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ON CLASSES OF FUNCTIONS GENERATING ABSOLUTELY CONTINUOUS VARIATIONAL MEASURES

Abstract

It is proved that for a wide class of bases in $\mathbb{R}^m$ a function generates a σ-finite absolutely continuous variational measure if and only if this function belongs to ACG_δ-class. It is also shown that under some additional assumptions on a basis, σ-finiteness follows from the absolute continuity of the variational measure.

1 Introduction.

Descriptive characterizations of the Henstock-Kurzweil type integrals in terms of absolute continuity of variational measure were given in several recent papers (see [1, 2, 4, 7, 8, 10, 17, 20, 29, 30]). In the case of the one-dimensional full interval basis this characterization is known (see [2]) to be equivalent to the classical descriptive definition of the Denjoy-Perron integral in terms of the ACG^*-class. However the definition of this class relies heavily on the order structure of the real line and so it is difficult to extend this definition to higher dimensions and to more general bases. As an alternative, the notion of ACG_δ-class and its generalization to the case of more general bases was considered. This version of the generalized absolute continuity was introduced by Henstock

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and investigated in [11, 12] in dimension one and in [15, p.58] and in [9] for the regular interval basis in $\mathbb{R}^m$ (see also [28]). The ACG_δ-class for dyadic basis was considered in [13, 3, 30]. In all those cases the class of ACG_δ-functions coincides with the class of indefinite Henstock-Kurzweil integrals with respect to the corresponding basis, which in turn coincides also with the class of functions generating absolutely continuous variational measure with respect to the considered basis. But there exist bases for which ACG_δ-classes are not equivalent to the respective classes of the indefinite integrals (a special case of the so called $\mathcal{P}$-adic basis can be taken as an example, see [5]). So the problem of a direct comparison of the ACG_δ-class and the class of functions generating absolutely continuous variational measure with respect to such bases arises.

In the present paper we prove that for a wide class of bases in $\mathbb{R}^m$ a function generates a σ-finite absolutely continuous variational measure if and only if this function belongs to ACG_δ-class. Moreover under some additional assumptions on a basis, σ-finiteness follows from absolute continuity of the variational measure. So, for such a basis, we obtain that the ACG_δ-class and the respective class of functions generating absolutely continuous variational measures are equivalent. We note that no differentiability assumptions or references to the Ward type theorem are involved in our proof.

2 Preliminaries.

By $\mathbb{R}^m$, $m \geq 1$, we denote the m-dimensional Euclidean space. If $E \subset \mathbb{R}^m$, then $|E|$, $\overline{E}$, ∂E and $Int\ E$ denote the m-dimensional Lebesgue (outer) measure, the closure, the border and the interior of E, respectively. If $|E| = 0$ then the set E is called *negligible.* Almost everywhere ($a.e$) is always used in the sense of the Lebesgue measure. The term "measurable", unless specified otherwise, will refer to measurability in the Lebesgue sense. An open ball of radius r centered at x is denoted by $B(x, r)$.

A *derivation basis* (or simply *basis*) $\mathcal{B}$ in $\mathbb{R}^m$ is a subset of the product space $\mathbb{R}^m \times \Psi$, where Ψ is a fixed class of measurable bounded sets called *generalized intervals* or *$\mathcal{B}$-sets.*

For each positive function δ on $\mathbb{R}^m$, called a *gage*, we set

$$\mathcal{B}_\delta = \{(x, M) \in \mathcal{B} : M \subset B(x, \delta(x))\}. \tag{1}$$

To be exact, the basis (according to [26, 18]) in our case is the collection $\{\mathcal{B}_\delta\}_\delta$, where δ runs over the set of all gages.

