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ON WHITNEY SETS AND THEIR GENERALIZATION

Abstract

Using results and methods of G. Choquet (1944) and M. Laczkovich and G. Petruska (1984), we slightly generalize their results on "Whitney sets".

1 Introduction.

Let H be a connected subset of $\mathbb{R}^n$. Following Laczkovich and Petruska [3], we say that H is a Whitney set (a W-set) if there is a non-constant function $f : H \to \mathbb{R}$ such that

$$\lim_{x \to x_0, x \in H} \frac{|f(x) - f(x_0)|}{\|x - x_0\|} = 0 \tag{1}$$

holds for every $x_0 \in H$.

The existence of a W-set (with even stronger properties) follows from the well-known example by H. Whitney [1]. G. Choquet [2] constructed a similar example. Moreover, he gave two simple sufficient conditions for a C-set (a connected set that is not a W-set). Namely, he proved that every connected $H \subset \mathbb{R}^n$ with σ-finite 1-dimensional Hausdorff measure is a C-set and the following deeper result.

Theorem 1. *Let f be a continuous real function defined on an interval $I \subset \mathbb{R}$. Then its graph $\{[x, y];\ y = f(x),\ x \in I\}$ is a C-set.*

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Laczkovich and Petruska [3] proved a sufficient condition (based on an easier technique than the Choquet's one) for a curve in $\mathbb{R}^n$ to be a C-set, from which Theorem 1 follows.

Theorem 2. *Let $\varphi : [a,b] \to \mathbb{R}^n$ be continuous and let*

$$E = \Big\{\varphi(x);\ x \in [a,b),\ \lim_{y\to x+} \frac{\|\varphi(y)-\varphi(x)\|}{|y-x|} = \infty\Big\}.$$

If E has σ-finite 1-dimensional Hausdorff measure, then $\varphi([a,b])$ is a C-set.

We will show (Theorem 3) that an easy consequence can be immediately inferred from Theorem 1: if a curve in $\mathbb{R}^{n+1}$ has n components of bounded variations, then the image of the curve is a C-set.

We investigate also more general notions of $W^{(h)}$-sets and $C^{(h)}$-sets in $\mathbb{R}^n$ (see Definition 1), which were already examined (without using our terminology) by Choquet ([2], see Theorem 4 below).

Definition 1. Let $h : [0,\infty) \to [0,\infty)$ be an increasing function with $h(0) = 0$. A connected set $H \subset \mathbb{R}^n$ is said to be a $W^{(h)}$-set, if there is a function $f : H \to \mathbb{R}$ with the following properties

(i) f is not constant,

(ii)
$$\lim_{x\to x_0, x\in H} \frac{f(x)-f(x_0)}{h(\|x-x_0\|)} = 0 \tag{2}$$
holds for every $x_0 \in H$.

A connected subset of $\mathbb{R}^n$ is called a $C^{(h)}$-set if it is not a $W^{(h)}$-set.

If $k \in \mathbb{N}$, we write W^k-set and C^k-set instead of $W^{(h)}$-set and $C^{(h)}$-set, respectively, for $h(t) = t^k$. Thus, W-set and C-set are W^1-set and C^1-set, respectively.

Using methods from [3] we will prove a generalization of our Theorem 3 (see Corollary 3). We will also prove a more general sufficient condition for curves in $\mathbb{R}^n$ to be C^k-sets ($k \in \mathbb{N}$). This condition follows from Proposition 1 which is a natural generalization of Theorem 2 and can be proved analogously.

2 Results.

At first we present the following easy consequence of Theorem 1.

Theorem 3. *Let $\varphi : [\alpha,\beta] \to \mathbb{R}^{n+1}$ ($n \geqq 1,\ -\infty < \alpha < \beta < \infty$) be a curve with n components having bounded variations. Then $\varphi([\alpha,\beta])$ is a C-set.*

Proof. For simplicity we can assume that the first n components have bounded variations and denote $\psi = (\varphi_1, \dots, \varphi_n)$. It is well-known that by changing the variable of φ we may assume that ψ is Lipschitz.

