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ON THE HENSTOCK-FUBINI THEOREM FOR MULTIPLE STOCHASTIC INTEGRALS

Abstract

The generalized Riemann (or Henstock) approach to integration is well-known for its explicitness and directness. It has been used to give an alternative definition to the Itô integral and the multiple stochastic integral, see [1, 3, 8, 9, 11, 12, 13, 14]. In this paper we shall derive the Henstock-Fubini's Theorem for multiple stochastic integral based on the Henstock approach. We also show that the iterated multiple integral formula is a direct consequence of Henstock-Fubini's theorem..

1 Introduction.

The theory of multiple stochastic integral was first studied by N. Wiener in 1930. This study was later followed up in greater detail by K. Itô in early 1950's, see [5].

The Riemann approach is well-known for its explicitness and directness in studying integrals. It is impossible to define stochastic integrals using the Riemann approach, since the integrators have paths of unbounded variation, and the integrands are highly oscillatory. The deficiency of the Riemann approach is due to the uniform meshes used in the Riemann sums. Uniform mesh is unable to handle highly oscillatory integrands and integrators.

A way out of this apparent impasse of the Riemann approach was introduced by J. Kurzweil and R. Henstock independently in 1950s. They used non-uniform meshes (meshes that vary from point to point) in the definition of the Riemann-Stieltjes integral. This technically minor but conceptually

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important modification of the classical definition of Riemann leads to the integrals which are more general than the Riemann-Stieltjes integral and the Lebesgue-Stieltjes integral.It has been used to give alternative definitions to the Itô's integral, see [1, 3, 8, 9, 11, 12, 14], and the multiple Wiener integral, see [13]. The Henstock approach gives definitions that are equivalent to the classical integrals.

In this paper, we shall use the generalized Riemann approach (or Henstock approach) to study the general multiple stochastic integral and derive the Henstock-Fubini's Theorem for multiple stochastic integral over non-diagonals for deterministic functions. We shall derive the iterated multiple integral formula as a direct consequence of Henstock-Fubini's Theorem.

2 Setting.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $\mathbb{R} = (-\infty, \infty)$, $T = (a, b]$ and $T^m = (a, b] \times (a, b] \times \cdots \times (a, b]$, that is, m copies of $[a, b]$. An interval $I \subset T^m$ is said to be *left-open* if $I = \prod_{i=1}^{m} (a_i, b_i] \subset T^m$, where each $(a_i, b_i]$ is a left-open interval in T. Let $\mathcal{G}_m$ be the collection of all left-open intervals in T^m.

It is noted that T^m can be decomposed into two parts: the diagonal part $\mathcal{D}$ consisting of

$$\mathcal{D} = \{(x_1, x_2, x_3, \cdots, x_m) \in T^m : x_i = x_j \text{ for some } i \neq j\}$$

and the non-diagonal part $\mathcal{D}^c$ which consists of

$$\mathcal{D}^c = \{(x_1, x_2, \cdots, x_m) : x_i \neq x_j \text{ whenever } i \neq j\}.$$

In addition, the non-diagonal part can be decomposed into $m!$ connected sets contiguous to the diagonal in the following way:

Let S_m be the set of all permutation of m distinct objects. Hence there are $m!$ elements in S_m. For each $\pi \in S_m$, define

$$G_\pi = \{(x_1, x_2, \cdots, x_m) : x_{\pi(1)} < x_{\pi(2)} < x_{\pi(3)} < \cdots < x_{\pi(m)}\}.$$

Note that each set G_π is open in T^m. The $m!$ contiguous sets are disconnected from one another and disjoint from the diagonal.

The multiple stochastic integral over the non-diagonal set was dealt with in [13], which is a generalization of [5]. It was shown in the paper that the Henstock approach gives equivalent definition to the classical multiple Wiener integral.

For each $\pi \in S_m$, we shall denote the projection of G_π to T^p, where $p \leq m$, by $\mathrm{Proj}_p(G_\pi)$, that is,

$$\mathrm{Proj}_p(G_\pi) = \{(x_1, x_2, \cdots, x_p) \in \mathbb{R}^p : (x_1, x_2, \cdots, x_p, x_{p+1}, \cdots, x_m) \in G_\pi\}.$$

Let $I = \prod_{i=1}^{p} I_i = I_1 \times I_2 \times \cdots \times I_p \in \mathcal{G}_p$, where each $I_i \subset \mathbb{R}$ is of the form $(a_i, b_i]$. The interval I is said to be *non-diagonal* if $I \subset \mathrm{Proj}_p(G_\pi)$ for some $\pi \in S_m$.