If in the definition (1) we assume, that $(x, M) \in \mathcal{B}_\delta$ implies $x \in M$, then the basis $\mathcal{B}$ is called a *Perron basis.* Otherwise we call it a *McShane basis.*

For $E \subset \mathbb{R}^m$ we put $\mathcal{B}_\delta[E] = \{(x, M) \in \mathcal{B}_\delta : x \in E\}$ and call $\left\{\mathcal{B}_\delta[E]\right\}_\delta$ the *restriction of $\mathcal{B}$ on E* or simply the *basis $\mathcal{B}$ on E.* Throughout this paper we

shall assume that:
(a) *basis $\mathcal{B}$ is a Vitali basis, that is $\mathcal{B}_\delta[\{x\}] \neq \varnothing$ for any x and for any gage δ;*
(b) $|\partial M| = 0$ *and $Int\ M \neq \varnothing$ for each $\mathcal{B}$-set M.*

As typical examples of Perron bases satisfying conditions (a) and (b) we consider the *full interval basis*, with Ψ being the set of all nondegenerate closed subintervals of $\mathbb{R}^m$ or of some fixed compact interval $I_0 \subset \mathbb{R}^m$, and the *dyadic basis*, with Ψ being the set of all dyadic intervals $I=[\frac{p_1}{2^{k_1}}, \frac{p_1+1}{2^{k_1}}] \times \ldots \times [\frac{p_m}{2^{k_m}}, \frac{p_m+1}{2^{k_m}}]$, where $k_i = 0,1,2,\ldots,$ $p_i = 0,\ldots,2^{k_i}-1$ $(i=1,\ldots,m)$. Conditions (a) and (b) are still satisfied if we substitute intervals with figures in the above definitions, but condition (b) excludes from our consideration the basis with Ψ consisting of all bounded BV sets (see [19]). Another useful class of bases satisfying (a) and (b) to be mentioned here is the one composed by bases defined by so called local systems (see [24], [3] and [6]). In particular the approximate basis belongs to this class.

A finite subset π of $\mathcal{B}_\delta[E]$ is called *δ-fine partition on E* if for any pairs $(x,M),(y,L)\in\pi$ sets M and L are nonoverlapping, i.e. $Int\ M \cap Int\ L = \varnothing$. Having fixed a set $W\subset\mathbb{R}^m$, a Vitali basis $\mathcal{B}$ on W and a $\mathcal{B}$-set function F we define, for each $E\subset W$,

$$Var\big(\mathcal{B}_\delta, F, E\big) = \sup_{\pi} \sum_{(x,M)\in\pi} |F(M)|,$$

where π is δ-fine partition on E, and we put $V\big(\mathcal{B},F,E\big) = \inf_{\delta} Var\big(\mathcal{B}_\delta,F,E\big)$. Set functions $Var\big(\mathcal{B}_\delta,F,\cdot\,\big)$ and $V\big(\mathcal{B},F,\cdot\,\big)$, being defined on the family of all the subsets $E\subset W$, are called, respectively, the *δ-variation* and the *variational measure on W, generated by F, with respect to the basis $\mathcal{B}$*. It's easy to check that $V\big(\mathcal{B},F,\cdot\,\big)$ is a *metric outer measure*, so it is *σ-additive measure defined on all Borel subsets of W* (see [14, chapter 2, § 11]). Unless specified otherwise, *absolute continuity* of variational measure we understand in the sense of *absolute continuity with respect to Lebesgue measure*. The sentence *"variational measure is σ^*-finite on the set S"* means that $V\big(\mathcal{B},F,\cdot\,\big)$ is σ-finite on S as the outer measure, defined on all the subsets of S, i.e.

$$S = \bigcup_{n=1}^{\infty} S_n \text{ with } V\big(\mathcal{B},F,S_n\big) < +\infty \text{ for each } n. \tag{2}$$

The term *"σ-finite"* we reserve for the case when $V\big(\mathcal{B},F,\cdot\,\big)$ is considered as a measure defined on some σ-ring (or σ-algebra) $\mathcal{A}$, i.e. the condition (2) is valid under additional assumption $S_n\in\mathcal{A}$. Of course σ-finiteness and σ^*-finiteness are equivalent concepts for Borel regular measures. The Borel regularity can be easily established for variational measure related to the full interval basis in $\mathbb{R}$ (it is enough to modify the proof of Theorem 3.15 of [25]). But for a general case and in particular for the mentioned above bases defined by local systems, the problem of regularity seems to be open. That is why we prefer to avoid the assumption of regularity here, the more so that we do not need it in our proofs.

