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NO TRANSCENDENCE BASIS OF $\mathbb{R}$ OVER $\mathbb{Q}$ CAN BE ANALYTIC

Abstract

It has been proved by Sierpiński that no linear basis of $\mathbb{R}$ over $\mathbb{Q}$ can be an analytic set. Here we show that the same assertion holds by replacing "linear basis" with "transcendence basis". Furthermore, it is demonstrated that purely transcendental subfields of $\mathbb{R}$ generated by Borel bases of the same cardinality are Borel isomorphic (as fields). Following Mauldin's arguments, we also indicate, for each ordinal α such that $1 \leq \alpha < \omega_1$ ($2 \leq \alpha < \omega_1$), the existence of subfields of $\mathbb{R}$ of exactly additive (multiplicative, ambiguous) class α in $\mathbb{R}$.

1 Introduction.

Sierpiński showed in [9] that no linear basis of $\mathbb{R}$ over $\mathbb{Q}$ can be analytic (in particular, Borel). In this note, we prove the same statement for the so-called transcendence bases of $\mathbb{R}$ over $\mathbb{Q}$:

Theorem 1.1. *No transcendence basis of* $\mathbb{R}$ *over* $\mathbb{Q}$ *can be analytic.*[1]

Moreover, suggested by the reading of Le Gac's [6], in Section 3 we give an elementary proof for the following assertion:

Theorem 1.2. *Fields of reals generated by algebraically independent Borel sets of the same cardinality are Borel isomorphic (as fields).*[2]

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[1] The phrases "of $\mathbb{R}$" and "over $\mathbb{Q}$" shall be frequently omitted.

[2] By appealing to a deep result by Kallman [4], Le Gac shows that $\mathbb{Q}$-linear subspaces of $\mathbb{R}$ generated by Borel bases of the same cardinality are Borel isomorphic (as groups). Our approach, depending on Mauldin's [7], does not require Kallman's analysis.

Before proceeding further, let us fix the terminology according to Isaacs's book [3], to which we refer the reader for the necessary elements of field theory needed below.

A set $\mathcal{T} \subseteq \mathbb{R}$ is a transcendence basis if $\mathcal{T}$ is algebraically independent and maximal, in the sense of set-theoretic inclusion (by virtue of Zorn's Lemma, it does exist). Given F a subfield of $\mathbb{R}$, we put $F^* := F \setminus \{0\}$. $\operatorname{alg} F$ is the subfield of $\mathbb{R}$ ([3], theorem 17.5) consisting of the numbers algebraic over F, i.e., the roots of the polynomials in $F[X]$. If $x \in \operatorname{alg} F$, $\deg_F x$ stands for the degree of x over F. S_n denotes the symmetric group on $\{1, \dots, n\}$. Whenever $\mathcal{T}$ is a transcendence basis, $F := \mathbb{Q}(\mathcal{T})$ is a purely transcendental extension of $\mathbb{Q}$ in $\mathbb{R}$ and

$$\mathbb{R} = \operatorname{alg} F = \bigcup_{n=1}^{\infty} F_n, \tag{1}$$

where $F_n := \{x \in \mathbb{R} : \deg_F x \leq n\}$.

We refer the reader to chapter 8 of [1] for the elements of the theory of analytic and borelian subsets of Polish spaces needed below.

2 Proof of Theorem 1.1.

The proof consists in showing that whenever $\mathcal{A}$ is an algebraically independent, analytic set of reals, the field $\operatorname{alg} \mathbb{Q}(\mathcal{A})$ is analytic and of Lebesgue measure zero.[3] In case $\mathcal{A}$ is a transcendence basis, by (1) this clearly leads to the absurd conclusion that $\mathbb{R}$ itself is Lebesgue null.

Suppose $\mathcal{A}$ algebraically independent and of analytic type. Defined, for every $n \in \mathbb{N}$,

$$\mathcal{A}_n := \{(x_1, \dots, x_n) \in \mathcal{A}^n : x_i \neq x_j \text{ for } i \neq j\}$$

($\mathcal{A}^n$ denoting the cartesian product of n copies of $\mathcal{A}$), we have

$$F := \mathbb{Q}(\mathcal{A}) = \bigcup_{n \in \mathbb{N}} \bigcup_{R} R(\mathcal{A}_n), \tag{2}$$

where, for any n, R ranges over all the field $\mathbb{Q}(X_1, \dots, X_n)$ of rational functions in n indeterminates over $\mathbb{Q}$, i.e., $R = P/Q$ with $P, Q \in \mathbb{Q}[X_1, \dots X_n]$, $Q \neq 0$. Note that each R above is well-defined on $\mathcal{A}_n$ and continuous. Consequently, F is analytic, inasmuch as it is the union of denumerably many continuous images of analytic subsets of Polish spaces.