Definition 2.1. Let $X : \mathcal{G}_p \times \Omega \to \mathbb{R}$ and $Y : \mathcal{G}_q \times \Omega \to \mathbb{R}$ such that $E(X^2(I)) < \infty$ and $E(Y^2(J)) < \infty$ for all $I \in \mathcal{G}_p$, $J \in \mathcal{G}_q$ and $p + q = m$. Then X and Y are said to be *uncorrelated of second order* if

$$E\left(X(I^{(1)})X(I^{(2)})Y(J^{(1)})Y(J^{(2)})\right) = E\left(X(I^{(1)})X(I^{(2)})\right)E\left(Y(J^{(1)})Y(J^{(2)})\right)$$

whenever $I^{(i)} \times J^{(j)} \subset G_\pi$ for some fixed $\pi \in S_m$, $i = 1, 2$, $j = 1, 2$.

Definition 2.2. Let $X : \mathcal{G}_p \times \Omega \to \mathbb{R}$. Then X is said to satisfy the *orthogonal property* if $E\left(X(J)X(K)\right) = 0$ for all disjoint pair of intervals $J, K \in \mathcal{G}_p$ and which are in $\mathrm{Proj}_p(G_\pi)$ for some fixed $\pi \in S_m$.

3 Multiple Stochastic Integral.

We begin this section by defining the non-uniform division of T^m that we shall take.

Definition 3.1. Let δ be a positive function defined on the closure of T^m, which we shall denote by $\overline{T^m}$, $\xi = (\xi_1, \xi_2, \ldots, \xi_m) \in \overline{T^m}$ and $I = \prod_{i=1}^{m} I_i$ be an interval of T^m. An *interval-point pair* (I, ξ) is said to be δ-fine if $I_k \subset [\xi_k - \delta(\xi), \xi_k + \delta(\xi)]$ for each $k = 1, 2, 3, \cdots, m$.

Note that ξ_k may or may not be in I_k for each $k = 1, 2, 3, \cdots, m$. A finite collection D of interval-point pairs $\{(I^{(i)}, \xi^{(i)}) : i = 1, 2, 3, \ldots, n\}$ is said to be a δ-fine division of T^m if

(i) $I^{(i)}$, $i = 1, 2, 3, \ldots, n$, are disjoint left-open intervals of T^m;

(ii) $\bigcup_{i=1}^{n} I^{(i)} = (0, 1]^m$.

We remind the readers that from Definition 3.1, a δ-fine division consists of a δ-fine interval-point pairs $\{(I^{(i)}, \xi^{(i)}) : i = 1, 2, 3, \ldots, n\}$. We remark that for any given positive function δ on T^m, a δ-fine division of T^m exists, which can be proved using continued bisection.

Notation 3.2. Let $f : T^m \times \Omega \to \mathbb{R}$, $Z : \mathcal{G}_m \times \Omega \to \mathbb{R}$ and δ be a given positive function on T^m. We shall denote the Riemann sum $(D) \sum_{i=1}^{n} f(\xi^{(i)}, \omega) Z(I^{(i)}, \omega)$ by $S(f, \delta, D, Z)$ if $D = \{(I^{(i)}, \xi^{(i)}) : i = 1, 2, 3, \ldots, n\}$ is a δ-fine division of T^m.

Definition 3.3. The function $Z : \mathcal{G}_m \times \Omega \to \mathbb{R}$ is said to be *additive* if for each $\omega \in \Omega$,

$$Z(I \cup J, \omega) = Z(I, \omega) + Z(J, \omega)$$

whenever $I, J \in \mathcal{G}_m$ and $I \cup J \in \mathcal{G}_m$ while I and J are disjoint.

Definition 3.4. A function $f : T^m \times \Omega \to \mathbb{R}$ is said to be multiple stochastic integrable to a function $M(f)$ with respect to an additive function $Z : \mathcal{G}_m \times \Omega \to \mathbb{R}$ on the interval T^m if for every $\varepsilon > 0$, there exists a positive function δ on T^m such that

$$E\left(|S(f, \delta, D, Z) - M(f)|^2\right) < \varepsilon$$

whenever $D = \{(I^{(i)}, x^{(i)}) : i = 1, 2, 3, \ldots, n\}$ is a δ- fine division of T^m.

Here Z is the integrator and f is the integrand. $M(f)$ is the primitive function of f with respect to Z on T^m.

Definition 3.5. The integral of f over any subinterval $I \subset T^m$ is as defined in Definition 3.4 above, except that the positive function δ is defined on I instead of the entire T^m.

Lemma 3.6. *(see, for example, [13, Lemma 3.3]) Let δ be a positive function on T^m and $\{D_k\}$ be a finite family of δ-fine divisions of T^m. Then there exists a partition $\{A_1, A_2, \ldots, A_q\}$ of $[0, 1]$ and a finite family of δ-fine divisions of T^m denoted by $\{D'_k\}$ such that each interval of any D'_k is of the form $A_{l_1} \times A_{l_2} \times \cdots \times A_{l_m}$ and each D'_k is a refinement of D_k. Furthermore,*

$$S(f_0, \delta, D_k) = S(f_0, \delta, D'_k)$$

for all k.

Definition 3.7. A finite collection of δ-fine division of T^m of the form D'_k (in Lemma 3.6) is said to be a *standard* δ-fine division of T^m, that is, all the partitions of $\{D'_k\}$ have the same division points on T.

