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SOME TYPES OF CONVERGENCE AND RELATED BAIRE SYSTEMS

Abstract

We study several kinds of convergence of sequences of real numbers with indices $\overline{m} \in \mathbb{N}^n$, and the respective versions for uniform and pointwise convergence of sequences of real functions. We obtain a theorem about generalized Baire classes of real functions. This theorem is related to a result by Katetov from 1972. We add some comments and applications connected with ideal convergence, a generalization of statistical convergence investigated by several authors.

1 Convergence of Double Sequences.

We denote $\mathbb{N} = \{1, 2, \ldots\}$. Let us consider the following four types of convergence of a double sequence $(x_{mn})_{(m,n)\in\mathbb{N}^2}$ of reals to an $x \in \mathbb{R}$:

$$x_{mn} \to^{(1)} x \iff (\forall\, \varepsilon > 0)(\exists\, k \in \mathbb{N})(\forall\, m \geq k)(\forall n \in \mathbb{N}) \quad |x_{mn} - x| < \varepsilon,$$
$$x_{mn} \to^{(2)} x \iff (\forall\, \varepsilon > 0)(\exists\, k \in \mathbb{N})(\forall\, m \geq k)(\forall\, n \geq k) \quad |x_{mn} - x| < \varepsilon,$$
$$x_{mn} \to^{(3)} x \iff (\forall\, \varepsilon > 0)(\forall\, m \in \mathbb{N})(\exists k \in \mathbb{N})(\forall\, n \geq k) \quad |x_{mn} - x| < \varepsilon,$$
$$x_{mn} \to^{(4)} x \iff (\forall \varepsilon > 0)(\exists k \in \mathbb{N})(\forall m \geq k)(\exists\, p \in \mathbb{N})(\forall\, n \geq p) \quad |x_{mn} - x| < \varepsilon.$$

In each of these cases, x will be called the limit of the sequence $(x_{mn})_{(m,n)\in\mathbb{N}^2}$ with respect to convergence $\to^{(k)}$ where $k \in \{1, 2, 3, 4\}$. Of course, one

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may define three other types of convergence associated with those given above with numbers $1, 2, 4$, where the roles of indices m and n are interchanged. This however is analogous and does not seem of much interest. One can easily check that the limit of a sequence with respect to any convergence $\to^{(k)}$ is unique, provided it exists. Observe that

$$\begin{aligned}(x_{mn} \to^{(1)} x) \Rightarrow (x_{mn} \to^{(2)} x) \Rightarrow (x_{mn} \to^{(4)} x)\\(x_{mn} \to^{(1)} x) \Rightarrow (x_{mn} \to^{(3)} x) \Rightarrow (x_{mn} \to^{(4)} x).\end{aligned} \tag{1}$$

It is not hard to verify that each of (all possible) remaining implications is false, for the respectively chosen x_{mn} and x in $\mathbb{R}$.

The usual convergence $x_{mn} \to x$ means that, for every $\varepsilon > 0$, the set $\{(m, n) \in \mathbb{N}^2 : |x_{mn} - x| \geq \varepsilon\}$ is finite. Obviously $x_{mn} \to x$ implies that $x_{mn} \to^{(1)} x$ but the converse can be false.

Now, if E is a fixed nonempty set, we may consider a pointwise or uniform convergence with index $k \in \{1, 2, 3, 4\}$ of a sequence $(f_{mn})_{(m,n)\in\mathbb{N}^2}$ of functions $f_{mn} : E \to \mathbb{R}$ to a function $f : E \to \mathbb{R}$. This means that we add the quantifier $(\forall\, x \in E)$ at the beginning or at the end of the formula defining $\to^{(k)}$, and $|x_{mn} - x| < \varepsilon$ should be replaced by $|f_{mn}(x) - f(x)| < \varepsilon$. We then write $f_{mn} \to^{(k)} f$ or $f_{mn} \rightrightarrows^{(k)} f$ (on E). Observe that the respective versions of (1) hold if one considers pointwise or uniform convergence.

