqrt{\lambda}\xi). \tag{3.3}$$

Similarly as in [9] this implies that:

Claim 3. *If we can prove that ν_λ^m is absolutely continuous with L^2 density for Lebesgue almost every $\lambda \in (\frac{1}{m}, \frac{1}{\sqrt{m}})$ then the same holds on the bigger interval $(\frac{1}{m}, 1)$.*

PROOF. If ν_λ is absolute continuous with L^2 density for almost all $\lambda \in (\frac{1}{m}, \frac{1}{\sqrt{m}})$ then $\hat{\nu}_\lambda^m \in L^2$ holds for almost all $\lambda \in (\frac{1}{m}, \frac{1}{\sqrt{m}})$. Using (3.3), this implies that $\hat{\nu}_\lambda^m \in L^1$ for almost all $\lambda \in (\frac{1}{\sqrt{m}}, \frac{1}{\sqrt[4]{m}})$. Since $\hat{\nu}_\lambda^m$ is bounded this implies that $\hat{\nu}_\lambda^m \in L^2$ for almost all $\lambda \in (\frac{1}{\sqrt{m}}, \frac{1}{\sqrt[4]{m}})$. Iterating this argument we get the statement of the Claim. $\square$

4 Checking Transversality.

Our aim in this section is to establish transversality on the interval $\left(\frac{1}{m}, \frac{1}{\sqrt{m}}\right)$.

A considerable part of the interval $(\frac{1}{m}, \frac{1}{\sqrt{m}})$ can be covered by the following lemma. It appears in [8, Corrolary 5.2].

Lemma 2. *Let $m \geq 2$. We write*

$$x_m := \frac{1}{1+\sqrt{m-1}}.$$

We can find $\delta > 0$ such that the δ-transversality condition holds for $\mathcal{B}_m$ on the interval $[0, x]$ for every $x < x_m$.

Corollary 2. *It is immediate from Theorem 3 (i) and Lemma 2 that ν_λ^m is absolute continuous with L^2 density for almost all $\lambda \in \left(\frac{1}{m}, \frac{1}{1+\sqrt{m-1}}\right)$.*

Latter we need the following two corollaries of the lemma:

Corollary 3. *For every $m \geq 4$ and $2 \leq k \leq 2m$ there exist $\delta > 0$ such that the δ-transversality holds for $\mathcal{B}_k$ on the interval $[0, 1/m]$.*

PROOF. This follows immediately from Lemma 2 since

$$\frac{1}{m} < \frac{1}{1+\sqrt{2m-1}} \leq \frac{1}{1+\sqrt{k-1}}.$$

$\square$

Corollary 4. *For every $m \geq 5$ there exist $\delta > 0$ such that the δ-transversality holds for $\mathcal{B}_{3m}$ on the interval $[0, 1/m]$.*

Proof. Since
$$\frac{1}{m} < \frac{1}{1+\sqrt{3m-1}}$$
holds for all $m \geq 5$, Lemma 2 implies the statement of the Corollary. □

Although $x_m = \frac{1}{1+\sqrt{m-1}}$ is very close to $\frac{1}{\sqrt{m}}$ for big m, unfortunately $x_m = \frac{1}{1+\sqrt{m-1}} < \frac{1}{\sqrt{m}}$ for every m. The following lemma is very important since it covers the gap between x_m and $\frac{1}{\sqrt{m}}$ for all m big enough.

Lemma 3. *Let $m \geq 3$ be arbitrary. Put*
$$A(m) := \frac{2m^2+1}{3m^3}.$$
Then ν_λ^m is absolute continuous with L^2 density for a.a. $\lambda \in \left(\sqrt{A(m)}, \frac{1}{\sqrt{m}}\right)$

Notice that for m big enough $\sqrt{A(m)}$ is approximately $\frac{1}{\sqrt{m}} \cdot 0.81649..$, so for big m, $\sqrt{A(m)} < x_m$ must hold. The proof follows an argument from [8].

Proof. Let $\eta_\lambda^m := \nu_\lambda^m * \nu_\lambda^m$. Then η_λ^m is the distribution of $Z_\lambda^m := \sum_{n=0}^{\infty} \theta_n \lambda_n$, where the random variables θ_n are i.i.d. and for $k \leq m-1$, $Prob(\theta_n = k) = \frac{k+1}{m^2}$. For $m \leq k \leq 2m-2$, $Prob(\theta_n = k) = \frac{2m-1-k}{m^2}$. So, for every n
$$\sum_{k=0}^{2m-2} Prob^2(\theta_n = k) = A(m).$$
We know from Corollary 3 that for $m \geq 4$ and for some $\delta > 0$, δ-transversality holds for $\mathcal{B}_{2m-1}$ on $[0, \frac{1}{m}]$. Lemma 2 implies that $\mathcal{B}_5$ is δ-transversal on the interval $[0, 1/3]$. Thus we obtain from Theorem 3 (i) that η_λ^m is absolute continuous with L^2 density for almost all $\lambda \in \left(A(m), \frac{1}{m}\right)$, $m \geq 3$. This means that $\hat{\eta}_\lambda^m \in L^2$ for almost all $\lambda \in \left(A(m), \frac{1}{m}\right)$. From the definition of η_λ^m this implies that $\hat{\nu}_\lambda^m \in L^4$ for a.a. $\lambda \in \left(A(m), \frac{1}{m}\right)$. Thus from (3.3) we obtain that $\hat{\nu}_\lambda^m \in L^2$ for a.a. $\lambda \in (\sqrt{A(m)}, \frac{1}{\sqrt{m}})$. This completes the proof of our Lemma. □

