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A NEW PROOF OF A LIOUVILLE-TYPE THEOREM FOR POLYHARMONIC FUNCTIONS

Abstract

We give a new and simple proof that every polyharmonic function on $\mathbb{R}^n$ which is bounded is constant.

Liouville's theorem states that if a function is holomorphic and bounded on all $\mathbb{C}$, then it is constant. It is also well-known that a similar result holds for harmonic functions: if a function is harmonic and bounded on all $\mathbb{R}^n$, then it is a constant (see [6] or [2, p.31]). More generally, this is true for every polyharmonic function of degree m, that is, a function f with $\Delta^m f = 0$, where $\Delta := \sum_{i=1}^n \partial^2/\partial x_i^2$ is the laplacian. The first proof of this fact was given, it seems, by Nicolesco in 1932 [7, p.136]; there he starts from Pizetti's formula [8, p.182] to get an integral mean value characterization of polyharmonic functions, from which he derives Liouville's theorem.

The aim of this note is to give a short and new proof of this result, assuming the result for harmonic functions and starting again from Pizetti's formula (in itself a little gem which deserves to be better known).

Pizetti's formula. Let U be an open set in $\mathbb{R}^n$, $m \in \mathbb{N}$, $f \in C^{2m}(U)$. Take $x \in U$ and $r > 0$ such that the closed ball with center x and radius r is in U. Write $\mathcal{M}(f,x,r)$ the mean value of f on the sphere with center x and radius r. Then

$$\mathcal{M}(f,x,r) = \sum_{j=0}^{m-1} \Delta^j f(x) \cdot a_j r^{2j} \; + \; R_m(f,x,r),$$

where $a_j := 2^{-2j}\Gamma(n/2)/(j!\,\Gamma(j+n/2))$ and the remainder satisfies

$$|R_m(f,x,r)| \leq \sup_{\|y-x\|\leq r} |\Delta^m f(y)| \cdot a_m r^{2m}.$$

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As Pizetti himself remarks, the formula reduces to a Taylor expansion when f is radial. Moreover the well-known mean value theorem for harmonic functions is a special case.

Proof. Since Pizetti's original proof (for $n = 3$) is not easily accessible, we will give it here, but amended so that it is valid for any $n \geq 3$.

First, take $0 < \varepsilon < r$. Applying Green's formula

$$\int_\Omega (f \cdot \Delta g - g \cdot \Delta f)(z)\, dz = \int_{\partial\Omega} (f \cdot \partial_\nu g - g \cdot \partial_\nu f)(y)\, d\sigma(y)$$

with $g(z) := r^{2-n} - \|z - x\|^{2-n}$ and $\Omega = \Omega_\varepsilon := \{z \in \mathbb{R}^n : \varepsilon < \|z - x\| < r\}$, we get, since $\operatorname{grad} g(z) = (n-2)\|z-x\|^{-n}(z-x)$ and $\Delta g = 0$ on $\mathbb{R}^n \setminus \{x\}$,

$$\begin{aligned} &- \int_{\Omega_\varepsilon} \left(\frac{1}{r^{n-2}} - \frac{1}{\|z-x\|^{n-2}} \right) \Delta f(z)\, dz \\ &= \frac{n-2}{r^{n-1}} \int_{\partial B(x,r)} f(y)\, d\sigma(y) \;-\; \frac{n-2}{\varepsilon^{n-1}} \int_{\partial B(x,\varepsilon)} f(y)\, d\sigma(y) \\ &\quad - \left(\frac{1}{r^{n-2}} - \frac{1}{\varepsilon^{n-2}} \right) \int_{\partial B(x,\varepsilon)} \partial_\nu f(y)\, d\sigma(y). \end{aligned}$$

If we let ε tend to 0, we obtain

$$\lim_{\varepsilon \to 0_+} \frac{n-2}{\varepsilon^{n-1}} \int_{\partial B(x,\varepsilon)} f(y)\, d\sigma(y) = (n-2)\omega_n f(x)$$