Proposition 1. *Let $k \in \{1, 2, 3, 4\}$. Assume that $E \neq \emptyset$ and that for $f : E \to \mathbb{R}$, and $f_{mn} : E \to \mathbb{R}$, $(m, n) \in \mathbb{N}^2$, we have $f_{mn} \rightrightarrows^{(k)} f$ on E. Then there is a subsequence of $(f_{mn})_{(m,n)\in\mathbb{N}^2}$ which is uniformly convergent to f on E in the usual sense.*

Proof. Since the version of (1) holds for convergences $\rightrightarrows^{(k)}$, we may prove the assertion only for $k = 4$. Let $f_{mn} \rightrightarrows^{(4)} f$ on E. Thus

$$(\forall\, \varepsilon > 0)(\exists\, k \in \mathbb{N})(\forall\, m \geq k)(\exists\, p \in \mathbb{N})(\forall n \geq p)(\forall\, x \in E)\ \ |f_{mn}(x) - f(x)| < \varepsilon.$$

Consider $\varepsilon = 1/r$, $r \in \mathbb{N}$, and choose $k = k_r$ such that

$$(\forall\, m \geq k)(\exists\, p \in \mathbb{N})(\forall n \geq p)(\forall\, x \in E)\ \ |f_{mn}(x) - f(x)| < \varepsilon.$$

Then pick $m = k_r$ and choose $p = p_r$ such that the respective formula holds. Next pick $n = p_r$ and thus

$$(\forall\, x \in E)\ \ |f_{k_r p_r}(x) - f(x)| < \frac{1}{r}. \tag{2}$$

We can ensure that $k_r < k_{r+1}$ and $p_r < p_{r+1}$ for all $r \in \mathbb{N}$. Now, from (2) it follows that $f_{k_r p_r} \rightrightarrows f$ on E. □

Assume that E is a metric space. From Proposition 1 we conclude that every convergence $\rightrightarrows^{(k)}$ preserves continuity.

2 Generalized Baire classes.

Since the usual pointwise convergence does not preserve continuity, the same holds if one considers each of the pointwise convergences $\to^{(k)}$. It is natural to study the iteration process generated by a given convergence starting from the class of continuous functions, which leads to generalized Baire classes. So, assume that we have an abstract kind of "pointwise" convergence $\to^*$ for sequences of functions from a metric space (X,d) into $\mathbb{R}$. In Section 1 we proposed definitions of convergence for double sequences. Now, let us go further and consider the case of sequences $(f_{\overline{m}})_{\overline{m}\in\mathbb{N}^n}$ where $n\in\mathbb{N}$ is fixed and $f_{\overline{m}}: X\to\mathbb{R}$ for $\overline{m}=(m_1,\dots,m_n)\in\mathbb{N}^n$. Let $\to^*$ be a fixed kind of convergence for sequences of type $(f_{\overline{m}})_{\overline{m}\in\mathbb{N}^n}$.

We define a Baire system $\{B^*_\alpha\}_{\alpha<\omega_1}$ of functions from X to $\mathbb{R}$ as follows. Let B^*_0 consist of all continuous functions from X to $\mathbb{R}$, and for $0<\alpha<\omega_1$ let B^*_α consist of all functions f from X to $\mathbb{R}$ such that there is a sequence $(f_{\overline{m}})_{\overline{m}\in\mathbb{N}^n}$ with $f_{\overline{m}}\to^* f$ and $f_{\overline{m}}\in\bigcup_{\gamma<\alpha}B^*_\gamma$ for each $\overline{m}\in\mathbb{N}^n$. As before $f_{\overline{m}}\to f$ (the usual convergence) means that the set $\{\overline{m}\in\mathbb{N}^n : |f_{\overline{m}}(x)-f(x)|\geq\varepsilon\}$ is finite for any $\varepsilon>0$ and $x\in X$. We shall always assume that, for all $f_{\overline{m}}$ and f, we have

$$(f_{\overline{m}}\to f)\Rightarrow(f_{\overline{m}}\to^* f), \tag{3}$$

which means that the convergence is weaker than $\to$. The family $\{B_\alpha\}_{\alpha<\omega_1}$ will stand for the classical Baire system (generated by $\to$). From (3) it follows that $B_\alpha\subset B^*_\alpha$ for all $\alpha<\omega_1$ (clearly $B_0=B^*_0$). We will study the question how much B^*_α can be bigger than B_α. Note that the definition of $\{B_\alpha\}_{\alpha<\omega_1}$ does not depend on the choice of n since the condition defining $f_{\overline{m}}\to f$, $\overline{m}\in\mathbb{N}^n$, remains true if we renumber $\mathbb{N}^n$ into $\mathbb{N}^r$ (via a bijection), for any $r\in\mathbb{N}$. Hence the classical system $\{B_\alpha\}_{\alpha<\omega_1}$ may be considered with pointwise convergence of single sequences of type $(f_k)_{k\in\mathbb{N}}$.