[3]In this connection, we wish to quote a recent result due to Edgar and Miller [2]: the Hausdorff dimension of any analytic, proper subring (in particular, subfield) of $\mathbb{R}$ is 0.

Let us now show that the analyticity of F implies that every F_n (F_n as in (1)) is analytic as well. To this aim, for every $n \in \mathbb{N}$ define

$$P_n : F^n \times F^* \times \mathbb{R} \subseteq \mathbb{R}^{n+2} \to \mathbb{R} \qquad (a_0, \ldots, a_{n-1}, a_n, x) \mapsto \sum_{i=0}^{n} a_i x^i,$$

$$\pi_n : \mathbb{R}^{n+2} \to \mathbb{R} \qquad (a_0, \ldots, a_n, x) \mapsto x.$$

Moreover, put $E_n := \pi_n(P_n^{-1}(\{0\}))$.

Evidently, E_n consists of those reals that are roots of some polynomial in $F[X]$ having degree equal to n. Hence, for every $n \in \mathbb{N}$: $F_n = \bigcup_{i=1}^{n} E_i$.

By applying proposition 8.2.6 in [1] twice (note that, in view of our initial assumption, $F^n \times F^* \times \mathbb{R}$ is analytic in $\mathbb{R}^{n+2}$) we conclude that all the E_n –thus also the F_n– are analytic. A fortiori, Lebesgue measurable ([1], theorem 8.4.1).

It remains to check that each F_n is Lebesgue null: suppose, on the contrary, that there exists a certain F_n with positive Lebesgue measure. Then, by Steinhaus's Theorem (see proposition 1.4.8 in [1]) there must exist $\delta > 0$ such that

$$B(0, \delta) \subseteq \operatorname{diff}(F_n) := \{x - y : x, y \in F_n\},$$

which is in contrast with the following couple, valid for every n:

$$\operatorname{diff}(F_n) \subseteq F_{n^2} \quad \text{and} \quad \overline{\mathbb{R} \setminus F_{n^2}} = \mathbb{R}.$$

Indeed, the former is just a consequence of the elementary algebraic fact:

$$\deg_F x \leq m \quad \text{and} \quad \deg_F y \leq n \quad \Longrightarrow \quad \deg_F(x - y) \leq mn.$$

Concerning the latter, for every $n \in \mathbb{N}$ and $q \in \mathbb{Q}^*$ we have the following:

$$n = \deg_{\mathbb{Q}} \sqrt[n]{2} = \deg_{\mathbb{Q}} q \sqrt[n]{2} = \deg_F q \sqrt[n]{2},$$

due to both Eisenstein's Criterion (theorem 16.21 in [3]) and the fact that F is a purely transcendental extension of $\mathbb{Q}$ –a polynomial that is irreducible over $\mathbb{Q}$ cannot be reduced over any purely transcendental extension of $\mathbb{Q}$: apply this to $X^n - 2 \in \mathbb{Q}[X]$–. Consequently, for every $q \in \mathbb{Q}^*$ and $n \in \mathbb{N}$, letting $N := n^2 + 1$ we have $q \sqrt[N]{2} \in \mathbb{R} \setminus F_{n^2}$.

3 Proof of Theorem 1.2.

It consists in combining and adapting Le Gac's [6] and Mauldin's [7] ideas to the field theoretical case.

Assume that $\mathcal{A}$ and $\mathcal{A}'$ both are algebraically independent, Borel sets in $\mathbb{R}$ such that $\operatorname{card} \mathcal{A} = \operatorname{card} \mathcal{A}' = \mathfrak{c}$ (the case $\operatorname{card} \mathcal{A} = \operatorname{card} \mathcal{A}' \leq \aleph_0$ is obvious and of no interest).[4] On the basis of theorem 8.3.6 in [1], there exists a Borel isomorphism $g : \mathcal{A} \to \mathcal{A}'$. Clearly, this is extended uniquely to a field (algebraic) isomorphism $G : \mathbb{Q}(\mathcal{A}) \to \mathbb{Q}(\mathcal{A}')$. We are going to show that G is a Borel isomorphism as well.