Proof. [The proof of the Theorem 2 for $m \geq 16$] An elementary calculation shows that
$$\sqrt{A(m)} < \frac{1}{1+\sqrt{m-1}}. \tag{4.1}$$

holds for $m \geq 16$. Since $\left(\frac{1}{m}, \frac{1}{\sqrt{m}}\right) = \left(\frac{1}{m}, \frac{1}{1+\sqrt{m-1}}\right) \cup \left(\sqrt{A(m)}, \frac{1}{\sqrt{m}}\right)$. In Lemma 2 we checked the δ-transversality for $\lambda \in \left(\frac{1}{m}, \frac{1}{1+\sqrt{m-1}}\right)$. This implies by Theorem 3 (i) the ν_λ^m is absolute continuous with L^2 density for a.a. $\left(\frac{1}{m}, \frac{1}{1+\sqrt{m-1}}\right)$. The other interval $\left(\sqrt{A(m)}, \frac{1}{\sqrt{m}}\right)$ was settled in Lemma 3. This completes the proof for $m \geq 16$. □

The remaining cases will be checked in three groups:

Remark 5. *We would like to remark that upper bounds for the transversality interval in the case of $m = 3, 4$ were found in [8, Corollary 5.2].*

PROOF. [The proof of Theorem 2 for $m = 3, 4, 5$.] If we can prove that for $m \in \{3, 4, 5\}$ there is a (*) function h for $\mathcal{B}_m$ such that for some $x > \sqrt{A(m)}$

$$h(x) > 0 \text{ and } h'(x) < 0 \tag{4.2}$$

then we can take $\delta = \min\{h(x), -h'(x)\}$. We know from Lemma 1 and Remark 4 that this implies the existence of a $\tau > 0$ such that the τ-transversality condition holds for $\mathcal{B}_m$ on $[0, \sqrt{A(m)}]$. Using Theorem 3 this follows that ν_λ^m is absolute continuous with L^2 density for almost all $\lambda \in \left(\frac{1}{m}, \sqrt{A(m)}\right)$. The statement of the Theorem 2 for a.a. $\lambda \in \left(\sqrt{A(m)}, \frac{1}{\sqrt{m}}\right)$ was proved in Lemma 3. So, to complete the proof for $m = 3, 4, 5$ we only have to show (*) functions satisfying (4.2) for some $x > \sqrt{A(m)}$.

$m = 3$ $h(x) = 1 - 2x - 1.8x^2 + 2\sum_{i=3}^{\infty} x^i$,

$x = 0.484323 > \sqrt{A(3)} = 0.4843221048\ldots$. Then

$$h(x) = 0.497443\ldots > 0 \text{ and } h'(x) = -0.129874\ldots < 0$$

$m = 4$ $h(x) = 1 - 3x - 0.5x^2 + 3\sum_{i=3}^{\infty} x^i$,

$x = 0.414559 > \sqrt{A(4)} = 0.414578098794\ldots$. Then

$$h(x) = 0.003548\ldots > 0 \text{ and } h'(x) = -0.148952\ldots < 0$$

$m = 5$ $h(x) = 1 - 4x + 1.2x^2 + 4\sum_{i=3}^{\infty} x^i$,

$x = 0.368783 > \sqrt{A(5)} = 0.368781778291715\ldots$. Then

$$h(x) = 0.005898\ldots > 0 \text{ and } h'(x) = -0.259036\ldots < 0.$$

□

Using Corollary 2 and Lemma 3 we only have to deal with the interval $\left(\frac{1}{1+\sqrt{m-1}}, \sqrt{A(m)}\right)$. Unfortunately we cannot prove transversality on this interval for $\mathcal{B}_m, m \in \{6, 7, 9, 11, 13\}$. To make a step further, we have to invoke an idea from [9].