(where $\omega_n := \int_{S^{n-1}} d\sigma(y)$) by continuity of f at x, and

$$\lim_{\varepsilon \to 0_+} \left(\frac{1}{r^{n-2}} - \frac{1}{\varepsilon^{n-2}} \right) \int_{\partial B(x,\varepsilon)} \partial_\nu f(y)\, d\sigma(y) = 0$$

because $|\int_{\partial B(x,\varepsilon)} \partial_\nu f(y) d\sigma(y)| \leq \sup_{\|z-x\| \leq r} \|\operatorname{grad} f(z)\| \cdot \omega_n \varepsilon^{n-1}$. So

$$\begin{aligned} &\frac{n-2}{r^{n-1}} \int_{\partial B(x,r)} f(y)\, d\sigma(y) \;-\; (n-2)\omega_n f(x) \\ &= \int_{B(x,r)} \left(\frac{1}{\|z-x\|^{n-2}} - \frac{1}{r^{n-2}} \right) \Delta f(z)\, dz \\ &= \int_0^r \int_{S^{n-1}} \left(\frac{1}{\rho^{n-2}} - \frac{1}{r^{n-2}} \right) \Delta f(x + \rho u)\, \rho^{n-1}\, d\sigma(u)\, d\rho \end{aligned}$$

and therefore

$$\mathcal{M}(f, x, r) = f(x) + \frac{1}{n-2} \int_0^r \left(\rho - \frac{\rho^{n-1}}{r^{n-2}} \right) \mathcal{M}(\Delta f, x, \rho)\, d\rho. \tag{1}$$

Now, given a continuous function φ on $\mathbb{R}_+$, we write $\mathcal{I}_0\varphi := \varphi$ and define inductively $\mathcal{I}_k\varphi$ on $\mathbb{R}_+$ by

$$\mathcal{I}_k\varphi(t) := \frac{1}{n-2}\int_0^t \Big(\rho - \frac{\rho^{n-1}}{t^{n-2}}\Big)\mathcal{I}_{k-1}\varphi(\rho)\,d\rho$$

for $k \in \mathbb{N}$ (note that $\rho - \rho^{n-1}/t^{n-2} \geq 0$ for every $0 \leq \rho \leq t$). When φ is the constant function 1, a straightforward recurrence shows that

$$\mathcal{I}_k 1(t) = \frac{\Gamma(n/2)}{k!\,\Gamma(k+n/2)}\Big(\frac{t}{2}\Big)^{2k} = a_k t^{2k}.$$

Hence, m inductive applications of (1) will give

$$\mathcal{M}(f,x,r) = \sum_{j=0}^{m-1} \Delta^j f(x)\cdot a_j r^{2j} \ + \ \mathcal{I}_m\mathcal{M}(\Delta^m f, x, \rho)(r).$$

The conclusion follows from the estimate $|\mathcal{I}_m\varphi(t)| \leq \sup_{0\leq s\leq t}|\varphi(s)|\cdot\mathcal{I}_m 1(t)$, which is also easily obtained by recurrence. □

Pizetti's formula in $\mathbb{R}^2$ is proved using $g(z) := \ln\|z-x\| - \ln r$.

Theorem 1. *Let $m \in \mathbb{N}$ and $f \in C^\infty(\mathbb{R}^n)$ with $\Delta^m f = 0$ and f bounded on all $\mathbb{R}^n$. Then f is constant.*

Proof. By induction on m. The case $m = 1$ is the classical result for harmonic functions. Suppose then $m \geq 2$ and the assertion true for $m-1$. By Pizetti's formula we have

$$a_{m-1}\Delta^{m-1}f(x) = \mathcal{M}(f,x,r)\cdot r^{2-2m} - \sum_{j=0}^{m-2}\Delta^j f(x)\cdot a_j r^{2j+2-2m}$$

for all $x \in \mathbb{R}^n$ and all $r > 0$. Letting r tend to infinity, we get, since $\mathcal{M}(f,x,r)$ is bounded, $\Delta^{m-1}f(x) = 0$. By the induction hypothesis, f is constant. □

There is a different proof of Pizetti's formula in [3, pp.286–289]. From Pizetti's formula follows the spherical and, by integration, the volume mean value property of harmonic functions, which is the only result used in [6]. Hence, we could say that Pizetti's formula is essentially the only tool necessary to our proof. In contrast, other recent Liouville-type theorems for polyharmonic functions (e.g. [1], [4], [5]), because less elementary than our statement, need several facts: the mean value property of harmonic functions and also the Almansi expansion of polyharmonic functions in all three papers, the analyticity of harmonic functions in [4] and [5], and even some properties of spherical harmonics in [1].

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