In view of Lemma 3.6, we shall assume that all finite collections of δ-fine divisions of T^m that we consider in Definition 3.4 and 3.5 are all standard divisions.

4 Basic Properties of the Integral.

First we shall state some standard properties of multiple stochastic integrals over T^m. For the first few standard properties, we omit the proofs as they are classical in the theory of Henstock integration theory, see for example[2, 4, 6, 7].

Proposition 4.1. *The multiple stochastic integral of f with respect to Z, if it exists, is unique.*

Proposition 4.2. *Let f and g be multiple stochastic integrable with respect to Z on T^m, and let their integrals be denoted by $M(f)$ and $M(g)$ respectively, and let $k \in \mathbb{R}$ be fixed. Then kf and $f+g$ are integrable, with*

- **(a)** $M(f+g) = M(f) + M(g)$;
- **(b)** $M(kf) = kM(f)$.

Proposition 4.3. *Let f be multiple stochastic integrable with respect to X and Y on T^m, and let their respective integrals be denoted by $M_X(f)$ and $M_Y(f)$ respectively. Then f is multiple stochastic integrable with respect to $X+Y$, and, moreover,*

$$M_{X+Y}(f) = M_X(f) + M_Y(f).$$

Theorem 4.4. *(***Cauchy's Criterion***) A function $f : T^m \times \Omega \to \mathbb{R}$ is multiple stochastic integrable on T^m with respect to Z if and only if given $\varepsilon > 0$, there exists a positive function δ on $\overline{T^m}$ such that*

$$E\left(\left|S(f,\delta,D_1,Z) - S(f,\delta,D_2,Z)\right|^2\right) < \varepsilon$$

whenever D_1, D_2 are standard δ-fine divisions of T^m.

Theorem 4.5. *A stochastic process $f : T^m \times \Omega \to \mathbb{R}$ is a multiple stochastic integrable function on T^m with respect to Z if and only if there exists a sequence $\{\delta_n\}$ of positive functions on T^m, $n = 1, 2, 3, \ldots$, with $\delta_{n+1}(\xi) < \delta_n(\xi)$ for all $n = 1, 2, 3, \ldots$, such that $M(f)$ is the limit of $S(f, \delta_n, D_n, Z)$ under L_2-norm.*

PROOF. Suppose f is multiple stochastic integrable on T^m to $M(f)$. For $n = 1, 2, \ldots$, there exists $\delta_n(\xi) > 0$ on T^m such that the inequality in Definition 3.4 holds with $\varepsilon = \frac{1}{n}$. For each $n = 1, 2, 3, \ldots$, fix a δ_n-fine division D_n. We may assume that $\delta_{n+1}(\xi) < \delta_n(\xi)$ for each n. Hence we have

$$\lim_{n\to\infty} E\left(\left|S(f,\delta_n,D_n,Z) - M(f)\right|^2\right) = 0$$

thereby completing the necessity part of the proof

Conversely, if there exists a decreasing sequence $\{\delta_n(\xi)\}$ of positive functions defined on $\overline{T^m}$ such that for any δ_n-fine division D_n of T^m we have

$$\lim_{n\to\infty} E\left(|S(f,\delta_n,D_n,Z)-M(f)|^2\right)=0.$$

Suppose that f is not multiple stochastic integrable on T^m to $M(f)$. Then there exists $\varepsilon>0$ such that for every positive function δ, there exists a δ-fine division D of T^m such that

$$E\,|S(f,\delta,D,Z)-M(f)|^2\ge\varepsilon.$$

Hence, for each δ_n there exists a δ_n-fine division D_n such that

$$E\,|S(f,\delta_n,D_n,Z)-M(f)|^2\ge\varepsilon,$$

leading to a contradiction. Therefore f is multiple stochastic integrable to $M(f)$. □

5 Henstock-Fubini's Theorem.

Recall in Section 2, we let $\mathcal{G}_s$ to denote the class of all left-open intervals in T^m for any positive integer s. In the remaining part of this section let p and q be two positive integers such that $p+q=m$. Also, let $X:\mathcal{G}_p\times\Omega\to\mathbb{R}$ and $Y:\mathcal{G}_q\times\Omega\to\mathbb{R}$ and $Z(I\times J)=X(I)Y(J)$ for all $I\in\mathcal{G}_p$ and $J\in\mathcal{G}_q$. Further we shall assume that X and Y are uncorrelated of second order (see Definition 2.1) and that both X and Y have the orthogonal properties (see Definition 2.2).