Definition 1. Let $\mathcal{B}$ be a Vitali derivation basis on X. $\mathcal{B}$-set function F is said to be $\mathcal{B}AC_\delta(X)$-*function* if for any $\varepsilon > 0$ there exist a gage $\delta : X \to (0, +\infty)$ and a real number $\eta > 0$ such that $\sum\limits_{i=1}^{p} |F(M_i)| < \varepsilon$ for each δ-fine partition $\{(\xi_i, M_i)\}_{i=1}^{p}$ on the set X satisfying the inequality $\sum\limits_{i=1}^{p} |M_i| < \eta$. $\mathcal{B}$-set function F is said to be $\mathcal{B}ACG_\delta(X)$-*function* if there exist sets X_n such that $X = \bigcup\limits_{n=1}^{+\infty} X_n$ with F being $\mathcal{B}AC_\delta(X_n)$-function for each $n = 1, 2, \ldots$.

3 Absolute Continuity of Variational Measure and $\mathcal{B}ACG_\delta$-Functions.

In this section we compare the class of $\mathcal{B}ACG_\delta$-functions with the class of functions generating absolutely continuous variational measure. We need the following version of the known result (see [21]).

Proposition 1. *Let ν be a finite σ-additive measure defined on σ-ring $\mathcal{R}$ and μ be the outer measure defined on $\mathcal{R}$. Measure ν is absolutely continuous with respect to μ iff for any $\varepsilon > 0$ there exists a number $\delta > 0$ such that $\nu(E) < \varepsilon$ for each set $E \in \mathcal{R}$ with $\mu(E) < \delta$.*

We characterize now $\mathcal{B}ACG_\delta(X)$-functions in terms of variational measures. The next lemma is a crucial step in this direction.

Lemma 1. *Let $\mathcal{B}$ be a derivate basis on $X \subset \mathbb{R}^m$ satisfying conditions (a) and (b), and let F be a $\mathcal{B}$-set function. Then under assumption $V\big(\mathcal{B}, F, X\big) < +\infty$ the variational measure $V\big(\mathcal{B}, F, \cdot\big)$ is absolutely continuous on X iff F is $\mathcal{B}AC_\delta(X)$-function.*

PROOF. The necessity part for the case of the full interval basis was in fact proved in [17]. Following the lines of the proof [17, lemma 6.2], we adjust it for our more general case of the basis, dropping also the assumption of measurability of the set X. Note first that if $|X| = 0$ then $V\big(\mathcal{B}, F, X\big) = 0$ and F is a $\mathcal{B}AC_\delta(X)$-function by the definition. Suppose now that $|X| > 0$ and fix

$\varepsilon > 0$. Since $V(\mathcal{B}, F, X) < +\infty$, there exists a gage $\delta : X \to (0, +\infty)$ such that

$$Var(\mathcal{B}_\delta, F, X) < V(\mathcal{B}, F, X) + \frac{\varepsilon}{3}\,. \tag{3}$$

$V(\mathcal{B}, F, \cdot)$ being a metric outer measure, is σ-additive measure on σ-algebra $\mathfrak{B}_X$ of all Borel sets in the metric space X. Proposition 1 implies that there exists $\eta \in \left(0, \frac{|X|}{2}\right)$ such that

$$V(\mathcal{B}, F, Y) < \frac{\varepsilon}{3} \tag{4}$$

for any set $Y \in \mathfrak{B}_X$ with $|Y| < \eta$. Let $\pi = \{(\xi_i, M_i)\}_{i=1}^p$ be any δ-fine partition on X satisfying condition $\sum\limits_{i=1}^p |M_i| < \eta$. As $|\partial M_i| = 0$ by condition (b), then $\sum\limits_{i=1}^p \left|\overline{M}_i\right| < \eta$. The set $X \cap \left(\bigcup\limits_{i=1}^p \overline{M}_i\right)$ is Borel in X and $\left|X \cap \left(\bigcup\limits_{i=1}^p \overline{M}_i\right)\right| \leqslant$ $\leqslant \sum\limits_{i=1}^p \left|\overline{M}_i\right| < \eta$. We get

$$V\left(\mathcal{B}, F, X \cap \left(\bigcup_{i=1}^p \overline{M}_i\right)\right) \overset{(4)}{<} \frac{\varepsilon}{3}\,. \tag{5}$$