Let $\{B^{(k)}_\alpha\}_{\alpha<\omega_1}$ denote the Baire system generated by pointwise convergence $\to^{(k)}$ $(k=1,2,3,4)$ defined in Section 1.

Proposition 2. *For any $k\in\{1,2,3\}$ and $\alpha<\omega_1$ we have $B^{(k)}_\alpha=B_\alpha$.*

Proof. We have $B_\alpha\subset B^{(k)}_\alpha$ for all $k\in\{1,2,3\}$ and $\alpha<\omega_1$. Thus by (1) it suffices to consider the cases $k=2$ and $k=3$, and show that $B^{(k)}_\alpha=B_\alpha$ for all $\alpha<\omega_1$. But this is an easy consequence of the following two observations: if $f_{mn}\to^{(2)} f$ on X, then $f_{nn}\to f$ on X, and if $f_{mn}\to^{(3)} f$ on X, then $f_{1n}\to f$ on X. $\square$

Convergence $\to^{(4)}$ is more interesting. First, we will generalize it to the case of convergence for sequences of reals with n indices. Thus let $n\in\mathbb{N}$. We say that $x_{\overline{m}} \to_n x$ (where $x, x_{\overline{m}} \in \mathbb{R}$ for $\overline{m} \in \mathbb{N}^n$) if the following formula holds

$$(\forall\, \varepsilon > 0)(\exists\, k_1 \in \mathbb{N})(\forall\, m_1 \geq k_1) \ldots (\exists\, k_n \in \mathbb{N})(\forall\, m_n \geq k_n) \quad |x_{\overline{m}} - x| < \varepsilon.$$

Here, as before, $\overline{m} = (m_1, \ldots, m_n)$. Note that $\to_1$ means the usual convergence and $\to_2$ is identical with $\to^{(4)}$. Now, we can define pointwise convergence $f_{\overline{m}} \to_n f$ (where $f, f_{\overline{m}} : X \to \mathbb{R}, \overline{m} \in \mathbb{N}^n$) in the standard manner. The Baire system generated by $\to_n$ will be written as $\{B_\alpha(n)\}_{\alpha<\omega_1}$. This system, as before, starts from $B_0(n)$, the family of all continuous real functions on X. We can define an analogous system generated by $\to_n$ and starting from an arbitrary family $\mathcal{F}$ of real functions defined on X. This system will be denoted by $\{(B_\alpha(n))(\mathcal{F})\}_{\alpha<\omega_1}$ (hence $(B_0(n))(\mathcal{F}) = \mathcal{F}$). Analogously we understand $\{(B_\alpha)(\mathcal{F})\}_{\alpha<\omega_1}$ if $\to_n$ is replaced by $\to$.

Theorem 1. *For every $n \in \mathbb{N}$ and any $\alpha < \omega_1$, we have*

$$B_\alpha(n) = \begin{cases} B_\alpha & \text{if } \alpha \text{ is a limit number} \\ B_{\gamma+kn} & \text{if } \alpha = \gamma + k, \ \gamma \text{ is a limit number and } k \in \mathbb{N}. \end{cases}$$

The proof will be broken into several lemmas. If $\overline{m} = (m_1, \ldots, m_n) \in \mathbb{N}^n$, we write $\overline{m}|0 = 0$ and $\overline{m}|k = (m_1, \ldots, m_k)$ for $k = 1, \ldots, n$.

Lemma 1. *Let $n \in \mathbb{N}, n \geq 2$, and $x_{\overline{m}} \in \mathbb{R}, \overline{m} \in \mathbb{N}^n$. For a fixed $\overline{m} \in \mathbb{N}^n$ we define inductively*

$$x_{\overline{m}|(k-1)} = \lim_{m_k \to \infty} x_{\overline{m}|k} \text{ for } k = n, n-1, \ldots, 1\,. \tag{4}$$

and assume that these limits exist. Then $x_{\overline{m}} \to_n x_0$.