Lemma 5.1. *Let $g:T^p\times\Omega\to\mathbb{R}$ be multiple stochastic integrable on T^p with respect to X. Suppose we let $M(I)$ for each $I\in\mathcal{G}_p$ denote the integral of g with respect to X on each subinterval I, then*

(i) *M has the orthogonal property;*

(ii) *$E\,(X(J)M(I))=0$ whenever I and J are disjoint and in $\mathrm{Proj}_p(G_\pi)$ for a fixed $\pi\in S_m$;*

(iii) *$cX-M$ has the orthogonal property, where c is a real constant; and*

(iv) *Y and $cX-M$ are uncorrelated of second order, where c is a real constant.*

PROOF. Let I and J be disjoint and in $\mathrm{Proj}_p(G_\pi)$. Given $\varepsilon>0$ choose a positive function δ to be the corresponding function as in Definition 3.5 on the intervals I and J, with the corresponding integrals $M(I)$ and $M(J)$. For such a chosen positive function δ it is clear that

$$E\left[S(g,\delta,D(I),X)S(g,\delta,D(J),X)\right]=0,$$

where $D(I)$ and $D(J)$ are δ-fine divisions of I and J respectively. This is because (a) X has the orthogonal property and (b) that if $\{I,J\}$ are disjoint and in $\mathrm{Proj}_p(G_\pi)$, then any pair $\{U,V\}$ of intervals such that $U\subset I$ and $V\subset J$ are disjoint and in $\mathrm{Proj}_p(G_\pi)$.

To prove (i): By Lemma 4.5 and the fact that $E[fh]=\lim_{n\to\infty}E[f_nh_n]$ whenever we have $\lim_{n\to\infty}E(f_n-f)^2=0$ and $\lim_{n\to\infty}E(h_n-h)^2=0$,

$$E\left(M(I)M(J)\right)=\lim_{n\to\infty}E\left[S(g,\delta_n,D(I),X)S(g,\delta_n,D(J),X)\right]=0$$

thereby proving (i).

To prove (ii): By using the similar reasoning as in (i) above,

$$E\left[X(J)M(I)\right]=\lim_{n\to\infty}E\left[X(J)S(g,\delta_n,D(I),X)\right]=0$$

thereby completing the proof of (ii).

To prove (iii): Using (i) and (ii) above, we have

$$\begin{aligned}E\left([cX-M](I)[cX-M](J)\right)&=c^2E\left(X(I)X(J)\right)+E\left(M(I)M(J)\right)\\&\quad-cE\left(X(I)M(J)\right)-cE\left(X(J)M(I)\right)\\&=0.\end{aligned}$$

To prove (iv): Let $I^{(i)}\times J^{(j)}\subset G_\pi$, $i=1,2, j=1,2$. It is also clear that, since X and Y are uncorrelated of second order, we have

$$\begin{aligned}&E\left[Y(I^{(1)})Y(I^{(2)})X(J^{(1)})S(g,\delta_n,D(J^{(2)}),X)\right]\\&=E\left[Y(I^{(i)})Y(I^{(2)})\right]E\left[X(J^{(1)})S(g,\delta_n,D(J^{(2)}),X)\right].\end{aligned}$$

Consequently, $cX-S_n$ and Y are uncorrelated of second order, where

$$S_n(I)=S(g,\delta_n,D(I),X)$$

for any interval I and any positive integer n, where δ_n are given in Lemma

4.5.

$$\begin{aligned}
&E\left\{Y(I^{(1)})Y(I^{(2)})\left(cX(J^{(1)})-M(J^{(1)})\right)\left(cX(J^{(2)})-M(J^{(2)})\right)\right\}\\
&=\lim_{n\to\infty}E\left\{Y(I^{(1)})Y(I^{(2)})\left(cX(J^{(1)})-S_n(J^{(1)})\right)\times\right.\\
&\qquad\qquad\left.\left(cX(J^{(2)})-S_n(J^{(2)})\right)\right\}\\
&=\lim_{n\to\infty}E\left\{Y(I^{(1)})Y^{(2)})\right\}E\left\{\left(cX(J^{(1)})-S_n(J^{(1)})\right)\right.\\
&\qquad\qquad\quad\left.\left(cX(J^{(2)})-S_n(J^{(2)})\right)\right\}\\
&=\lim_{n\to\infty}E\left\{\left(cX(J^{(1)})-S_n(J^{(1)})\right)\left(cX(J^{(2)})-S_n(J^{(2)})\right)\right\}\times\\
&\qquad\qquad E\left\{Y(I^{(1)})Y(I^{(2)})\right\}\\
&=E\left\{\left(cX(J^{(1)})-M(J^{(1)})\right)\left(cX(J^{(2)})-M(J^{(2)})\right)\right\}\times\\
&\qquad\qquad\quad E\left\{Y(I^{(1)})Y(I^{(2)})\right\}
\end{aligned}$$

thereby completing our proof of the entire lemma. $\square$

Lemma 5.2. *Let δ be a positive function on T^p. Then*

$$E\left|(D)\sum_{i=1}^{n}a_iX(I^{(i)})\right|^2=E\left((D)\sum_{i=1}^{n}a_i^2X^2(I^{(i)})\right)$$

for any standard δ-fine partial division $D=\{(I^{(i)},x^{(i)}):i=1,2,\ldots,n\}$ for which all the intervals $I^{(i)}$ and the points $x^{(i)}$ are from $\mathrm{Proj}_p(G_\pi)$. The same result applies to Y on T^q.

