Real Analysis Exchange
Vol. 30(1), 2004/2005, pp. 323–328

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ON THE DUAL SPACE OF BV-INTEGRABLE FUNCTIONS IN EUCLIDEAN SPACE

Abstract

The dual space (with respect to the Alexiewicz norm) of the class of $\mathcal{BV}$-integrable functions on a compact cell $\prod_{i=1}^{m}[a_i, b_i] \subset \mathbb{R}^m$ is shown to be isometrically isomorphic to the space of finite signed Borel measures on $\prod_{i=1}^{m}[a_i, b_i)$, and the usual integral representation theorem holds. An example is also given to show that Lipschitz functions are not part of this dual space. This answers a question of Thierry De Pauw.

1 Introduction.

It is well known that the divergence of a differentiable vector field need not be Lebesgue integrable. In [9] a new extension of the Lebesgue integral (the so-called v-integral in that paper) was introduced to overcome this drawback. This integral, now commonly referred to as the $\mathcal{BV}$-integral, has been studied extensively [1, 2, 3, 4, 10, 11].

Let $\mathcal{R}(E)$ be the space of $\mathcal{BV}$-integrable functions on a compact cell $E := \prod_{i=1}^{m}[a_i, b_i]$ in $\mathbb{R}^m$, and $\|\cdot\|_A$ denotes the Alexiewicz norm [3, Section 3]. In this paper we answer a question of Thierry De Pauw [3, Question 3.1]. More precisely, we prove that the dual of $(\mathcal{R}(E), \|\cdot\|_A)$ is isometrically isomorphic to the space of finite signed Borel measures on $\prod_{i=1}^{m}[a_i, b_i)$, and that the usual integral representation theorem holds. Moreover, we give an example of a real-valued Lipschitz function H on $[0,1]\times[0,1]$ such that H is not contained in the dual of $(\mathcal{R}([0,1]\times[0,1]), \|\cdot\|_A)$.

Key Words: Finite signed Borel measure BV-integral.
Mathematical Reviews subject classification: Primary 46E99; Secondary 26E99.
Received by the editors September 22, 2003
Communicated by: Peter Bullen

2 Main Results.

All integrals will be assumed to be $\mathcal{BV}$-integrals, unless specified otherwise.

In view of the continuity of the indefinite $\mathcal{BV}$-integral, [3, p.196], for each $f \in \mathcal{R}(E)$, we may define a continuous function F_f on E by

$$F_f(x_1,\dots,x_m) := \begin{cases} \int_{\prod_{i=1}^m [x_i,b_i]} f(t_1,\dots,t_m)\ d(t_1,\dots,t_m) & \text{if } a_i \le x_i < b_i \text{ for all } i \in \{1,2,\dots,m\}, \\ 0 & \text{if } x_i = b_i \text{ for some } i \in \{1,2,\dots,m\}. \end{cases}$$

It is easy to check that

$$\frac{1}{2^m}\|f\|_A \le \sup_{(x_1,\dots,x_m)\in E} |F_f(x_1,\dots,x_m)| \le \|f\|_A \tag{1}$$

for each $f \in \mathcal{R}(E)$.

We can now state and prove our integral representation theorem involving the multiple Riemann-Stieltjes integral, [7, p.157].

Lemma 2.1. *T is a bounded linear functional on $(\mathcal{R}(E), \|\cdot\|_A)$ if and only if there exists a unique finite signed Borel measure ν on $\prod_{i=1}^m [a_i,b_i)$ such that*

$$T(f) = \int_E F_f(t_1,\dots,t_m)\ d\big(\nu(\prod_{i=1}^m [t_i,b_i))\big) \tag{2}$$

for each $f \in \mathcal{R}(E)$. Moreover, $\|T\| = \|\nu\|$.

PROOF. Suppose that T is a bounded linear functional on $(\mathcal{R}(E), \|\cdot\|_A)$. For any given $f \in \mathcal{R}(E)$, the function F_f is continuous on E. Hence we may modify the proof of [8, Proposition 3] to obtain a unique finite Borel measure ν on $\prod_{i=1}^m [a_i,b_i)$ such that $T(f)$ can be expressed as a Lebesgue-Stieltjes integral and $\|T\| = \|\nu\|$. More precisely, for each $f \in \mathcal{R}(E)$, we have

$$T(f) = \int_{\prod_{i=1}^m [a_i,b_i)} F_f\ d\nu.$$

It remains to prove that the above Lebesgue-Stieltjes integral can be expressed as a Riemann-Stieltjes integral and (2) holds. Let $[c,d) := [c_1,d_1) \times$

$\cdots \times [c_m, d_m)$, where $c = (c_1, \dots, c_m)$ and $d = (d_1, \dots, d_m)$. If μ is a finite positive Borel measure on $[a, b)$, then the following inequality

$$\big\{\inf_{t\in[u,v)} F_f(t)\big\}\mu\big([u,v)\big) \leq \int_{[u,v)} F_f \; d\mu \leq \big\{\sup_{t\in[u,v)} F_f(t)\big\}\mu\big([u,v)\big) \tag{3}$$

holds whenever $[u, v) \subseteq [a, b)$ and $u_i < v_i$ for all $i = 1, 2, \dots, m$. In particular, (3) holds whenever $\mu = |\nu| - \nu$ or $\mu = |\nu|$. It is not difficult to check that (2) holds. Since the converse is obvious, the proof is complete. $\square$

Our next lemma, which sharpens [3, Lemma 3.2] and [6, Theorem 4.7] can be formulated as

Lemma 2.2. *If T is a bounded linear functional on $\big(\mathcal{L}^1(E), \|\cdot\|_A\big)$, then there exists a finite signed Borel measure ν on $\prod_{i=1}^{m}[a_i, b_i)$ such that*

$$T(f) = \int_E f\Big(\nu\big(\prod_{i=1}^{m}[a_i, \cdot)\big)\Big)$$

for each $f \in \mathcal{L}^1(E)$. Moreover, $\|T\| = \|\nu\|$.