PROOF. Let $\varepsilon > 0$. By (4) we choose a $k_1 \in \mathbb{N}$ such that $|x_{(m_1)} - x_0| < \varepsilon/n$ for each $m_1 \geq k_1$. Then for each $m_1 \in \mathbb{N}$ we choose a $k_2 \in \mathbb{N}$ such that $|x_{(m_1,m_2)} - x_{m_1}| < \varepsilon/n$ for each $m_2 \geq k_2$, and so on. Thus

$$|x_0 - x_{\overline{m}}| \leq \sum_{k=1}^{n} |x_{\overline{m}|(k-1)} - x_{\overline{m}|k}| < n\frac{\varepsilon}{n} = \varepsilon$$

for the respective $m_1, \ldots, m_n$. Hence $x_{\overline{m}} \to_n x_0$. □

Corollary 1. *Let $\mathcal{F} \subset \mathbb{R}^X$. Then $(B_n)(\mathcal{F}) \subset (B_1(n))(\mathcal{F})$ for each $n \in \mathbb{N}$.*

Lemma 2. *Let $n \in \mathbb{N}, n \geq 2$. Assume that $x_0 \in \mathbb{R}$ and $x_{\overline{m}} \in \mathbb{R}$ for $\overline{m} \in \mathbb{N}^n$, and let $x_{\overline{m}} \to_n x_0$. Fix $\overline{m} \in \mathbb{N}^n$, and for $k = n, n-1, \ldots, 2$, we define inductively*

$$x_{\overline{m}|(k-1)} = \text{ either } \liminf_{m_k \to \infty} x_{\overline{m}|k} \text{ or } \limsup_{m_k \to \infty} x_{\overline{m}|k}. \tag{5}$$

Then $\lim_{m_1\to\infty} x_{(m_1)} = x_0$.

Proof. Consider the formula defining $x_{\overline{m}} \to_n x_0$. In inequality $|x_{\overline{m}} - x| < \varepsilon$, according to the rule (5), we take step by step $\liminf_{m_k\to\infty}$ or $\limsup_{m_k\to\infty}$ for $k = n, n-1, \dots, 2$. At the end, we obtain $|x_{(m_1)} - x_0| \le \varepsilon$ for all sufficiently large $m_1 \in \mathbb{N}$. $\square$

For a family $\mathcal{F} \subset \mathbb{R}^X$, let

$$\underline{\mathrm{Lim}}(\mathcal{F}) = \{f \in \mathbb{R}^X : (\exists\, (f_m)_{m\in\mathbb{N}} \in \mathcal{F}^{\mathbb{N}})(\forall\, x \in X) f(x) = \liminf_{m\to\infty} f_m(x)\},$$

$$\overline{\mathrm{Lim}}(\mathcal{F}) = \{f \in \mathbb{R}^X : (\exists\, (f_m)_{m\in\mathbb{N}} \in \mathcal{F}^{\mathbb{N}})(\forall\, x \in X) f(x) = \limsup_{m\to\infty} f_m(x)\}.$$

Then define classes $\underline{B}_n(\mathcal{F})$, $\overline{B}_n(\mathcal{F})$, $n \in \mathbb{N} \cup \{0\}$, as follows

$$\underline{B}_0(\mathcal{F}) = \overline{B}_0(\mathcal{F}) = \mathcal{F},$$

$$\underline{B}_n(\mathcal{F}) = \begin{cases} \underline{\mathrm{Lim}}\, \underline{B}_{n-1}(\mathcal{F}) & \text{if } n \text{ is odd} \\ \overline{\mathrm{Lim}}\, \underline{B}_{n-1}(\mathcal{F}) & \text{if } n \text{ is even,} \end{cases}$$

$$\overline{B}_n(\mathcal{F}) = \begin{cases} \overline{\mathrm{Lim}}\, \overline{B}_{n-1}(\mathcal{F}) & \text{if } n \text{ is odd} \\ \underline{\mathrm{Lim}}\, \overline{B}_{n-1}(\mathcal{F}) & \text{if } n \text{ is even.} \end{cases}$$

From the above definitions and Lemma 2 we derive

Corollary 2. *Let* $\mathcal{F} \subset \mathbb{R}^X$. *Then* $(B_1(n))(\mathcal{F}) \subset \underline{B}_n(\mathcal{F}) \cap \overline{B}_n(\mathcal{F})$ *for each* $n \in \mathbb{N}$.