Define the function $g : [\alpha, \beta] \to \mathbb{R}$ by $g(t) = \varphi_{n+1}(t)$. Now if f is a function with the property (1) for $H = \varphi([\alpha, \beta])$ we define the function $\widetilde{f} : \mathrm{graph}(g) \to \mathbb{R}$ as

$$\widetilde{f}(t, g(t)) = f(\varphi(t)).$$

Since the following inequality ($x = (t, g(t))$, $y = (s, g(s))$, we use the l_1-norm)

$$\left| \frac{\widetilde{f}(x) - \widetilde{f}(y)}{\|x - y\|} \right| = \left| \frac{\widetilde{f}(t, g(t)) - \widetilde{f}(s, g(s))}{|t - s| + |g(t) - g(s)|} \right|$$

$$\leqq \left| \frac{f(\varphi(t)) - f(\varphi(s))}{|\psi(t) - \psi(s)| + |\varphi_{n+1}(t) - \varphi_{n+1}(s)|} \right| = \left| \frac{f(\varphi(t)) - f(\varphi(s))}{\|\varphi(t) - \varphi(s)\|} \right|$$

holds, it is not difficult to verify that $\widetilde{f}$ satisfies property (1) for $H = \mathrm{graph}(g)$. Using Theorem 1 we can conclude that $\widetilde{f}$ is constant and thus f is constant. □

The following theorem is contained in [2] with the proof for the case $h(t) = t$. It was also noted that the proof for the general case is quite similar so we will just recapitulate the basic steps of the proof for a general function h. (For the notion of the Hausdorff h measure see [7].)

Theorem 4. *Let h be an increasing function with $h(0) = 0$. If $E \subset \mathbb{R}^n$ has σ-finite Hausdorff h measure, then $f(E)$ is a Lebesgue null set if*

$$\lim_{x \to x_0, x \in E} \frac{f(x) - f(x_0)}{h(\|x - x_0\|)} = 0$$

holds for every $x_0 \in E$.

Proof. Denote by $\mathcal{H}^h$ the Hausdorff h measure. We may clearly assume that there is $\alpha > 0$ such that $\mathcal{H}^h(E) < \alpha < \infty$. Let $\varepsilon > 0$ and define

$$H_n = \left\{ x \in E;\ \left(y \in E \ \&\ 0 < \|y - x\| < \frac{1}{n} \right) \Rightarrow \frac{|f(y) - f(x)|}{h(\|y - x\|)} < \varepsilon \right\}.$$

We have that $H_n \subset H_{n+1}$ for all $n \in \mathbb{N}$.

There is a cover $\{U_{n,i}\}_{i=1}^{\infty}$ of H_n such that $\mathrm{diam}(U_{n,i}) < 1/n$ holds for all $i \in \mathbb{N}$ and $\sum_{i=1}^{\infty} h(\mathrm{diam}(U_{n,i})) \leqq \alpha$. Since for all $x, y \in U_{n,i} \cap H_n$ $|f(x) - f(y)| < \varepsilon h(\|x - y\|)$ holds we get $\mathrm{diam}(f(U_{n,i} \cap H_n)) \leqq \varepsilon h(\mathrm{diam}(U_{n,i} \cap H_n))$. Hence we obtain

$$\lambda^*(f(H_n)) \leqq \sum_{i=1}^{\infty} \lambda^*(f(U_{n,i} \cap H_n)) \leqq \sum_{i=1}^{\infty} \varepsilon h(\mathrm{diam}(U_{n,i} \cap H_n)) \leqq \varepsilon\alpha.$$

Using the regularity of λ^* and $f(H_n) \nearrow f(E)$ we arrive at

$$\lambda^*(f(E)) \leqq \varepsilon\alpha$$

for all $\varepsilon > 0$ which implies $\lambda^*(f(E)) = 0$. □

To prove Theorem 5 we need the following lemma that was inspired by Lemma 1 in [4].

Lemma 1. *Let $\varphi : [\alpha, \beta] \to \mathbb{R}^n$ $(-\infty < \alpha < \beta < \infty)$ be a curve, $L \subset\subset \mathbb{R}^d$, $L \neq \{0\}$, $Z = L^\perp$ and Π_L, Π_Z orthogonal projections onto L, Z. Denote by M the set*

$$\Big\{t_0 \in [\alpha, \beta);\ \exists_{\delta \in (0, \beta - t_0)}\ \forall_{t \in (t_0, t_0 + \delta)}\ \|\Pi_Z(\varphi(t) - \varphi(t_0))\| \geqq \|\Pi_L(\varphi(t) - \varphi(t_0))\|\Big\}.$$

Then $\varphi(M)$ has σ-finite $\dim Z$-dimensional Hausdorff measure.