Consider the set $Z = X \setminus \left(\bigcup\limits_{i=1}^p \overline{M}_i\right)$. Since $\eta \in \left(0, \frac{|X|}{2}\right)$, then $|Z| > 0$. Define a gage $\delta_0 : Z \to (0, +\infty)$ having the following properties:
(i) $\delta_0(x) \leqslant \delta(x)$ for each $x \in Z$;
(ii) $B(x, \delta_0(x)) \cap \left(\bigcup\limits_{i=1}^p \overline{M}_i\right) = \varnothing$ for each $x \in Z$;
(iii) $Var\left(\mathcal{B}_{\delta_0}, F, Z\right) < +\infty$ (since $V(\mathcal{B}, F, X) < +\infty$).

By condition (iii) there exists δ_0-fine partition $\mathrm{P} = \{(y_i, L_i)\}_{i=1}^q$ on Z such that

$$\sum_{i=1}^q |F(L_i)| > Var\left(\mathcal{B}_{\delta_0}, F, Z\right) - \frac{\varepsilon}{3}\,. \tag{6}$$

By condition (ii) none of L_i intersect $\bigcup\limits_{i=1}^p M_i$. Then $\mathrm{P} \cup \pi$ is δ-fine partition on X (condition (i)). As $V(\mathcal{B}, F, \cdot)$ is an outer measure we finally get

$$\sum_{i=1}^p |F(M_i)| \overset{(3)}{<} V(\mathcal{B}, F, X) - \sum_{i=1}^q |F(L_i)| + \frac{\varepsilon}{3} \overset{(6)}{<} V(\mathcal{B}, F, X) - Var\left(\mathcal{B}_{\delta_0}, F, Z\right) +$$

$$+ \frac{2\varepsilon}{3} \leqslant V(\mathcal{B}, F, X) - V(\mathcal{B}, F, Z) + \frac{2\varepsilon}{3} \leqslant V\left(\mathcal{B}, F, X \cap \left(\bigcup_{i=1}^p \overline{M}_i\right)\right) + \frac{2\varepsilon}{3} \overset{(5)}{<} \varepsilon.$$

It means that F is $\mathcal{B}AC_\delta(X)$-function.

To prove the sufficiency let E be an arbitrary negligible subset of X. Then for each n it is possible to choose a function $\delta'_n : E \to (0,+\infty)$, a number $\eta_n > 0$ and an open set G_n covering E so that

$$\mu(G_n) < \eta_n \tag{7}$$

and

$$\sum_{i=1}^{p_n} |F(M_i)| < \frac{1}{n} \tag{8}$$

for each δ'_n-fine partition $\pi_n = \{(\xi_i, M_i)\}_{i=1}^{p_n}$ on E satisfying the inequality $\sum\limits_{i=1}^{p_n} |M_i| < \eta_n$. Put $\delta_n(x) = \min\big(\delta'_n(x), \rho(x, \mathbb{R}^m \setminus G_n)\big)$ on E. Then by (7) any δ_n-fine partition $\mathrm{P}_n = \{(\zeta_i, M_i)\}_{i=1}^{s_n}$ on E satisfies the condition $\sum\limits_{i=1}^{s_n} |M_i| < \eta_n$. Thus (8) is valid for P_n. Hence $Var\big(\mathcal{B}_{\delta_n}, F, E\big) \leqslant \frac{1}{n}$, and so $V\big(\mathcal{B}, F, E\big) = 0$. $\square$

Remark 1. Note that the proof of the sufficiency part of Lemma 1 does not require finiteness of $V\big(\mathcal{B}, F, \cdot\big)$ on X. Therefore condition $F \in \mathcal{B}AC_\delta(X)$ always implies absolute continuity of variational measure $V\big(\mathcal{B}, F, \cdot\big)$ on X.