PROOF. This follows from the fact that X has orthogonal property (see Definition 2.2). $\square$

Lemma 5.3. *Let δ be a positive function on T^p and δ^1 on T^q, and $p+q=m$. Suppose that $D^1=\{(K^{(j)},y^{(j)}):j=1,2,3,\ldots,r\}$ is a standard δ^1-fine partial division of T^q and $D_j=\{(I_j^{(i)},x_j^{(i)}):i=1,2,3,\ldots,n(j)\}, j=1,2,3,\ldots,r$, are standard δ-fine partial divisions of T^p such that $K^{(j)}\times I_j^{(i)}$ are in G_π for all i,j. Then*

(a) $$\sum_{j=1}^{r}\left(E[Y^2(K^{(j)})]E\left|\sum_{i=1}^{n(j)}a_{ij}X(I_j^{(i)})\right|^2\right)=E\left(\left|\sum_{j=1}^{r}\sum_{i=1}^{n(j)}a_{ij}X(I_j^{(i)})Y(K^{(j)})\right|^2\right)$$

(b) $$\sum_{j=1}^{r}\left(E(Y^2(K^{(j)})E\left|\sum_{i=1}^{n(j)}(a_{ij}X(I_j^{(i)})-M(I_j^{(i)}))\right|^2\right)$$
$$= E\left(\left|\sum_{j=1}^{r}\sum_{i=1}^{n(j)}(a_{ij}X(I_j^{(i)})-M(I_j^{(i)}))Y(K^{(j)})\right|^2\right)$$

where M is as given in Lemma 5.1.

PROOF. To prove Part (a) of the Lemma, we use the fact that X and Y are uncorrelated of second order and that each has orthogonal increment,

$$\begin{aligned}
&\sum_{j=1}^{r} E[Y^2(K^{(j)})]E\left|\sum_{i=1}^{n(j)} a_{ij}X(I_j^{(i)})\right|^2 \\
&= \sum_{j=1}^{r} E(Y^2(K^{(j)}))E\left(\sum_{i=1}^{n(j)} a_{ij}^2 X^2(I_j^{(i)})\right) \\
&= \sum_{j=1}^{r}\sum_{i=1}^{n(j)} a_{ij}^2 E\left(Y^2(K^{(j)})\right)E\left(X^2(I_j^{(i)})\right) \\
&= \sum_{j=1}^{r}\sum_{i=1}^{n(j)} a_{ij}^2 E\left(X^2(I_j^{(i)})Y^2(K^{(j)})\right) \\
&= \sum_{j=1}^{r}\sum_{i=1}^{n(j)} E\left[a_{ij}X(I_j^{(i)})Y(K^{(j)})\right]^2 \\
&= E\left|\sum_{j=1}^{r}\sum_{i=1}^{n(j)} a_{ij}X(I_j^{(i)})Y(K^{(j)})\right|^2 .
\end{aligned}$$

To prove (b) of the same Lemma, from part (iii) of Lemma 5.1 on the orthogonality of $cX - M$, we have

$$E\left(\sum_{i=1}^{n}\left(a_{ij}X(I_j^{(i)})-M(I_j^{(i)})\right)\right)^2 = \sum_{i=1}^{n} E\left(a_{ij}X(I_j^{(i)})-M(I_j^{(i)})\right)^2 .$$

Hence, as in part (a) above,

$$
\begin{aligned}
&\sum_{j=1}^{r}\left(E(Y^2(K^{(j)})E\left|\sum_{i=1}^{n}(a_{ij}X(I_j^{(i)})-M(I_j^{(i)}))\right|^2\right)\\
&=\sum_{j=1}^{r}\left(E(Y^2(K^{(j)}))\sum_{i=1}^{n}E\left(a_{ij}X(I_j^{(i)})-M(I_j^{(i)})\right)^2\right)\\
&=\sum_{j=1}^{r}\sum_{i=1}^{n}\left(E(Y^2(K^{(j)}))E(a_{ij}X(I_j^{(i)})-M(I_j^{(i)}))^2\right)\\
&=\sum_{j=1}^{r}\sum_{i=1}^{n}E\left[Y^2(K^{(j)})\left(a_{ij}X(I_j^{(i)})-M(I_j^{(i)})\right)^2\right]\\
&=E\left(\left|\sum_{j=1}^{r}\sum_{i=1}^{n}(a_{ij}X(I_j^{(i)})-M(I_j^{(i)}))Y(K^{(j)})\right|^2\right)
\end{aligned}
$$

thereby completing the proof. □

Definition 5.4. A function $f : T^q \to \mathbb{R}$ is said to be of *quadratic variation zero* with respect to Y if for any $\varepsilon > 0$ there exists $\delta(\xi) > 0$ on T^q such that

$$\left|(D)\sum_{i=1}^{n} f(y^{(i)})E\left\{Y^2(I^{(i)})\right\}\right| < \varepsilon$$

whenever $D = \{(I^{(i)}, y^{(i)}) : i = 1, 2, \ldots, n\}$ is a standard δ-fine partial division of T^q. A subset $F \subset T^q$ is said to be of *quadratic variation zero with respect to* Y if the indicator function 1_F is of quadratic variation zero with respect to Y.