Proof. Denote

$$M_n = \{t_0 \in M;\ \forall_{t \in (t_0, t_0 + \frac{1}{n}) \cap [\alpha, \beta)}\ \|\Pi_Z(\varphi(t) - \varphi(t_0)\| \geqq \|\Pi_L(\varphi(t) - \varphi(t_0))\|\}$$

and split every M_n into $\{M_{n,m}\}_{m=1}^\infty$ such that $\operatorname{diam}(M_{n,m}) < 1/n$. Then $M = \bigcup_{n=1}^\infty M_n = \bigcup_{n,m=1}^\infty M_{n,m}$ and for every $s, t \in M_{n,m}$ the inequality

$$\|\Pi_Z(\varphi(s) - \varphi(t)\| \geqq \|\Pi_L(\varphi(s) - \varphi(t))\|$$

holds.

Now we can define the function $f_{n,m}$ on the set $\Pi_Z(\varphi(M_{n,m}))$. If $p \in \Pi_Z(\varphi(M_{n,m}))$ then there is precisely one point $x_p \in \varphi(M_{n,m})$ such that $\Pi_Z(x_p) = p$. We define $f_{n,m}(p) = x_p$. If $q \in \Pi_Z(\varphi(M_{n,m}))$ and $x_q \in \varphi(M_{n,m})$ such that $\Pi_Z(x_q) = q$ and $\varphi(s) = x_p$ and $\varphi(t) = x_q$ $(s, t \in M_{n,m})$ then

$$\begin{aligned}\|x_p - x_q\| &= \|\Pi_Z(x_p - x_q) + \Pi_L(x_p - x_q)| \\ &\leqq \|\Pi_Z(\varphi(s) - \varphi(t))\| + \|\Pi_L(\varphi(s) - \varphi(t))\| \leqq 2\|\Pi_Z(\varphi(s)) - \Pi_Z(\varphi(t))\|.\end{aligned}$$

Hence $f_{n,m}$ is Lipschitz. The countable union of $\{f_{n,m}(\Pi_Z(\varphi(M_{n,m})))\}_{n,m=1}^\infty$ (which has σ-finite $\dim Z$-dimensional Hausdorff measure) covers $\varphi(M)$. □

As we already noted, Proposition 1 was stated in [3, Theorem 4] for $h(t) = t$. To prove the proposition, we need to use the following preparatory statements from [3].

Lemma 2. *Let f be continuous on $[a, b]$ and put*

$$L = \{x \in [a, b);\ f'_+(x) > 0\}.$$

Then $\lambda^*(f(L)) \geqq f(b) - f(a)$.

Corollary 1. *If $\lambda(f(L))=0$, then f is decreasing on $[a,b]$.*

Corollary 2. *Let g be continuous and*

$$N=\Big\{x\in[a,b);\ \liminf_{y\to x+}\frac{g(y)-g(x)}{y-x}>0 \text{ or } \limsup_{y\to x+}\frac{g(y)-g(x)}{y-x}<0\Big\}.$$

If $\lambda(f(N))=0$, then f is constant on $[a,b]$.

In the following proof we will recall some basic steps of the proof of Theorem 2 adapted to the more general setting of the Hausdorff h measure.

Proposition 1. *Let $h:[0,\infty)\to[0,\infty)$ be an increasing function with $h(0)=0$, let $\varphi:[a,b]\to\mathbb{R}^n$ be a curve and*

$$E=\Big\{\varphi(x);\ x\in[a,b),\ \lim_{y\to x+}\frac{h(\|\varphi(y)-\varphi(x)\|)}{|y-x|}=\infty\Big\}.$$

If E has σ-finite Hausdorff h measure, then $\varphi([a,b])$ is a $C^{(h)}$-set.

Proof. Denote $H=\varphi([\alpha,\beta])$. Let $g:H\to\mathbb{R}$ satisfy property (2).

Define $f(x)=g(\varphi(x))$ for $x\in[\alpha,\beta]$ and set N as in Corollary 2. If $x\in[\alpha,\beta)$ and $\varphi(x)\notin E$, then $\lim_{y\to x+}\frac{h(\|\varphi(y)-\varphi(x)\|)}{|y-x|}=\infty$ does not hold. Thus there is $K>0$ and a sequence $\{x_n\}_{n=1}^{\infty}$, $x_n>x$, $x_n\to x$ such that $\frac{h(\|\varphi(x_n)-\varphi(x)\|)}{|x_n-x|}<K$. If $\varphi(x_n)\neq\varphi(x)$, then

$$\frac{|f(x_n)-f(x)|}{|x_n-x|}=\frac{|g(\varphi(x_n))-g(\varphi(x))|}{h(\|\varphi(x_n)-\varphi(x)\|)}\,\frac{h(\|\varphi(x_n)-\varphi(x)\|)}{|x_n-x|}.$$

Hence $\frac{|f(x_n)-f(x)|}{|x_n-x|}$ tends to zero as $n\to\infty$ which easily implies $x\notin N$. We obtained $\varphi(N)\subset E$, hence $\lambda^*(f(N))=\lambda^*(g(\varphi(N)))\leqq\lambda^*(g(E))=0$, where the last equality follows from Theorem 4. Corollary 2 gives us the conclusion. $\square$

Now we are ready to prove our main theorem.