To this aim, we firstly note that for every $n \in \mathbb{N}$ the map g induces a Borel isomorphism $g_n : \mathcal{A}_n \to \mathcal{A}'_n$ defined as follows:

$$g_n(x_1, \ldots, x_n) := (g(x_1), \ldots, g(x_n)).$$

Secondly, we introduce the following definition: we call a set $X \subseteq \mathcal{A}_n$ *transversal* in $\mathcal{A}_n$ if for any $(x_1, \ldots, x_n) \in \mathcal{A}_n$ there exists an unique $\sigma \in S_n$ for which $(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \in X$. For example, the Borel set

$$\mathcal{B}_n := \{(x_1, \ldots, x_n) \in \mathcal{A}_n : x_1 < \ldots < x_n\}$$

is transversal in $\mathcal{A}_n$. We leave to the reader the easy task to prove that, for any $n \in \mathbb{N}$ and any Borel X transversal in $\mathcal{A}_n$, the restriction map $g_{n_{|X}} : X \to X' := g_n(X)$ is a Borel isomorphism, and that X' is transversal in $\mathcal{A}'_n$.

Furthermore, for every $n \in \mathbb{N}$ let us agree to denote with $\mathfrak{R}_n$ the set of all the *proper* rational functions in $\mathbb{Q}(X_1, \ldots, X_n)$, i.e., the set

$$\mathfrak{R}_n := \mathbb{Q}(X_1, \ldots, X_n) \setminus \left(\bigcup_{i=1}^{n} \mathbb{Q}(X_1, \ldots, \widehat{X}_i, \ldots, X_n) \right),$$

the symbol $\widehat{X}_i$ meaning that the indeterminate X_i is omitted.

This done, we may reformulate (2) in this way ($\mathcal{B}_n$ and $\mathcal{B}'_n$ as above):

$$F := \mathbb{Q}(\mathcal{A}) = \mathbb{Q} \bigcup \left(\bigcup_{n \in \mathbb{N}} \bigcup_{R \in \mathfrak{R}_n} R(\mathcal{B}_n) \right) \tag{3}$$

and, analogously,

$$F' := \mathbb{Q}(\mathcal{A}') = \mathbb{Q} \bigcup \left(\bigcup_{n \in \mathbb{N}} \bigcup_{R \in \mathfrak{R}_n} R(\mathcal{B}'_n) \right). \tag{4}$$

Indeed, if we have $z = R(x_1, \ldots, x_n)$ for certain $z \in \mathbb{R}$, $(x_1, \ldots, x_n) \in \mathcal{A}_n$ and $R = R(X_1, \ldots, X_n) \in \mathfrak{R}_n$, then $z = \widetilde{R}(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$, where $\sigma \in S_n$ is such that $(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \in \mathcal{B}_n$ and $\widetilde{R} = \widetilde{R}(X_1, \ldots, X_n) := R(X_{\sigma^{-1}(1)}, \ldots, X_{\sigma^{-1}(n)}) \in \mathfrak{R}_n$.

[4] It is a well-established fact that algebraically independent, uncountable Borel subsets of $\mathbb{R}$ do exist: see [8] or [5], for instance.

Lemma 3.1. *Suppose $\mathcal{A}$ algebraically independent over $\mathbb{Q}$, $\mathcal{X}$ and $\mathcal{Y}$ subsets of $\mathcal{A}$. Then*

$$\mathbb{Q}(\mathcal{X} \cap \mathcal{Y}) = \mathbb{Q}(\mathcal{X}) \cap \mathbb{Q}(\mathcal{Y}).$$

Proof. Put $\mathcal{Z} := \mathcal{X} \cap \mathcal{Y}$ and consider the following chain of equalities:

$$\mathbb{Q}(\mathcal{X}) \cap \mathbb{Q}(\mathcal{Y}) = (\mathbb{Q}(\mathcal{Z})(\mathcal{X} \setminus \mathcal{Y})) \cap (\mathbb{Q}(\mathcal{Z})(\mathcal{Y} \setminus \mathcal{X})) = \mathbb{Q}(\mathcal{Z}).$$