Remark 2. The case of the dyadic basis was considered in [30, Lemma 7].

Theorem 1. *Let $\mathcal{B}$ be a derivate basis on X satisfying conditions (a) and (b) and let F be a $\mathcal{B}$-set function generating a σ^*-finite variational measure $V\big(\mathcal{B}, F, \cdot\big)$. Then $V\big(\mathcal{B}, F, \cdot\big)$ is absolutely continuous on X iff F is $\mathcal{B}ACG_\delta(X)$-function.*

PROOF. σ^*- finiteness of $V\big(\mathcal{B}, F, \cdot\big)$ implies that $X = \bigcup\limits_{n=1}^{\infty} X_n$, with $V\big(\mathcal{B}, F, X_n\big) < +\infty$. If $V\big(\mathcal{B}, F, \cdot\big)$ is absolutely continuous on X, then using Lemma 1 we conclude that F is $\mathcal{B}AC_\delta(X_n)$-function for each n. Hence F is $\mathcal{B}ACG_\delta(X)$-function. Conversely, let F be $\mathcal{B}ACG_\delta(X)$-function. By definition there exists the sequence $\{X_n\}_{n=1}^{\infty}$ such that $X = \bigcup\limits_{n=1}^{\infty} X_n$ and F is $\mathcal{B}AC_\delta(X_n)$-function for each n. By Remark 1 we obtain absolute continuity of the variational measure $V\big(\mathcal{B}, F, \cdot\big)$ on each X_n and therefore, by subadditivity of $V\big(\mathcal{B}, F, \cdot\big)$, on X. $\square$

4 Absolute Continuity Implies σ-Finiteness.

In this section we impose two additional assumptions on basis $\mathcal{B}$:
(c) for each $\mathcal{B}$-set M and for each $x \in M$, the pair (x, M) belongs to $\mathcal{B}$;
(d) each point of ∂M is a point of positive lower density of M.

Bases satisfying condition (c) are sometimes referred to as *Busemann–Feller bases* (abbreviated as *BF-bases*).

We prove now that for a Perron basis defined on a measurable set and satisfying conditions $(a)-(d)$ a priori assumption of σ^*-finiteness in Theorem 1 can be dropped because it follows from the absolute continuity of $V(\mathcal{B}, F, \cdot)$.

Theorem 2. *Let $\mathcal{B}$ be a Perron derivation basis on a closed set E satisfying conditions $(a)-(d)$ and let F be a $\mathcal{B}$-set function. If the variational measure $V(\mathcal{B}, F, \cdot)$ is σ-finite on each negligible Borel set $B \subset E$ then it is σ-finite on E.*

PROOF. Let Q be the set of all points $x \in E$ for which $V(\mathcal{B}, F, \cdot)$ is not σ-finite on $E \cap O$ for any open ball O containing x. Since (E, τ_E) with topology τ_E induced by topology of $\mathbb{R}^m$ is obviously a Lindelöf space, then $V(\mathcal{B}, F, \cdot)$ is σ-finite on $E \setminus Q$. So we are to prove that Q is empty. If not, we can easily deduce that Q is closed and $|Q| > 0$ (for a proof of a similar fact in dimension one see [4, theorem 3.1]). Let P be the set of all density points of Q belonging to Q. In this notation we prove the following lemmas.