Theorem 5. *Let $\varphi:[\alpha,\beta]\to\mathbb{R}^{n+k}$ ($-\infty<\alpha<\beta<\infty$, $n,k\geqq 1$) be a curve. Denote*

$$E=\Big\{t\in[\alpha,\beta);\ \lim_{s\to t+}\frac{\|\varphi(s)-\varphi(t)\|^k}{|s-t|}=\infty\Big\}$$

and suppose that for all $t_0\in E$ there are $\delta>0$ ($t_0+\delta<\beta$), $M>0$ and n natural numbers $1\leqq i_1<\cdots<i_n\leqq n+k$ such that the inequality $|\varphi_{i_l}(t)-\varphi_{i_l}(t_0)|\leqq M|t-t_0|^{1/k}$ ($l=1,\dots,n$) holds for all $t\in(t_0,t_0+\delta)$. Then $\varphi([\alpha,\beta])$ is a C^k-set.

PROOF. As the limit in the set described in the theorem does not depend on the norm in $\mathbb{R}^{n+k}$ (as well as in Theorem 2), we will (without loss of generality) use the l_1-norm.

Let $t_0 \in [\alpha, \beta)$ such that

$$\lim_{t \to t_0+} \frac{\|\varphi(t) - \varphi(t_0)\|^k}{|t - t_0|} = \infty.$$

Due to the assumptions and continuity of φ there are $\delta(t_0) > 0$, $M(t_0) > 0$ and n natural numbers $1 \leqq i_1(t_0) < \cdots < i_n(t_0) \leqq n + k$ such that

- $t_0 + \delta(t_0) < \beta$ and
- for all $t \in (t_0, t_0 + \delta(t_0))$ the inequality

  $$|\varphi_{i_l(t_0)}(t) - \varphi_{i_l(t_0)}(t_0)| \leqq M(t_0)|t - t_0|^{1/k}, l = 1, \ldots, n$$

  holds. (Denote the elements of $\{1, \ldots, n + k\} \setminus \{i_1(t_0), \ldots, i_n(t_0)\}$ by $1 \leqq j_1(t_0) < \cdots < j_k(t_0) \leqq n + k$.)
- for all $t \in (t_0, t_0 + \delta(t_0))$ the inequality $K|t - t_0| \leqq \|\varphi(t) - \varphi(t_0)\|^k$ holds, with $K = M(t_0)^k C^k(n^k + n)$ (C is a positive number such that $(|x_1| + \cdots + |x_{n+1}|)^k \leqq C^k(|x_1|^k + \cdots + |x_{n+1}|^k)$ holds for all $x \in \mathbb{R}^{n+1}$).

Then for $t \in (t_0, t_0 + \delta(t_0))$ we obtain

$$\begin{aligned} K|t - t_0| &\leqq (|\varphi_1(t) - \varphi_1(t_0)| + \cdots + |\varphi_{n+k}(t) - \varphi_{n+k}(t_0)|)^k \\ &\leqq C^k(|\varphi_{i_1(t_0)}(t) - \varphi_{i_1(t_0)}(t_0)|^k + \cdots + |\varphi_{i_n(t_0)}(t) - \varphi_{i_n(t_0)}(t_0)|^k + \\ &\quad + (|\varphi_{j_1(t_0)}(t) - \varphi_{j_1(t_0)}(t_0)| + \cdots + |\varphi_{j_k(t_0)}(t) - \varphi_{j_k(t_0)}(t_0)|)^k) \\ &\leqq C^k n M(t_0)^k |t - t_0| + C^k(|\varphi_{j_1(t_0)}(t) - \varphi_{j_1(t_0)}(t_0)| + \cdots + |\varphi_{j_k(t_0)}(t) - \varphi_{j_k(t_0)}(t_0)|)^k. \end{aligned}$$

Thus

$$\begin{aligned} &|\varphi_{j_1(t_0)}(t) - \varphi_{j_1(t_0)}(t_0)| + \cdots + |\varphi_{j_k(t_0)}(t) - \varphi_{j_k(t_0)}(t_0)| \\ &\geqq |t - t_0|^{1/k} M(t_0) n \geqq |\varphi_{i_1(t_0)}(t) - \varphi_{i_1(t_0)}(t_0)| + \cdots + |\varphi_{i_n(t_0)}(t) - \varphi_{i_n(t_0)}(t_0)|. \end{aligned} \tag{3}$$