Lemma 5.5. *Let $f(x) > 0$ on $F \subset T^q$, then the subset F is of quadratic variation zero if and only if $f 1_F$ is of zero quadratic variation.*

PROOF. Let $Q_i = \{x : 0 < f(x) \leq i\}$ for each $i = 1, 2, \ldots$. Then it is easy to verify that Lemma 5.5 holds true for each $F \bigcap Q_i$ for each i. Consequently, we get the required result of Lemma 5.5. □

Definition 5.6. The function $Y : \mathcal{G}_q \times \Omega \to \mathbb{R}$ is said to be of bounded quadratic variation if its quadratic variation, denoted by $V(Y)$, is finite and where

$$V(Y) = \inf_{\delta(\xi)>0} \sup_{D} \sum E(Y^2(I^{(i)}))$$

the supremum is taken over all δ-fine partial division D of T^q and the infimum is taken over all $\delta(\xi) > 0$ defined on T^q.

For the remaining of this section, as we involve stochastic integrals of different integrators, we shall state the integrator on the prefix, for example, the stochastic integral of f with respect to X on T^m will be written as $M_m^X(f)$ to avoid confusion.

Before we prove Henstock-Fubini's Theorem, we need one fundamental lemma:

Lemma 5.7. *(See, for example, [4, p.162 Lemma 17.1]). Let δ be a positive function defined on $T^p \times T^q$. For each $y \in T^q$, define $\delta_{1y}(x) = \frac{1}{\sqrt{2}}\delta(x,y)$ on T^p. For each $y \in T^q$, let*

$$D_1(y) = \{(I_y^{(i)}, x^{(i)}) : i = 1, 2, \ldots, n(y)\}$$

be a δ_{1y}-fine division of T^p. Define

$$\delta_2(y) = \min\left\{\frac{1}{\sqrt{2}}\delta(x^{(i)}, y) : i = 1, 2, \ldots, n(y)\right\}.$$

Then $(I_y^{(i)} \times J, (x^{(i)}, y))$ is δ-fine for each i if (J, y) is δ_2-fine on T^q.

We are now ready to prove the Henstock-Fubini's theorem.

Theorem 5.8. (Henstock-Fubini's Theorem). *Let $T^m = T^p \times T^q$ and $F \subset G_\pi$ for some fixed $\pi \in S_m$. Suppose $f : T^m \to \mathbb{R}$ and that $f1_F$ is multiple stochastic integrable on T^m with respect to Z. Then*

(a) *for each $y \in T^q$, except possibly on a set of quadratic variation zero with respect to Y, $f(\cdot, y)1_F(\cdot, y)$ is multiple stochastic integrable on T^p with respect to X; and*

(b) *$M_m^Z(f1_F) = M_q^Y M_p^X(f1_F)$ if Y has bounded quadratic variation $V(Y)$, where the symbol $M_m^Z(f1_F)$ denotes the multiple stochastic integral of $f1_F$ with respect to the integrator Z on T^m.*

PROOF. The idea of the following proof follows closely that of [4, p.150 Theorem 5.1.2] for classical integration theory.

Given $\varepsilon > 0$ there exists a positive function $\delta(x, y) > 0$ on $T^p \times T^q$ such that

$$E\left(|S(f1_F, \delta, D_1, Z) - S(f1_F, \delta, D_2, Z)|^2\right) < \varepsilon$$

whenever D_1 and D_2 are standard δ-fine divisions of $T^p \times T^q$. Let N be a subset of T^q consisting of $y \in T^q$ such that $f(\cdot, y)1_F(\cdot, y)$ is not integrable on T^p. For each $y \in N$, let $\delta_y(x) = \delta_{1y}(x) = \dfrac{1}{\sqrt{2}}\delta(x,y)$ given as in Lemma 5.7. By Cauchy criteria (Theorem 4.4), for the non-existence of integral of $f(\cdot,y)1_F(\cdot,y)$ on T^p, for each $y \in N$, there exists $Q(y) > 0$ and two standard δ_y-fine divisions of T^p; namely,

$$\begin{aligned} D_1'(y) &= \{(I_y^{(i)}, x_y^{(i)}) : i = 1,2,3,\ldots,s(y)\} \\ D_2'(y) &= \{(J_y^{(i)}, u_y^{(i)}) : i = 1,2,3,\ldots,l(y)\} \end{aligned}$$

such that

$$\begin{aligned} 0 < Q(y) \le E\Bigg(\Big|(D_1'(y))\sum f(x_y^{(i)},y)1_F(x_y^{(i)},y)X(I_y^{(i)}) \\ -(D_2'(y))\sum f(u_y^{(i)},y)1_F(u_y^{(i)},y)X(J_y^{(i)})\Big|^2\Bigg). \end{aligned} \tag{5.1}$$

We may assume that $s(y) = l(y)$, $I_y^{(i)} = J_y^{(i)}$ for all i.