The first one is always true, independently of our assumption on $\mathcal{A}$ (for, obviously, $\mathcal{X} = \mathcal{Z} \cup (\mathcal{X} \setminus \mathcal{Y})$ and $\mathcal{Y} = \mathcal{Z} \cup (\mathcal{Y} \setminus \mathcal{X})$). The second holds inasmuch as $\mathcal{X} \setminus \mathcal{Y}$ and $\mathcal{Y} \setminus \mathcal{X}$ are disjoint and $\mathcal{A} \setminus \mathcal{Z}$ –in particular, the set $(\mathcal{X} \setminus \mathcal{Y}) \cup (\mathcal{Y} \setminus \mathcal{X})$– is algebraically independent over $\mathbb{Q}(\mathcal{Z})$, by lemma 24.6 in [3]. □

Lemma 3.2. *Let $\mathcal{A}$ be algebraically independent. Suppose there exist $R \in \mathfrak{R}_n$ and $S \in \mathfrak{R}_m$, with $(x_1, \ldots, x_n) \in \mathcal{A}_n$ and $(y_1, \ldots, y_m) \in \mathcal{A}_m$ such that $R(x_1, \ldots, x_n) = S(y_1, \ldots, y_m)$. Then, $m = n$, $R = S$ and there exists $\sigma \in S_n$ such that $y_i = x_{\sigma(i)}$ for every $i = 1, \ldots, n$.*

Proof. We have

$$z := R(x_1, \ldots, x_n) = S(y_1, \ldots, y_m) \in \mathbb{Q}(x_1, \ldots, x_n) \cap \mathbb{Q}(y_1, \ldots, y_m).$$

By Lemma 3.1 there exist distinct $z_1, \ldots, z_k \in \mathcal{A}$ and $T \in \mathbb{Q}(X_1, \ldots, X_k)$ such that

$$\{z_1, \ldots, z_k\} = \{x_1, \ldots, x_n\} \cap \{y_1, \ldots y_m\}$$

and

$$z = T(z_1, \ldots, z_k) \in \mathbb{Q}(z_1, \ldots, z_k) = \mathbb{Q}(x_1, \ldots, x_n) \cap \mathbb{Q}(y_1, \ldots, y_m).$$

Up to rearranging the x_i's and the y_i's, we may assume $z_i = x_i = y_i$ for $i = 1, \ldots, k$. Then, from

$$T(x_1, \ldots, x_k) - R(x_1, \ldots, x_n) = 0 = T(y_1, \ldots, y_k) - S(y_1, \ldots, y_m),$$

the algebraic independence of $\mathcal{A}$ and the fact that R and S are proper, we infer both $k = m = n$ and $T = S = R$. □

Lemma 3.3. *Every rational map $R : \mathcal{B}_n \to R(\mathcal{B}_n)$ in (3) is injective. The union in (3) is disjoint. (Identical propositions hold for (4).)*

Proof. Let us assume there exists $z \in \mathbb{R}$ such that $z = R(x_1, \ldots, x_m) = S(y_1, \ldots, y_n)$ for certain $(x_1, \ldots, x_m) \in \mathcal{B}_m$ and $(y_1, \ldots, y_n) \in \mathcal{B}_n$, $R \in \mathfrak{R}_m$ and $S \in \mathfrak{R}_n$. By Lemma 3.2, $m = n$ and $R = S$. By definition of $\mathcal{B}_n$, $y_i = x_i$ for every $i = 1, \ldots, n$. This proves both the assertions. □

In virtue of proposition 8.3.5 and theorem 8.3.7 in [1] and of Lemma 3.3, for any $n \in \mathbb{N}$ and $R \in \mathfrak{R}_n$ the set $R(\mathcal{B}_n)$ turns out to be borelian and Borel isomorphic to $R(\mathcal{B}'_n)$ via the composite map

$$G_{|R(\mathcal{B}_n)} : R(\mathcal{B}_n) \to \mathcal{B}_n \to \mathcal{B}'_n \to R(\mathcal{B}'_n).$$

Hence, we infer that F and F' are both Borel sets, and finally that $G : F \to F'$ is a Borel isomorphism. This concludes the proof.

Incidentally, the existence of algebraically independent, perfect subsets of $\mathbb{R}$ [8], [5] allows us to establish the following

Theorem 3.4. *There is a purely transcendental subfield of $\mathbb{R}$ of exactly additive class 1 in $\mathbb{R}$. For each ordinal α such that $2 \leq \alpha < \omega_1$, there exists a purely transcendental subfield of $\mathbb{R}$ of exactly additive (multiplicative, ambiguous) class α in $\mathbb{R}$.*

Mutatis mutandis, the proof is that of Mauldin: we omit it and refer the reader to theorem 1 in [7].

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