PROOF. [The proof of the Theorem 2 for $m = 6, 7, 9, 11, 13$] Let

$$Z_\lambda^m := \sum_{n \in I} \theta_n \lambda^n,$$

where $I := \{i \in \mathbb{N} : i \neq 3k + 2 \text{ for any } k \in \mathbb{N}\}$ and θ_n i.i.d. with

$$Prob(\theta_n = 0) = \ldots = Prob(\theta_n = m-1) = \frac{1}{m}.$$

We write η_λ^m for the distribution of Z_λ^m. (This Z_λ^m and η_λ^m are different from the ones defined in an earlier proof in this paper.) It follows from (3.2) that

$$|\hat{\eta}_\lambda^m(x)| \geq |\hat{\nu}_\lambda^m(x)|,$$

for all $x \in \mathbb{R}$. Therefore, to prove that $\hat{\nu}_\lambda^m(x) \in L^2$ almost surely on a parameter interval, it is enough to point out the same for $\hat{\eta}_\lambda^m(x)$. Notice that

$$m^{-2/3} < \frac{1}{1 + \sqrt{m-1}} \tag{4.3}$$

holds for all $m \geq 6$. So, if we can prove that for some $\delta > 0$, δ-transversality holds for $\mathcal{B}_{m,I}$ on the interval $\left(m^{-2/3}, x\right)$ for some $x > \sqrt{A(m)}$ then Theorem 3 (ii) implies that $\hat{\eta}_\lambda^m(x) \in L^2$ for a.a. $\lambda \in \left(m^{-2/3}, x\right)$ and by (4.3), in this way we have covered the missing interval $\left(\frac{1}{1+\sqrt{m-1}}, \frac{1}{\sqrt{m}}\right)$. So according to Lemma 1 what we are left to do it is to construct for every $m \in \{6, 7, 9, 11, 13\}$ a (*) function $h_1(x)$ for $\mathcal{B}_{m,I}$ and another (*) function $h_2(x)$ for $\mathcal{B}_{m,\sigma I}$ such that

$$h_i(x) > 0 \text{ and } h_i'(x) < 0, \quad i = 1, 2.$$

Then taking $\delta := \min\{h_i(x), |h_i'(x)|, i = 1, 2\} > 0$ we get from Lemma 1 that there exists a $\tau > 0$ such that τ-transversality holds for the interval $\left(m^{-2/3}, x\right)$. Then Theorem 3 (ii) completes the proof for the m's considered here. Now we construct the required (*) functions for $m \in \{6, 7, 9, 11, 13\}$:

$$h_{1,m}(x) = 1 - 3x + (m-1)\sum_{i \in I} x^i = 1 - 3x + (m-1)\frac{x^3 + x^4}{1 - x^3}$$

and

$$h_{2,m}(x) = 1 - 5x^2 + (m-1) \sum_{i \in \sigma I} x^i = 1 - 5x^2 + (m-1) \frac{x^3 + x^5}{1 - x^3}.$$

Then $h_{1,m}(x)$ is a (*) function for $\mathcal{B}_{m,I}$ and $h_{2,m}(x)$ is a (*) function for $\mathcal{B}_{m,\sigma I}$.

m	$x > \sqrt{A(m)}$	$h_{1,m}(x)$	$h'_{1,m}(x)$	$h_{2,m}(x)$	$h'_{2,m}(x)$
6	0.3357	0.255492	– 0.364468	0.655278	– 1.19327
7	0.311	0.310949	– 0.387734	0.720471	– 0.964625
9	0.274	0.392061	– 0.439015	0.805257	– 0.628602
11	0.247	0.449788	– 0.494282	0.857287	– 0.392609
13	0.227	0.493266	– 0.527637	0.8917	– 0.208441

□

We are left to deal with $m \in \{8, 10, 12, 14, 15\}$. In each of these cases, we use the following observation, which follows easily from Claim 2.

Remark 6.

$$|\hat{\nu}_\lambda^{qm}(x)| \leq |\hat{\nu}_\lambda^{m}(x)|$$

holds for all $q \in \mathbb{N}, q \geq 1$ and all x. Thus $\hat{\nu}_\lambda^m \in L^2$ implies $\hat{\nu}_\lambda^{qm} \in L^2$.

Proof. [The proof of Theorem 2 for $8, 10, 12, 14$.] Let $m \in \{4, 5, 6, 7\}$. Then we have already proved that $\hat{\nu}_\lambda^m \in L^2$ for a.a. $\lambda \in \left(\frac{1}{m}, \frac{1}{\sqrt{m}}\right)$. Using Remark 6 this implies that also

$$\hat{\nu}_\lambda^{2m} \in L^2 \text{ for a.a. } \lambda \in \left(\frac{1}{m}, \frac{1}{\sqrt{m}}\right). \tag{4.4}$$

So what we are left to do it is to prove that ν_λ^{2m} has almost surely L^2 density on the interval $(\frac{1}{2m}, \frac{1}{m})$. However in Corollary 3 we have pointed out that for some $\delta > 0$, δ-transversality hold for $\mathcal{B}_{2m}$ on $\left[0, \frac{1}{m}\right]$. By Theorem 3 (i) this completes the proof. □

Proof. [The proof of Theorem 2 for 15.] We argue exactly as above for $m = 5$ and use Corollary 4. □

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