It is known that the family of Borel sets in X can be expressed as $\bigcup_{\alpha<\omega_1} \Sigma^0_\alpha = \bigcup_{\alpha<\omega_1} \Pi^0_\alpha$, where $\Sigma^0_\alpha \subset \Sigma^0_\gamma$, $\Pi^0_\alpha \subset \Pi^0_\gamma$ for any $\alpha < \gamma < \omega_1$, and $\Sigma^0_1 =$ open sets, $\Sigma^0_2 = F_\sigma$ sets, etc., and $\Pi^0_1 =$ closed sets, $\Pi^0_2 = G_\delta$ sets etc. Recall [5, 24.3] that for each $\alpha < \omega_1$ we have: $f \in B_\alpha$ if and only if $f^{-1}(U) \in \Sigma^0_{\alpha+1}$, for every open set $U \subset \mathbb{R}$. Of course we can only use open sets U of the form $(-\infty, c)$ and (c, ∞) for $c \in \mathbb{R}$. We will write $f^{-1}(c,\infty) = [f > c]$, $f^{-1}(-\infty, c) = [f < c]$. The sets $[f \le c]$, $[f \ge c]$ have similar meaning. Since the complements of sets from Σ^0_α are in Π^0_α and vice versa, $f \in B_\alpha$ if and only if $[f \le c]$, $[f \ge c]$ are in $\Pi^0_{\alpha+1}$ for every $c \in \mathbb{R}$.

Lemma 3. *Let* $\beta < \omega_1$ *and* $n \in \mathbb{N}$ *be arbitrary.*

(a) *For each* $f \in \underline{B}_n(B_\beta)$ *and every* $c \in \mathbb{R}$, *we have*

$$[f > c] \in \begin{cases} \Sigma^0_{\beta+n+1} & \textit{if } n \textit{ is odd} \\ \Sigma^0_{\beta+n+2} & \textit{if } n \textit{ is even} \end{cases}$$

$$[f \ge c] \in \begin{cases} \Pi^0_{\beta+n+1} & \textit{if } n \textit{ is even} \\ \Pi^0_{\beta+n+2} & \textit{if } n \textit{ is odd.} \end{cases}$$

(b) For each $f \in \overline{B}_n(B_\beta)$ and every $c \in \mathbb{R}$, we have

$$[f < c] \in \begin{cases} \Sigma^0_{\beta+n+1} & \text{if } n \text{ is odd} \\ \Sigma^0_{\beta+n+2} & \text{if } n \text{ is even} \end{cases}$$

$$[f \le c] \in \begin{cases} \Pi^0_{\beta+n+1} & \text{if } n \text{ is even} \\ \Pi^0_{\beta+n+2} & \text{if } n \text{ is odd.} \end{cases}$$

PROOF. Observe that for any $\beta < \omega_1, n \in \mathbb{N}$, and $f \in \mathbb{R}^X$ we have

$$f \in \underline{B}_n(B_\beta) \Leftrightarrow -f \in \overline{B}_n(B_\beta) \tag{6}$$

which follows from the definitions of $\underline{B}_n(B_\beta)$ and $\overline{B}_n(B_\beta)$. This shows that it suffices to demonstrate (a). Let $\beta < \omega_1$ be fixed. Assume that $n = 1$ and $f \in \underline{B}_1(B_\beta)$. Thus $f = \liminf_{m\to\infty} f_m$ for some $(f_m)_{m\in\mathbb{N}} \in (B_\beta)^{\mathbb{N}}$. We have

$$[f > c] = \bigcup_{r\in\mathbb{N}} \bigcup_{p\in\mathbb{N}} \bigcap_{m\ge p} [f_m \ge c - \frac{1}{r}].$$

Hence $[f > c] \in \Sigma^0_{\beta+2}$, as desired. Also, we have

$$[f \ge c] = \bigcap_{r\in\mathbb{N}} [f > c - \frac{1}{r}]$$

which yields $[f \ge c] \in \Pi^0_{\beta+3}$, as desired. Now, let $n > 1$ and assume that the assertion (a) has been proved for $n - 1$. Let n be even. So $f = \limsup_{m\to\infty} f_m$ for $(f_m)_{m\in\mathbb{N}} \in (\underline{B}_{n-1})(B_\beta)$. We have

$$[f \ge c] = \bigcap_{r\in\mathbb{N}} \bigcap_{p\in\mathbb{N}} \bigcup_{m\ge p} [f_m > c - \frac{1}{r}].$$