In fact the inequality (1) can be extended to all $y \in T^q \backslash N$ with $Q(y) = 0$ by choosing any δ_y-fine division $D_1'(y) = \{(I_y^{(i)}, x_y^{(i)}) : i = 1,2,\ldots,s(y)\}$ and setting $D_2'(y) = D_1'(y)$ for all $y \in T^q \backslash N$. Next we shall prove that Q is of quadratic variation zero.

Apply Lemma 5.7 to $D_1'(y)$ and $D_2'(y)$ and let

$$\delta_2(y) = \min\left\{\frac{1}{\sqrt{2}}\delta(x^{(i)},y), \frac{1}{\sqrt{2}}\delta(u_y^{(j)},y) : i = 1,2,\ldots,s(y), j = 1,2,\ldots,l(y)\right\}.$$

Denote δ_2 by δ' and let $D_j = \{(K^{(j)}, y^{(j)}) : j = 1,2,\ldots,r\}$ be a standard δ'-fine division of T^q. By Lemma 5.7, $\left(I_{y(j)}^{(i)} \times K^{(j)}, (x_{y(j)}^{(i)}, y^{(j)})\right)$ and $\left(J_{y(j)}^{(i)} \times K^{(j)}, (u_{y(j)}^{(i)}, y^{(j)})\right)$ are δ-fine for all i and j. Since F is contained in one contiguous set, which is open, assume that $I_{y(j)}^{(i)} \times K^{(j)}$ lies completely in G_π whenever $(x_{y(j)}^{(i)}, y^{(j)}) \in F$ for the particular choice of $\delta(\xi)$ on $T^p \times T^q$.

By (1) and using Lemma 5.3(a), we have

$$\begin{aligned}
0 &\le \sum_j \left[Q(y^{(j)})E(Y^2(K^{(j)}))\right] \\
&\le \sum_j E(Y^2(K^{(j)}))E\left(\Big|S(f1_F, \delta_{y(j)}, D_1'(y^{(i)}), X)\right. \\
&\qquad \left. -S(f1_F, \delta_{y(j)}, D_2'(y^{(j)}), X)\Big|^2\right) \\
&= E\left|\sum_j \Big(S(f1_F, \delta_{y(j)}, D_1'(y^{(j)}), X) - S(f1_F, \delta_{y(j)}, D_2'(y^{(j)}), X)\Big) Y(K^{(j)})\right)^2\Bigg| \\
&< \varepsilon,
\end{aligned}$$

thus the function Q is of quadratic variation zero with respect to Y. Hence the set N is of zero quadratic variation with respect to Y.

Next we shall prove (b). Without loss of generality assume that the function given by $M_p^X(f(\cdot,y)1_F(\cdot,y))$ exists for all $y \in T^q$. For each $\varepsilon > 0$ there exists $\delta(x,y) > 0$ on $T^p \times T^q$ such that

$$E\left((D)\left|\sum_i f(x^{(i)}, y^{(i)})1_F(x^{(i)}, y^{(i)})Z(I^{(i)}) - M_m^Z(f1_F)\right|^2\right) < \varepsilon \tag{5.2}$$

whenever $D = \{(I^{(i)}, (x^{(i)}, y^{(i)}) : i = 1, 2, \ldots, n\}$ is a standard δ-fine division of $T^p \times T^q$. For each $y \in T^q$, let δ_{1y} be as given in Lemma 5.7. We may assume that for any δ_{1y}-fine division $D_1(y) = \{(I_y^{(i)}, x_y^{(i)}) : i = 1, 2, \ldots, n(y)\}$ of T^p we have

$$E\left|(D_1(y))\sum f(x_y^{(i)}, y)1_F(x_y^{(i)}, y)X(I_y^{(i)}) - M_p^X(f(\cdot,y)1_F(\cdot,y))\right|^2 < \varepsilon.$$

Let $\delta_2(y)$ be also as that given in Lemma 5.7. By the same Lemma, the division given by $\{(I_y^{(i)} \times K^{(j)}), (x_y^{(i)}, y^{(j)}) : i = 1, 2, \ldots, n(y^{(j)}), j = 1, 2, \ldots, r\}$ is a standard δ-fine division of $T^p \times T^q$ whenever $D_2 = \{(K^{(j)}, y^{(j)}) : j = 1, 2, \ldots, r\}$ is a standard δ_2-fine division of T^q. Therefore

$$E\left(\left|\sum_j \sum_i f(x_{y(j)}^{(i)}, y^{(j)})1_F(x_{y(j)}^{(i)}, y^{(j)})X(I_y^{(i)})Y(K^{(j)}) - M_m^Z(f1_F)\right|^2\right) < \varepsilon.$$