Denote the linear subspace of $\mathbb{R}^{n+k}$ generated by vectors $\{e_{i_1(t_0)}, \ldots, e_{i_n(t_0)}\}$ by L and set $Z = L^{\perp}$. Then for $t \in (t_0, t_0 + \delta(t_0))$ (with Π_L, Π_Z orthogonal projections onto L, Z) (3) can be rewritten as

$$\|\Pi_Z(\varphi(t) - \varphi(t_0))\| \geqq \|\Pi_L(\varphi(t) - \varphi(t_0))\|.$$

The union of the sets

$$A_{p_1,\dots,p_n} := \{t_0 \in E;\ p_1 = i_1(t_0), \dots, p_n = i_n(t_0)\}$$

over all combinations of natural numbers $1 \leqq p_1 < \cdots < p_n \leqq n+k$ covers the set E.

By Lemma 1 the set $\varphi(A_{n_1,\dots,n_n})$ has σ-finite k-dimensional Hausdorff measure and so has the set E. Now we can conclude by employing Proposition 1 with $h(t) = t^k$. □

To rewrite a "nonlocal version" of the previous theorem, we need some results on "α-variations" (cf. e.g. [5], [6]).

Definition 2. Let f be defined on $A \subset \mathbb{R}$. For $\alpha \geqq 0$ we denote by $V_\alpha(f, A)$ the least upper bound of the sums

$$\sum_{i=1}^{n} |f(b_i) - f(a_i)|^\alpha,$$

where $\{[a_i, b_i]\}_{i=1}^n$ is an arbitrary finite system of non-overlapping intervals with $a_i, b_i \in A$ $(i = 1, \dots, n)$. We call $V_\alpha(f, A)$ α-variation.

We will need a property that was proved by L. C. Young in Theorem (4.2) of [5]. Without explicitly stating it, he showed that if f is a real function (defined on a compact interval I) of bounded α-th power variation (for $\alpha > 1$), then there is a continuous increasing function h (from I onto itself) such that $f \circ h$ is $1/\alpha$-Hölder.

Now we can state the corollary of Theorem 5.

Corollary 3. *Let $f : [\alpha, \beta] \to \mathbb{R}^{n+k}$ $(n, k \geqq 1)$ be a curve with n components of bounded k-variations. Then $\varphi([\alpha, \beta])$ is a C^k-set.*

Proof. The case of $k = 1$ is handled by Theorem 3 so let $k > 1$. For simplicity (without loss of generality) we can assume that the first n components have bounded k-variations and denote $\psi = (\varphi_1, \dots, \varphi_n)$. We will prove that there is a homeomorphism h of $[\alpha, \beta]$ such that $\psi \circ h$ is $1/k$-Hölder. As remarked above, due to [5] there are continuous increasing functions $h_1, \dots, h_n$ from $[\alpha, \beta]$ onto itself and positive constants K_j such that $|\psi_j(h_j(s)) - \psi_j(h_j(t))| \leqq K_j |s-t|^{1/k}$ holds for all $\alpha \leqq s, t \leqq \beta$ $(j = 1, \dots, n)$.

Define $h := ({h_1}^{-1} + \cdots + {h_n}^{-1})^{-1}$. Let $\alpha \leqq s < t \leqq \beta$ and $j \in \{1, \dots, n\}$.

Then

$$|\psi_j(h(s)) - \psi_j(h(t))| \leqq K_j|{h_j}^{-1}(h(s)) - {h_j}^{-1}(h(t))|^{1/k} =$$
$$K_j|(({h_1}^{-1} + \cdots + {h_n}^{-1}) \circ h_j)^{-1}(s) - (({h_1}^{-1} + \cdots + {h_n}^{-1}) \circ h_j)^{-1}(t)|^{1/k} =$$
$$K_j|(({h_1}^{-1} + \cdots + {h_{j-1}}^{-1} + {h_{j+1}}^{-1} + \cdots + {h_n}^{-1}) \circ h_j + \mathrm{id})^{-1}(s) -$$
$$(({h_1}^{-1} + \cdots + {h_{j-1}}^{-1} + {h_{j+1}}^{-1} + \cdots + {h_n}^{-1}) \circ h_j + \mathrm{id})^{-1}(t)|^{1/k} \leqq K_j|s-t|^{1/k}.$$

Using Theorem 5 for the curve $\varphi \circ h$ we conclude that $\varphi([\alpha, \beta]) = \varphi(h([\alpha, \beta]))$ is a C^k-set. $\square$

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