By induction hypothesis, $[f_m > c - \frac{1}{r}] \in \Sigma^0_{\beta+n}$ and so $[f \ge c] \in \Pi^0_{\beta+n+1}$, as desired. Since

$$[f > c] = \bigcup_{r\in\mathbb{N}} [f \ge c + \frac{1}{r}],$$

we have $[f > c] \in \Sigma^0_{\beta+n+2}$, as desired. If n is odd, the proof is similar to the case $n = 1$. □

Corollary 3. *For any $\beta < \omega_1$ and $n \in \mathbb{N}$, we have $\underline{B}_n(B_\beta) \cap \overline{B}_n(B_\beta) \subset B_{\beta+n}$.*

PROOF. Fix $\beta < \omega_1$. Let $f \in \underline{B}_n(B_\beta) \cap \overline{B}_n(B_\beta)$. If n is odd, then $[f > c], [f < c] \in \Sigma^0_{\beta+n+1}$ for every $c \in \mathbb{R}$. If n is even, then $[f \ge c], [f \le c] \in \Pi^0_{\beta+n+1}$ for every $c \in \mathbb{R}$. Consequently, $f \in B_{\beta+n}$. □

Proof of Theorem 1. For $\alpha = 0$, the assertion is obvious. Let $\alpha > 0$ and assume that the assertion has been proved for all ordinals less than α. We will prove it for α. Assume first that $\alpha = \gamma + k$ where γ is a limit number and $k \in \mathbb{N}$. We have

$$B_{\gamma+kn} = B_n(B_{\gamma+(k-1)n}) \subset (B_1(n))(B_{\gamma+(k-1)n}) = (B_1(n))(B_{\gamma+(k-1)}(n)) \\ = B_{\gamma+k}(n) = B_\alpha(n),$$

by the induction hypothesis and Corollary 1. By Corollaries 2 and 3 we derive reverse inclusion:

$$B_\alpha(n) = (B_1(n))(B_{\gamma+(k-1)n}) \subset \underline{B}_n(B_{\gamma+(k-1)n}) \cap \overline{B}_n(B_{\gamma+(k-1)n}) \subset B_{\gamma+kn}.$$

If α is a limit number, by induction hypothesis we obtain

$$B_\alpha(n) = \bigcup_{\gamma<\alpha} B_\gamma(n) = \bigcup_{\gamma<\alpha} B_\gamma = B_\alpha. \qquad \square$$

3 Comments and Applications.

The core part of our main result, namely the equality $B_1(n) = B_n$, was proved by Katetov [3] and announced in [2], as one of a series of theorems concerning the so-called filter convergence. In our paper, we have presented a detailed proof that does not use filters explicitly. However, basic ideas of the both demonstrations are similar. We believe that our article, because of a different description, would be of some interest. While preparing our result, we were not aware of the existence of Katetov's work. Note that (as was observed by the Referee) our operator $\to_n$ means in fact the convergence with respect to the iterated product of the Fréchet filter on $\mathbb{N}$. A theory related to that created by Katetov was initiated again (independently) by Nuray and Ruckle [7], and by Kostyrko, Šalát and Wilczyński [4]. In their papers, filter convergence (or, equivalently ideal convergence) was viewed as a generalization of statistical convergence introduced by Fast [1] and investigated by many authors. In [4], Baire 1 classes generated by ideal convergence were studied, and this motivated us to find for each $n \in \mathbb{N}$, an ideal $\mathcal{I}$ of subsets of $\mathbb{N}$ for which $\mathcal{I}$-Baire 1 class coincides with B_n. An extension of this result to the case when $n \in \mathbb{N}$ is replaced by $\alpha < \omega_1$ was obtained by Komisarski [6] who then learnt about Katetov's result and informed us of it.

Now, let us give some terminology borrowed mainly from [4]. For $n \in \mathbb{N}$, let $\mathcal{I} \subset \mathcal{P}(\mathbb{N}^n)$ denote an ideal. We shall always assume that $\mathcal{I} \neq \mathcal{P}(\mathbb{N}^n)$ and $\bigcup \mathcal{I} = \mathbb{N}^n$, then we call $\mathcal{I}$ an ideal on $\mathbb{N}^n$. We say that a sequence $(x_{\overline{m}})_{\overline{m}\in\mathbb{N}^n}$ of reals $\mathcal{I}$-converges to an $x \in \mathbb{R}$, and write $x_{\overline{m}} \to_{\mathcal{I}} x$, if $\{\overline{m} \in \mathbb{N}^n : |x_{\overline{m}} - x| \geq \varepsilon\} \in$

