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ON CAUCHY TYPE CHARACTERIZATIONS OF CONTINUITY AND BAIRE ONE FUNCTIONS

Abstract

In the paper of Lee et al. an equivalent condition for a function f to be of the first Baire class has been established. This condition is of an $\varepsilon - \delta$ type, similarly as in Cauchy's definition of continuity of a function. In the first part of this paper we examine a problem whether it is possible to obtain other classes of functions by further modifications of the above condition. It turns out that, in some sense, the answer is negative. In the second part we consider a topological version of the condition of Lee et al.

1 Introduction.

Assume that (X, d_X) and (Y, d_Y) are metric spaces and $f : X \to Y$. It has been observed in [2] that the following condition is equivalent to the continuity of f:

given $\varepsilon > 0$, there exists a function $\delta : X \to (0, \infty)$ such that for all x_1, x_2 in X,

$$d_X(x_1, x_2) < \max\{\delta(x_1), \delta(x_2)\} \text{ implies that } d_Y(f(x_1), f(x_2)) < \varepsilon. \tag{1}$$

Subsequently it is shown in [2] that, under the assumption that X and Y are complete and separable metric spaces, if we replace the operator max

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by min in the prior condition, then we will get an equivalent definition of a function of the first Baire class.

The following question arises: If we substitute another operator for max in condition (1) – for example, the arithmetic mean or the geometric mean – will we get a new class of functions? We answer this question in the next section.

Let $M : (0,\infty)^2 \to (0,\infty)$. By $\mathcal{A}_M(X,Y)$ (or simply $\mathcal{A}_M$) we will denote the family of all functions $f \in Y^X$ satisfying the following condition:

given $\varepsilon > 0$, there exists a function $\delta : X \to (0,\infty)$ such that for all x_1, x_2 in X,

$$d_X(x_1,x_2) < M(\delta(x_1),\delta(x_2)) \text{ implies that } d_Y(f(x_1),f(x_2)) < \varepsilon. \tag{2}$$

Let $\mathfrak{C}(X,Y)$ (or in short $\mathfrak{C}$) denote the family of all continuous functions from X into Y, and $\mathfrak{B}_1(X,Y)$ (or $\mathfrak{B}_1$) denote the family of all Baire class one functions. In the sequel we will compare the families $\mathcal{A}_M$, $\mathfrak{B}_1$ and $\mathfrak{C}$.

We say that a function $M : (0,\infty)^2 \to (0,\infty)$ is *nondecreasing* if it is nondecreasing with respect to each variable separately.

2 A Comparison of Classes $\mathcal{A}_M$, $\mathfrak{B}_1$ and $\mathfrak{C}$.

Proposition 1. *Let M_1, $M_2 : (0,\infty)^2 \to (0,\infty)$. If for every function $\delta : X \to (0,\infty)$ the functional inequality*

$$M_1(\sigma(x_1),\sigma(x_2)) \le M_2(\delta(x_1),\delta(x_2)) \tag{3}$$

has a solution $\sigma_0 : X \to (0,\infty)$, then $\mathcal{A}_{M_1} \supseteq \mathcal{A}_{M_2}$.

Proof. Indeed, let $f \in \mathcal{A}_{M_2}$. Fix an $\varepsilon > 0$ and consider a function δ satisfying condition (2) with respect to M_2. Then the function σ_0 fulfills the same condition with respect to M_1. Hence $f \in \mathcal{A}_{M_1}$. □

Proposition 2. *For every function $M : (0,\infty)^2 \to (0,\infty)$ there exists a symmetric function $\hat{M}$ such that $\mathcal{A}_M = \mathcal{A}_{\hat{M}}$.*

Proof. For all $s,t \in (0,\infty)$ set

$$\hat{M}(s,t) := \max\{M(s,t), M(t,s)\}.$$

Then $\hat{M}$ is symmetric and $M \le \hat{M}$. Hence and by Proposition 1, $\mathcal{A}_M \supseteq \mathcal{A}_{\hat{M}}$.