Lemma 2. *For each open ball O with $O \cap P \neq \varnothing$,*

$$V(\mathcal{B}, F, P \cap O) = +\infty. \tag{9}$$

PROOF OF LEMMA 2. Using representation

$$E \cap O = (P \cap O) \cup \big((Q \setminus P) \cap O\big) \cup \big((E \setminus Q) \cap O\big)$$

for the ball O, we note that $V(\mathcal{B}, F, \cdot)$ is σ-finite on $(Q \setminus P) \cap O$ (since $Q \setminus P$ is negligible Borel set) and σ-finite on $(E \setminus Q) \cap O$ (because it is σ-finite on $E \setminus Q$). To avoid contradiction with the definition of Q we conclude that $V(\mathcal{B}, F, \cdot)$ is not σ-finite on $P \cap O$. This implies (9). □

Lemma 3. *Let M be a $\mathcal{B}$-set with $P \cap M \neq \varnothing$. Then $P \cap (Int\ M) \neq \varnothing$.*

PROOF OF LEMMA 3. If $P \cap (Int\ M) = \varnothing$, then for any ball O centered at $x \in P \cap \partial M$ we have

$$M \cap O = (\partial M \cap P \cap O) \cup (M \cap (O \setminus P))$$

and so the equality

$$\lim_{diam(O) \to 0} \frac{|M \cap O|}{|O|} = \lim_{diam(O) \to 0} \frac{|\partial M \cap P \cap O|}{|O|} + \lim_{diam(O) \to 0} \frac{|M \cap (O \setminus P)|}{|O|} = 0$$

leads to a contradiction with condition (d) imposed on $\mathcal{B}$-sets. This completes the proof of Lemma 3. □

Proceeding with the proof of Theorem 2, note that Lemma 2 implies $Var\big(\mathcal{B}_\delta, F, P\cap O\big)=+\infty$ for any ball O with $O\cap P\neq\varnothing$ and for any gage δ. Taking the ball O small enough and choosing δ so that all the $\mathcal{B}$-sets from $\mathcal{B}_\delta[O\cap P]$ are contained in O, we can construct a partition $\big\{(x_j^{(1)}, M_j^{(1)})\big\}_{j=1}^{p_1}$ on P such that

$$\sum_{j=1}^{p_1}\big|M_j^{(1)}\big|<\frac{1}{2}\quad\text{ and }\quad\sum_{j=1}^{p_1}\big|F(M_j^{(1)})\big|>2\,.$$

By Lemma 3 we find for each $M_j^{(1)}$ a point $z_j^{(1)}\in P\cap Int\ M_j^{(1)}$. Now considering small enough balls $O_j^{(1)}$ centered at $z_j^{(1)}$ such that $O_j^{(1)}\subset Int\ M_j^{(1)}$ and once again applying Lemma 2 and choosing an appropriate gage, we find a partition $\big\{(x_j^{(2)}, M_j^{(2)})\big\}_{j=1}^{p_2}$ on $\bigcup\limits_{j=1}^{p_1}\big(P\cap O_j^{(1)}\big)$ such that

- each $\overline{M}_j^{(2)}$ is contained in some $O_i^{(1)}\subset M_i^{(1)}$;
- each $O_i^{(1)}$ contains at least two $\overline{M}_j^{(2)}$;
- $\sum\limits_{j=1}^{p_2}\big|M_j^{(2)}\big|<\frac{1}{4}$;
- $\sum\limits_{M_j^{(2)}\subset O_i^{(1)}}\big|F\big(M_j^{(2)}\big)\big|>4$ for each $i=1,\dots,p_1$.

Proceeding by induction we construct a family of partitions $\Big\{\{(x_j^{(k)}, M_j^{(k)})\}_{j=1}^{p_k}\Big\}_{k=1}^{\infty}$ such that

(i) $P\cap M_j^{(k)}\neq\varnothing$;

(ii) each $\overline{M}_j^{(k)}$ is contained in some $M_i^{(k-1)}$;

(iii) each $M_i^{(k-1)}$ contains at least two $\overline{M}_j^{(k)}$;

(iv) $\sum\limits_{j=1}^{p_k}\big|M_j^{(k)}\big|<2^{-k}$;

(v) $\sum\limits_{M_j^{(k)}\subset M_i^{(k-1)}}\big|F\big(M_j^{(k)}\big)\big|>2^k$ for each $i=1,\dots,p_{k-1}$.