$\mathcal{I}$ for each $\varepsilon > 0$. (See [4].) Observe that the usual convergence $\to$ coincides with $\mathcal{I}$-convergence when $\mathcal{I}$ consists of finite sets. Also, $\mathcal{I}$-convergence can be defined in a different equivalent way if a dual filter $\mathcal{F} = \{\mathbb{N}^n \setminus A : A \in \mathcal{I}\}$ is used instead of $\mathcal{I}$. (See [3]) and [7].) Two ideals $\mathcal{I}$ on $\mathbb{N}^n$ and $\mathcal{J}$ on $\mathbb{N}^r$ are called equivalent (which is written as $\mathcal{I} \sim \mathcal{J}$), if there is a bijection $\varphi : \mathbb{N}^n \to \mathbb{N}^r$ such that $\mathcal{J} = \{\varphi(A) : A \in \mathcal{I}\}$. Plainly $\mathcal{I} \sim \mathcal{J}$ implies $x_{\overline{m}} \to_{\mathcal{I}} x \Leftrightarrow y_{\overline{k}} \to_{\mathcal{J}} x$ for any $(x_{\overline{m}})_{\overline{m} \in \mathbb{N}^n}$ and $x \in \mathbb{R}$ where $y_{\overline{k}} = x_{\varphi^{-1}(\overline{k})}$ for $\overline{k} \in \mathbb{N}^r$. If an ideal $\mathcal{I}$ on $\mathbb{N}^n$ is given, we define convergence $f_{\overline{m}} \to_{\mathcal{I}} f$ for $f_{\overline{m}}, f \in \mathbb{R}^X$ ($\overline{m} \in \mathbb{N}^n$) in a standard manner. One can consider the Baire system $\{B_\alpha^{\mathcal{I}}\}_{\alpha<\omega_1}$ generated by convergence $\to_{\mathcal{I}}$. In [4], some sufficient conditions for ideals $\mathcal{I}$ on $\mathbb{N}$ were found in order to have $B_1^{\mathcal{I}} = B_1$. One of ideals considered there (we call it $\mathcal{E}$) is associated with a fixed partition $\Delta = \{D_k : k \in \mathbb{N}\}$ of $\mathbb{N}$ into infinitely many infinite sets D_k. Namely, $\mathcal{E} = \mathcal{E}_\Delta$ consists of all sets $A \subset \mathbb{N}$ such that A intersects a finite number of sets in Δ. This ideal will be called a partition ideal. In an analogous way, we can define a partition ideal on $\mathbb{N}^n$ for $n \in \mathbb{N}$. It is easy to check that any two partition ideals $\mathcal{E}_1$ on $\mathbb{N}^n$ and $\mathcal{E}_2$ on $\mathbb{N}^r$ are equivalent.

Now, we derive the following corollary from Proposition 2.

Corollary 4. *If $n \in \mathbb{N}$ and $\mathcal{E}$ is a partition ideal on $\mathbb{N}^n$, then $B_\alpha^{\mathcal{E}} = B_\alpha$ for all $\alpha < \omega_1$.*

PROOF. Observe that convergences $\to^{(k)}$ for $k = 1, 2$ are generated by partition ideals on $\mathbb{N}^2$. Indeed, let $\mathcal{E}_1 \subset \mathcal{P}(\mathbb{N}^2)$ denote the ideal associated with the partition of $\mathbb{N}^2$ into rows $\{(m, n) \in \mathbb{N}^2 : n \in \mathbb{N}\}$, $m \in \mathbb{N}$. Thus $\to^{(1)}$ is identical with $\to_{\mathcal{E}_1}$. Similarly $\to^{(2)}$ is identical with $\to_{\mathcal{E}_2}$ where $\mathcal{E}_2$ on $\mathbb{N}^2$ is the ideal associated with the partition of $\mathbb{N}^2$ into sets $\{(m, k) : m \in \mathbb{N}\} \cup \{(k, n) : n \in \mathbb{N}\}, k \in \mathbb{N}$. By Proposition 2 we have $B_\alpha^{(i)} = B_\alpha$ for all $\alpha < \omega_1$ and $i = 1, 2$. Since any two partition ideals are equivalent and two equivalent ideals produce identical Baire classes, the assertion follows. $\square$

Note that a result similar to Corollary 4 was mentioned in [4] for the case when $n = 1, \alpha = 1$ and if X is a complete metric space.