Now assume that $f \in \mathcal{A}_M$. Let $\varepsilon > 0$ and let $\delta : X \to (0,\infty)$ be a function satisfying condition (2). Let points $x_1, x_2 \in X$ be such that $d_X(x_1,x_2) < \hat{M}(\delta(x_1),\delta(x_2))$. Then at least one of the following inequalities holds:

$$d_X(x_1,x_2) < M(\delta(x_1),\delta(x_2)) \text{ or } d_X(x_1,x_2) < M(\delta(x_2),\delta(x_1))$$

so, by the symmetry of d_X and d_Y, we may infer that $d_Y(f(x_1), f(x_2)) < \varepsilon$. Hence $f \in \mathcal{A}_{\hat{M}}$. $\square$

In the sequel, in virtue of Proposition 2, we may assume without loss of generality that a function M is always symmetric.

Given $r > 0$, we say that a space (X, d_X) is *r-chainable* if for each $u, v \in X$ there exists a finite sequence $(x_i)_{i=1}^n$ such that

$$x_1 = u, \;\; x_n = v \;\text{ and }\; d_X(x_i, x_{i+1}) < r \;\text{ for all }\; i = 1, \cdots, n-1.$$

Proposition 3. *Assume that $r := \inf\{M(s,t) : s,t > 0\} > 0$. If a space (X, d_X) is r-chainable and $f \in \mathcal{A}_M$, then f is constant.*

Proof. Let $\varepsilon > 0$ and let $\delta : X \to (0, \infty)$ satisfy condition (2). Let points $u, v \in X$ be such that $d_X(u,v) < r$. Then $d_X(u,v) < M(\delta(u), \delta(v))$, so $d_Y(f(u), f(v)) < \varepsilon$. Since ε was arbitrary, we obtain that $f(u) = f(v)$. Now, given $x, z \in X$, there exists a chain $(x_i)_{i=1}^n$ joining x and z such that $d_X(x_i, x_{i+1}) < r$ and then $f(x_i) = f(x_{i+1})$ for all $i = 1, \cdots, n-1$. Thus f is constant. $\square$

Remark 1. Every connected metric space is r-chainable for all $r > 0$ (see, e. g., [1, Exercise 6.1.C]).

Theorem 1. *Let a function $M : (0,\infty)^2 \to (0,\infty)$ be symmetric and nondecreasing. The following statements are equivalent:*

(i) for all metric spaces X and Y, $\mathfrak{C}(X,Y) \subseteq \mathcal{A}_M(X,Y)$;

(ii) $\mathfrak{C}(\mathbb{R},\mathbb{R}) \subseteq \mathcal{A}_M(\mathbb{R},\mathbb{R})$;

(iii) $\lim_{s,t\to 0^+} M(s,t) = 0$.

Proof. Implication (i) $\Rightarrow$(ii) is obvious.

(ii) $\Rightarrow$(iii): Suppose, on the contrary, that condition (iii) does not hold. Then

$$\inf\{M(s,t) : s,t > 0\} = \lim_{s,t\to 0^+} M(s,t) > 0.$$

Since $\mathbb{R}$ is connected, we obtain, by virtue of Proposition 3, that $\mathcal{A}_M$ consists only of constant functions. This contradicts condition (ii).

(iii) $\Rightarrow$(i): Let $f \in \mathfrak{C}(X,Y)$. Fix an $\varepsilon > 0$ and consider the function δ satisfying condition (1). Since $\lim_{s,t\to 0^+} M(s,t) = 0$, given $x \in X$ there exists $\sigma(x) > 0$ such that $M(\sigma(x), \sigma(x)) < \delta(x)$. Then the function σ satisfies the following inequality:

$$M(\sigma(x_1), \sigma(x_2)) < \max\{\delta(x_1), \delta(x_2)\},$$

so by Proposition 1, $\mathcal{A}_M(X,Y) \supseteq \mathcal{A}_{\max}(X,Y) = \mathfrak{C}(X,Y)$. $\square$