Applying Lemma 5.3(b) with $a_{ij} = f(x^{(i)}_{y(j)}, y^{(j)})1_F(x^{(i)}_{y(j)}, y^{(i)})$, we get

$$\begin{aligned}
&E\left(\left|\sum_j \sum_i f(x^{(i)}_{y(j)}, y^{(j)})1_F(x^{(i)}_{y(j)}, y^{(j)})X(I^{(i)}_{y(j)})\right.\right.\\
&\qquad\qquad \left.\left. - \sum_j M^X_p(f(\cdot, y^{(j)})1_F(\cdot, y^{(j)}))[Y(K^{(j)})]\right|^2\right)\\
&= \sum_j \{E[Y^2(K^{(j)})]E[\sum_i \{f(x^{(i)}_{y(j)}, y^{(i)})1_F(x^{(i)}_{y(j)})X(I^{(i)}_{y(j)})\\
&\qquad\qquad - M^X_p(f(\cdot, y^{(j)})1_F(\cdot, y^{(j)})\}]^2\}\\
&\leq \varepsilon V(Y),
\end{aligned}$$

where $V(Y)$ denotes the quadratic variation of Y, see Definition 5.6 above. Consequently

$$E\left|\sum_j M_p(f(\cdot, y^{(j)}))1_F(\cdot, y^{(j)})Y(K^{(j)}) - M^Z_m(f1_F)\right|^2 \leq 2\varepsilon V(Y) + 2\varepsilon$$

whenever $D_2 = \{(K^{(j)}, y^{(j)}) : j = 1, 2, \ldots, r\}$ is a standard δ_2-fine division of T^q. Hence

$$M^Z_m(f1_F) = M^Y_q M^X_p(f(\cdot, y)1_F(\cdot, y))$$

thereby completing the proof. $\square$

Theorem 5.9. (Iterated Wiener Integral). *Let G_π be an open connected set of T^m mentioned in Theorem 5.8 above. Suppose that $f : T^m \to \mathbb{R}$ and that $f1_{G_\pi}$ is multiple stochastic integrable on T^m with value $M_m(f1_{G_\pi})$. Then*

$$M_m(f1_{G_\pi}) = \int_0^1 \int_0^{t_{\pi(m)}} \ldots \int_0^{t_{\pi(2)}} f(t_1, t_2, \ldots, t_m)dW_{t_{\pi(1)}}dW_{t_{\pi(2)}} \ldots dW_{t_{\pi(m)}}.$$

PROOF. By Henstock-Fubini's Theorem (Theorem 5.8),

$$\begin{aligned}
&M_m(f1_{G_{\pi(m)}})\\
&= M_{m-1}(IM_1(f(t_{\pi(1)},\cdot)1_{G_\pi}(t_{\pi(1)},\cdot))\\
&= M_{m-1}\left(\int_0^1 f(t_{\pi(1)},\cdot)1_{G_\pi}(t_{\pi(1)},\cdot)dX_{t_{\pi(1)}}\right)\\
&= M_{m-1}\left(\int_0^{t_{\pi(2)}} f(t_{\pi(1)},\cdot)1_{G_\pi}(t_{\pi(1)},\cdot)dW_{t_{\pi(1)}}\right)\\
&= M_{m-2}\left(\int_0^{t_{\pi(3)}}\int_0^{t_{\pi(2)}} f(t_{\pi(1)},t_{\pi(2)},\cdot)1_{G_\pi}(t_{\pi(1)},t_{\pi(2)},\cdot)dW_{t_{\pi(1)}}dW_{t_{\pi(2)}}\right)\\
&= \qquad\qquad\qquad\qquad \cdots\\
&= \int_0^1\int_0^{t_{\pi(m)}}\cdots\int_0^{t_{\pi(3)}}\int_0^{t_{\pi(2)}} f(t_{\pi(1)},t_{\pi(2)},\cdots,t_{\pi(m)})\\
&\qquad 1_{G_\pi}(t_{\pi(1)},t_{\pi(2)}\cdots,t_{\pi(m)})dW_{t_{\pi(1)}}dW_{t_{\pi(2)}}\cdots dW_{t_{\pi(m)}}
\end{aligned}$$

thereby completing the proof. □

Remark 5.10. It is known that $f1_{G_\pi}$ is multiple stochastic integrable if and only if it is classical multiple stochastic integrable, see [**13**]. Therefore the above holds true for the classical multiple stochastic integral defined by Wiener-Itô.

6 Conclusion.

We have used Henstock approach to derive the Henstock-Fubini's Theorem for multiple stochastic integral, the idea of which was inspired by the classical integration theory approach. We further remark that Henstock's approach can also be used to study the integral over the diagonal of T^m, and the classical Hu-Meyer theorem, see for example [10], can be derived. This will appear as a paper later

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