We put

$$N=\bigcap_{k=1}^{\infty}\bigcup_{j=1}^{p_k}\overline{M}_j^{(k)}.$$

It follows from (iv) that N is a closed negligible set. Now we can complete the proof by repeating the arguments used in [22, theorem 1], [23, theorem 2] or [4, theorem 3.1]. □

If X is measurable then Theorem 2 allows to drop the condition of σ^*-finiteness of the variational measure in Theorem 1.

Theorem 3. *Let $\mathcal{B}$ be a Perron derivation basis on a measurable set $X \subset \mathbb{R}^m$ satisfying conditions $(a)-(d)$ and F be a $\mathcal{B}$-set function. Then the variational measure $V\big(\mathcal{B}, F, \cdot\big)$ is absolutely continuous on X iff F is $\mathcal{B}ACG_\delta(X)$-function and sets X_n, on which F is $\mathcal{B}AC_\delta(X_n)$-function, can be chosen so that the set X_1 is negligible and the sets X_n, for $n = 2, 3, \ldots,$ are Borel.*

PROOF. The set X can be represented as $X = K \cup X_1$ where K is F_σ-set and $\mu(X_1) = 0$. By Theorem 2 the variational measure $V\big(\mathcal{B}, F, \cdot\big)$ is σ-finite on K, i.e. there are Borel sets X_n such that $V\big(\mathcal{B}, F, X_n\big) < +\infty$ and $K = \bigcup_{n=2}^{\infty} X_n$. By Lemma 1 the function F is $\mathcal{B}AC_\delta(X_n)$-function for all $n = 1, 2, 3, \ldots$. Hence it is $\mathcal{B}ACG_\delta(X)$-function. Sufficiency follows from Remark 1 and subadditivity of variational measure. □

Remark 3. $\mathcal{B}ACG_\delta(X)$-class for the $\mathcal{P}$-adic basis was considered in [5]. It is obvious that the $\mathcal{P}$-adic basis is a Perron basis satisfying conditions $(a)-(d)$. So Theorem 3 gives the answer to the problem stated in [5, p. 586].

We apply now the above results to characterize the multiple Henstock-Kurzweil integral with respect to the full interval basis and to the dyadic basis. Denoting both of these integrals by $\mathcal{HK}_\mathcal{B}$-*integral* we get

Corollary 1. *Let $\mathcal{B}$ be the full interval basis or the dyadic basis defined on the fixed $\mathcal{B}$-interval I_0 and let F be an additive $\mathcal{B}$-interval function. The following statements are equivalent:*
1) *variational measure $V\big(\mathcal{B}, F, \cdot\big)$ is absolutely continuous on I_0;*
2) *F is $\mathcal{B}ACG_\delta(I_0)$–function;*
3) *F is the indefinite $\mathcal{HK}_\mathcal{B}$-integral of its ordinary $\mathcal{B}$-derivative $F'_\mathcal{B}(x)$ (in the sense of Saks, see [21, 30]).*

PROOF. 1)⇔2) follows from Theorem 3. The existence of the ordinary $\mathcal{B}$-derivative follows from Corollary **??** by standard methods using the Ward property (see [4, 10, 16, 27, 30]). The rest of the proof follows the lines of [17, 30]. □

Remark 4. Corollary 1 gives in fact a descriptive definition of $\mathcal{HK}_\mathcal{B}$-integral, which extends the one-dimensional Denjoy-Perron type definition to higher dimensions: a function f defined on a $\mathcal{B}$-interval I_0 is said to be *$\mathcal{HK}_{\mathcal{B}}$-integrable on I_0* if there exists an additive $\mathcal{B}ACG_\delta(I_0)$-function F such that the $\mathcal{B}$-ordinary derivative $F'_{\mathcal{B}}(x) = f(x)$ a.e. on I_0.

Remark 5. As we have already mentioned in the Introduction, for some bases the $\mathcal{B}ACG_\delta$-classes are not equivalent to the respective classes of the indefinite integrals and so the above type of a descriptive definition of the Henstock integral is not available for such bases. The so-called partial descriptive definition (see [19]), however, can be obtained in this case by including into the class of primitives only those $\mathcal{B}ACG_\delta$-functions which are $\mathcal{B}$-differentiable a.e.

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