Proposition 4. *Let a function $M : (0,\infty)^2 \to (0,\infty)$ be symmetric. If for every $s > 0$, $\inf\{M(s,t) : t > 0\} > 0$, then $\mathcal{A}_M(X,Y) \subseteq \mathfrak{C}(X,Y)$.*

Proof. By virtue of Proposition 1 we only need to show that for every function $\delta : X \to (0,\infty)$ there exists a function $\sigma : X \to (0,\infty)$ such that for all $x_1, x_2 \in X$,

$$\max\{\sigma(x_1), \sigma(x_2)\} \le M(\delta(x_1), \delta(x_2)).$$

For all $x \in X$ set

$$\sigma(x) := \inf\{M(\delta(x), t) : t > 0\}.$$

Then it is easy to verify that the function σ fulfills the above inequality. $\square$

Proposition 5. *Let a function $M : (0,\infty)^2 \to (0,\infty)$ be symmetric and nondecreasing. Let X and Y be complete and separable metric spaces. If there exists an $s_0 > 0$ such that*

$$\lim_{t \to 0^+} M(s_0, t) = 0,$$

then $\mathfrak{B}_1(X,Y) \subseteq \mathcal{A}_M(X,Y)$.

Proof. Fix a function $\delta : X \to (0,\infty)$. For all $x \in X$ set

$$\sigma(x) := \min\{s_0, s\},$$

where s is a positive number such that $M(s_0, s) < \delta(x)$. Then, for all $x_1, x_2 \in X$,

$$M(\sigma(x_1), \sigma(x_2)) \le M(\sigma(x_1), s_0) < \delta(x_1),$$

and analogously $M(\sigma(x_1), \sigma(x_2)) < \delta(x_2)$. Thus σ is a solution of the functional inequality

$$M(\sigma(x_1), \sigma(x_2)) < \min\{\sigma(x_1), \sigma(x_2)\}.$$

Hence and by Proposition 1 we get that $\mathfrak{B}_1(X,Y) \subseteq \mathcal{A}_M(X,Y)$. $\square$

Theorem 2. *Let a function $M : (0,\infty)^2 \to (0,\infty)$ be symmetric and nondecreasing. The following statements are equivalent:*

(i) for all metric spaces X and Y, $\mathcal{A}_M(X,Y) \subseteq \mathfrak{C}(X,Y)$;

(ii) $\mathcal{A}_M(\mathbb{R},\mathbb{R}) \subseteq \mathfrak{C}(\mathbb{R},\mathbb{R})$;

(iii) for every $s > 0$, $\lim_{t \to 0^+} M(s,t) > 0$.

Proof. Implication (i) $\Rightarrow$(ii) is obvious.

(ii) $\Rightarrow$(iii): Suppose, on the contrary, that condition (iii) is not satisfied. Then there exists an $s_0 > 0$ such that $\lim_{t\to 0^+} M(s_0,t) = 0$. By Proposition 5 we obtain that

$$\mathfrak{B}_1(\mathbb{R},\mathbb{R}) \subseteq \mathcal{A}_M(\mathbb{R},\mathbb{R}) \subseteq \mathfrak{C}(\mathbb{R},\mathbb{R})$$

which yields a contradiction.

(iii) $\Rightarrow$(i): Since M is nondecreasing, given $s > 0$ we have that

$$\inf\{M(s,t) : t > 0\} = \lim_{t\to 0^+} M(s,t) > 0.$$

Now Proposition 4 yields that $\mathcal{A}_M(X,Y) \subseteq \mathfrak{C}(X,Y)$. □

As an immediate consequence of Theorems 1 and 2 we get the following

Corollary 1. *Let a function $M : (0,\infty)^2 \to (0,\infty)$ be symmetric and nondecreasing. The following statements are equivalent:*

(i) for all metric spaces X and Y, $\mathcal{A}_M(X,Y) = \mathfrak{C}(X,Y)$;

(ii) $\mathcal{A}_M(\mathbb{R},\mathbb{R}) = \mathfrak{C}(\mathbb{R},\mathbb{R})$;

(iii) $\lim_{s,t\to 0^+} M(s,t) = 0$ and $\lim_{t\to 0^+} M(s,t) > 0$ for all $s > 0$.