In the proof of Corollary 4 we have observed that convergences $\to^{(k)}$ for $k = 1, 2$ are generated by ideals. We are going to do the same for convergences $\to^{(k)}$ with $k = 3, 4$, and for convergences $\to_n$ with $n \in \mathbb{N}$. Notice that ideals on $\mathbb{N}^n$, as subsets of $\mathcal{P}(\mathbb{N}^n)$, can be also viewed as subsets of the Cantor space $\{0, 1\}^{\mathbb{N}^n}$ since there is a natural bijection between $\mathcal{P}(\mathbb{N}^n)$ and $\{0, 1\}^{\mathbb{N}^n}$ via characteristic functions (indicators). So ideals, as subsets of $\{0, 1\}^{\mathbb{N}^n}$ can be closed, Borel, analytic etc.

Proposition 3. *(a) Let $k \in \{1, 2, 3\}$. Then convergence $\to^{(k)}$ is generated by a Borel ideal $\mathcal{I}$ on $\mathbb{N}^2$.*

(b) Let $n \in \mathbb{N}$. Then convergence $\to_n$ is generated by a Borel ideal on $\mathbb{N}^n$.

Proof. (a) Following [5, 23] we abbreviate $(\exists\, k)(\forall\, m \geq k)$ by $(\forall^\infty m)$, and $(\forall\, k)(\exists\, m \geq k)$ by $(\exists^\infty m)$. It is easy to check that, for $k \in \{1,2,3\}$, convergence $\to^{(k)}$ is generated by an ideal $\mathcal{I}^{(k)}$ on $\mathbb{N}^2$ defined as follows (when we identify $\mathcal{P}(\mathbb{N}^2)$ with $\{0,1\}^{\mathbb{N}^2}$):

$$\mathcal{I}^{(1)} = \{z \in \{0,1\}^{\mathbb{N}^2} : (\forall^\infty m)(\forall\, n)\ \ z(m,n) = 0\},$$

$$\mathcal{I}^{(2)} = \{z \in \{0,1\}^{\mathbb{N}^2} : (\forall^\infty m, n)\ \ z(m,n) = 0\},$$

$$\mathcal{I}^{(3)} = \{z \in \{0,1\}^{\mathbb{N}^2} : (\forall\, m)(\forall^\infty n)\ \ z(m,n) = 0\},$$

where $(\forall^\infty m, n)$ abbreviates $(\exists\, k)(\forall\, m, n \geq k)$. By the use of methods of [5, 23] we may conclude that $\mathcal{I}^{(1)}$, $\mathcal{I}^{(2)} \in \Sigma^0_2$ and $\mathcal{I}^{(3)} \in \Pi^0_3$.

(b) Observe that convergence $\to_n$ is generated by an ideal $\mathcal{J}_n$ on $\mathbb{N}^n$ defined as follows (we identify $\mathcal{P}(\mathbb{N}^n)$ with $\{0,1\}^{\mathbb{N}^n}$). For $n = 1$ we put

$$\mathcal{J}_1 = \{z \in \{0,1\}^{\mathbb{N}} : (\forall^\infty m_1)\ \ z(m_1) = 0\},$$

and for $n > 1$ we put

$$\mathcal{J}_{n+1} = \{z \in \{0,1\}^{\mathbb{N}^{n+1}} : (\forall^\infty m_1)\ \ z(m_1, \cdot) \in \mathcal{J}_n\}.$$

By a careful induction one can check that $\mathcal{J}_n \in \Sigma^0_{2n}$ for all $n \in \mathbb{N}$. (See [5, 23.5].) $\square$

The approach proposed by Katetov in [2, 3] seems more abstract and involved than ours, but it goes further than our considerations. For instance, Katetov showed that, under CH, the family B_{ω_1} (of all real-valued Borel functions on $\mathbb{R}$) can be expressed as the first Baire class generated by a filter convergence, with a filter $\mathcal{F} \subset \mathcal{P}(\mathbb{N})$. He proved even more since the domain $\mathbb{R}$ can be replaced by an arbitrary topological space. Note that $\mathcal{F}$ is the intersection of two ultrafilters on $\mathbb{N}$. It would be interesting to establish whether this result can be obtained within ZFC.

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