PROOF. By Hahn-Banach Theorem and Lemma 2.1, there exists a finite signed Borel measure ν such that (2) holds for each $f \in \mathcal{L}^1(E)$. For each $(x_1, \dots, x_m) \in E$, we put $g(x_1, \dots, x_m) = \nu\big(\prod_{i=1}^{m}[a_i, x_i)\big)$. Then

$$\begin{aligned} \nu\big(\prod_{i=1}^{m}[x_i, b)\big) &= g(b_1, \dots, b_m) - \sum_i g(b_1, \dots, b_{i-1}, x_i, b_{i+1}, \dots, b_m) \\ &\quad + \sum_{i,j} g(b_1, \dots, b_{i-1}, x_i, b_{i+1}, \dots, b_{j-1}, x_j, b_{j+1}, \dots, b_m) \\ &\quad - \sum_{i,j,k} + \sum_{i,j,k,l} + \cdots + (-1)^{m-1} \sum_k g(x_1, \dots, x_{k-1}, b_k, x_{k+1}, \dots, x_m) \\ &\quad + (-1)^m g(x_1, \dots, x_m). \end{aligned} \tag{4}$$

It follows from (4) and the linearity of the Riemann-Stieltjes integral that (2) still holds when the term $\prod_{i=1}^{m}[t_i, b_i)$ is replaced by $(-1)^m \prod_{i=1}^{m}[a_i, t_i)$. An application of [7, Theorem 7.3.5] and Young's multiple integration by parts formula [12, equation (17)] completes the proof. $\square$

The next theorem, which is a substantial improvement of [3, Theorem 3.1], solves the first part of [3, Question 3.1].

Theorem 2.3. *T is a bounded linear functional on $\big(\mathcal{R}(E), \|\cdot\|_A\big)$ if and only if there exists a finite signed Borel measure ν on $\prod_{i=1}^{m}[a_i, b_i)$ such that $f\Big(\nu\big(\prod_{i=1}^{m}[a_i, \cdot)\big)\Big) \in \mathcal{R}(E)$ and*

$$T(f) = \int_E f\Big(\nu\big(\prod_{i=1}^{m}[a_i, \cdot)\big)\Big)$$

for each $f \in \mathcal{R}(E)$. Moreover, $\|T\| = \|\nu\|$.

PROOF. In view of Lemma 2.1 there exists a unique finite signed Borel measure ν so that (2) holds for each $f \in \mathcal{R}(E)$. An application of Lemma 2.2 shows that

$$T(f) = \int_E f\Big(\nu\big(\prod_{i=1}^{m}[a_i, \cdot)\big)\Big)$$

for each $f \in \mathcal{L}^1(E)$. By using Lemma 2.2 instead of [3, Lemma 3.2] the rest of the proof is similar to that of [3, Theorem 3.1]. The proof is complete. □

The next example answers the second part of [3, Question 3.1] negatively.

Example 2.4. There exists a real-valued Lipschitz function H on $[0,1]\times[0,1]$ such that H is not contained in the dual of $(\mathcal{R}(E), \|\cdot\|_A)$.

PROOF. Define a function h on $[0,1]\times[0,1]$ by

$$h(x,y) = \begin{cases} (1/xy)\sin(1/xy) & \text{if } xy \neq 0, \\ 0 & \text{if } xy = 0. \end{cases}$$

Since $h(\cdot, y) \notin \mathcal{L}^1([0,1])$ for all $y \in (0,1]$, it follows from the Fubini's theorem that $h \notin \mathcal{L}^1([0,1]\times[0,1])$.

For each $x \in [0,1]$ and $0 < \beta \leq 1$, $f(x,\cdot)$ is Lebesgue, and hence $\mathcal{BV}$-integrable on $[\alpha, \beta]$ for all $0 < \alpha < \beta$. An application of [10, Theorem 6.2.1] and [3, Remark 5.1(ii)] shows that $f(x,\cdot)$ is $\mathcal{BV}$-integrable on $[0,\beta]$ and

$$\left| \int_0^\beta h(x,y)\, dy \right| \leq 2\beta.$$

Define a function H on $[0,1]\times[0,1]$ by

$$H(s,t) = \int_0^s \left\{ \int_0^t h(x,y)\, dy \right\} dx.$$

Clearly, $|H(x_0, y_0) - H(x_1, y_0)| \leq 2(x_1 - x_0)y_0$ for each subcell $[x_0, x_1] \times [0, y_0]$ of $[0,1]\times[0,1]$. By Lebesgue's dominated convergence theorem, Fubini's theorem and [10, Theorem 6.2.1], we have

$$|H(x_0, y_0) - H(x_0, y_1)| = \lim_{s\to 0^+} \left| \int_{y_0}^{y_1} \left\{ \int_s^{x_0} h(x, y)\ dx \right\} dy \right| \leq 2x_0(y_1 - y_0)$$

whenever $0 < x_0 \leq 1$ and $0 < y_0 < y_1 \leq 1$. It is now easy to check that H is Lipschitz on $[0,1] \times [0,1]$. Recall that $h \notin \mathcal{L}^1([0,1] \times [0,1])$, so an application of [7, Theorem 7.2.3] shows that H is not contained in the dual of $(\mathcal{R}([0,1] \times [0,1]), \|\cdot\|_A)$. The proof is complete. $\square$

Added in proofs. By using Example 2.4 and [5, Theorem], it is easy to show that Corollary 8.1 and Proposition 8.2 in [3] hold.

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