In particular, if M is the arithmetic mean, then $\mathcal{A}_M(X,Y) = \mathfrak{C}(X,Y)$ for all metric spaces X and Y.

Theorem 3. *Let a function $M : (0,\infty)^2 \to (0,\infty)$ be symmetric and nondecreasing. The following statements are equivalent:*

(i) for all complete and separable metric spaces X and Y, $\mathfrak{B}_1(X,Y) \subseteq \mathcal{A}_M(X,Y)$;

(ii) $\mathfrak{B}_1(\mathbb{R},\mathbb{R}) \subseteq \mathcal{A}_M(\mathbb{R},\mathbb{R})$;

(iii) there exists an $s_0 > 0$ such that $\lim_{t\to 0^+} M(s_0,t) = 0$.

Proof. Implication (i) $\Rightarrow$(ii) is obvious.

(ii) $\Rightarrow$(iii): Suppose, on the contrary, that condition (iii) is not satisfied. Then, by Theorem 2, we have that $\mathcal{A}_M(\mathbb{R},\mathbb{R}) \subseteq \mathfrak{C}(\mathbb{R},\mathbb{R})$. Since $\mathfrak{B}_1(\mathbb{R},\mathbb{R}) \subseteq \mathcal{A}_M(\mathbb{R},\mathbb{R})$ by (ii), we get that $\mathfrak{B}_1(\mathbb{R},\mathbb{R}) \subseteq \mathfrak{C}(\mathbb{R},\mathbb{R})$ which yields a contradiction.

Finally implication (iii) $\Rightarrow$(i) follows directly from Proposition 5. □

Theorem 4. *Let a function $M : (0,\infty)^2 \to (0,\infty)$ be symmetric and nondecreasing. If X and Y are complete and separable metric spaces, then $\mathcal{A}_M(X,Y) \subseteq \mathfrak{B}_1(X,Y)$.*

PROOF. Fix a function $\delta : X \to (0, \infty)$. For all $x \in X$ set

$$\sigma(x) := M(\delta(x), \delta(x)).$$

Since M is nondecreasing, we have that the function σ satisfies then the inequality

$$\min\{\sigma(x_1), \sigma(x_2)\} \leq M(\delta(x_1), \delta(x_2)) \text{ for all } x_1, x_2 \in X.$$

Hence, by virtue of Proposition 1, $\mathcal{A}_M(X, Y) \subseteq \mathfrak{B}_1(X, Y)$. □

Now Theorems 3 and 4 yield the following

Corollary 2. *Let a function $M : (0, \infty)^2 \to (0, \infty)$ be symmetric and nondecreasing. The following statements are equivalent:*

(i) for all complete and separable metric spaces X and Y, $\mathcal{A}_M(X, Y) = \mathfrak{B}_1(X, Y)$;

(ii) $\mathcal{A}_M(\mathbb{R}, \mathbb{R}) = \mathfrak{B}_1(\mathbb{R}, \mathbb{R})$;

(iii) $\mathcal{A}_M(\mathbb{R}, \mathbb{R}) \supseteq \mathfrak{B}_1(\mathbb{R}, \mathbb{R})$;

(iv) there exists an $s_0 > 0$ such that $\lim_{t \to 0^+} M(s_0, t) = 0$.

In particular, if M is the geometric mean, then $\mathcal{A}_M(X, Y) = \mathfrak{B}_1(X, Y)$ for all complete and separable metric spaces X and Y.

We may summarize the results of this section in the following

Remark 2. Let a function $M : (0, \infty)^2 \to (0, \infty)$ be symmetric and nondecreasing. Let X and Y be metric spaces. The following alternative holds: either

(A) $\lim_{s,t \to 0^+} M(s, t) > 0$ and then $\mathcal{A}_M(X, Y)$ consists only of constant functions if X is connected,

or

(B) $\lim_{s,t \to 0^+} M(s, t) = 0$ and then $\mathcal{A}_M(X, Y) \supseteq \mathfrak{C}(X, Y)$.

Thus only case (B) is interesting. In case (B) we have the next alternative: either

(B1) $\lim_{t \to 0^+} M(s, t) > 0$ for all $s > 0$, and then $\mathcal{A}_M(X, Y) = \mathfrak{C}(X, Y)$,

or

(B2) $\lim_{t\to 0^+} M(s,t) = 0$ for some $s > 0$, and then $\mathcal{A}_M(X,Y) = \mathfrak{B}_1(X,Y)$ if both spaces X and Y are complete and separable.

Therefore none of symmetric and nondecreasing function M can generate a class $\mathcal{A}_M$ which would coincide neither with $\mathfrak{C}$, nor with $\mathfrak{B}_1$. It remains an open question whether we could obtain a reasonable class between $\mathfrak{C}$ and $\mathfrak{B}_1$ by using some non-monotonic function M.

3 A Topological Version of the Condition by Lee et al.

In the sequel we assume that (X,τ) is a topological space, (Y,d) is a metric space, and $f \in Y^X$. $C(f)$ denotes the set of all continuity points of a function f.

First observe that if X is a metric space, then condition (2) with $M := \min$ is equivalent to the following one:

given $\varepsilon > 0$, there exists a family $\{U_x : x \in X\}$ of open subsets of X such that $x \in U_x$ for all $x \in X$, and given $v, w \in X$,

$$v \in U_w \text{ and } w \in U_v \text{ imply that } d(f(v), f(w)) < \varepsilon.$$

This condition can also be considered under the assumption that (X,τ) is a topological space. Let $\mathfrak{B}(X,Y)$ denote the set of all functions from X into Y satisfying that condition.

Similarly let $\mathfrak{B}_*(X,Y)$ denote the set of all functions being the pointwise limits of sequences of continuous functions from X into Y.

We show that $\mathfrak{B}_* \subseteq \mathfrak{B}$. In fact, we can prove even stronger result:

Theorem 5. *Let (X,τ) be a topological space and (Y,d) be a metric space. Let $f_n, f \in Y^X$ and f be a pointwise limit of the sequence $(f_n)_{n=1}^{\infty}$. If*

$$\limsup_{n\to\infty} C(f_n) = X,$$

then $f \in \mathfrak{B}$.

Proof. Since

$$X = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} C(f_k),$$

we obtain that given $x \in X$, there exists a strictly increasing sequence $(k_n(x))_{n=1}^{\infty}$ of positive integers such that all functions $f_{k_n(x)}$ are continuous at the point x for all $n \in \mathbb{N}$.

Fix an $\varepsilon > 0$. Given $x \in X$, let n_x be an integer such that:

$$d(f(x), f_n(x)) < \varepsilon/3 \text{ for all } n \geq n_x;$$
$$f_{n_x} \text{ is continuous at the point x.}$$

The latter condition implies that there exists an open neighbourhood U_x of x such that

$$d(f_{n_x}(x), f_{n_x}(y) < \varepsilon/3 \text{ for all } y \in U_x.$$

Let $x, y \in X$ be such that $x \in U_y$ and $y \in U_x$. Assume, for example, that $n_x \le n_y$. Then

$$d(f(x), f(y)) \le d(f(x), f_{n_y}(x)) + d(f_{n_y}(x), f_{n_y}(y)) + d(f_{n_y}(y), f(y)) < \varepsilon.$$

Hence $f \in \mathfrak{B}$. $\square$

It is an easy excercise to show that if (X, d_X) and (Y, d_Y) are metric spaces and a sequence $(f_n)_{n=1}^{\infty}$ of mappings from X into Y converges uniformly to a mapping f, then f is continuous provided given $x \in X$ infinitely many functions f_n are continuous at x. In the metric setting Theorem 5 and [2, Theorem 1] yield the following counterpart of the above result for Baire one functions.

Corollary 3. *Let X and Y be complete and separable metric spaces. Let $f : X \to Y$ be a pointwise limit of some sequence $(f_n)_{n=1}^{\infty}$ having the property that given $x \in X$, infinitely many functions f_n are continuous at x. Then f is of the first Baire class.*

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