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ON INTEGRATION BY PARTS FOR STIELTJES-TYPE INTEGRALS OF BANACH SPACE-VALUED FUNCTIONS

Abstract

In this paper we discuss integration by parts for several generalizations of the Riemann-Stieltjes integral. In addition, we obtain new results on integration by parts for the Henstock-Stieltjes integral and its interior modification for Banach space-valued functions.

1 Introduction.

The present paper is devoted to certain problems related to theory of Stieltjes-type integrals of Banach-valued functions. Normally, two fundamental methods are employed in the theory to obtain an integral *wider* than the Riemann-Stieltjes integral. The first method is to refine the class of partitions and the second is to modify the Riemann-Stieltjes sums.

1.1 Notation and Terminology.

Let E be a subset of the real line. In the paper we use the following notation: I, $\overline{E}$, $\mathrm{Int}(E)$, ∂E, and $|E|$ denote an *interval* [1] of the real line, the *closure* of E, the *interior* of E, the *boundary* of E, and the *Lebesgue measure* of a measurable set E, respectively.

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[1] *Open*, *semi-open*, and *compact* non-degenerate intervals are considered in this paper. The type of intervals will be specified in concrete cases.

$\Delta f(I)$ denotes the *increment* of a vector-valued function f on I. For example, we write $\Delta f((u,v)) = f(v-) - f(u+)$. The increment is similarly defined for other types of intervals as well. Further, we assume that $\Delta f([t,t]) \equiv 0$ for a fixed point t. We also use the following notation to represent the *saltus* of f at t: $\Delta^- f(t) = f(t) - f(t-) \equiv -\Delta f([t,t))$, $\Delta^+ f(t) = f(t+) - f(t) \equiv -\Delta f((t,t])$, $\Delta f(t) = f(t+) - f(t-) \equiv -\Delta f((t,t))$.

$\mathbf{1}_E$ is the characteristic function (or indicator) of a set E.

A vector-valued function defined on I is said to be *regulated* on I, if it has discontinuities of the first kind only. In particular, the function possesses all the unilateral limits at each point of the interval I.

A positive function defined on a set E will be called a *gauge* on E.

Finally, in this paper $[a,b]$ is a fixed compact interval of the real line.

2 Background.

2.1 Stieltjes-type Integrals of Real-valued Functions.

We are concerned with two types of generalizations of the Riemann-Stieltjes integral. Firstly, in 1920, S. Pollard [34] proposed a generalized limit process and introduced a new integral concept. Pollard's integral is an additive interval function, but the integration by parts formula does not hold for it under the familiar hypothesis. Secondly, R. Henstock in 1955 and J. Kurzweil in 1957 independently introduced the so-called *Henstock-Kurzweil* (or *gauge*) integral (see [7, 16]). We will call this integral $\mathcal{H}$*-integral.* Then, in 1960, Henstock [8] introduced the idea of a *variational integral* ($V\mathcal{H}$*-integral*), showing that the $V\mathcal{H}$-integral coincides with the Perron type *Ward's integral* ($\mathcal{W}$*-integral*) (see [46]). In 1961, Henstock went on to prove [9] that the $V\mathcal{H}$-integral coincides with the $\mathcal{H}$-integral.

Pollard-type integrals. Let f and g be real-valued functions defined on $[a,b]$. We now define the Pollard integral as follows.

Definition 2.1 (S. Pollard [34]). The function f is said to be $\mathcal{P}$*-integrable* (resp. $\mathcal{P}^*$*-integrable*) with respect to g on $[a,b]$ if there exists a real number z such that for each $\epsilon > 0$ there is a finite subset D of $[a,b]$ such that

$$\left| \sum_{i=1}^{n} f(\tau_i)\, \Delta g([t_{i-1}, t_i]) - z \right| < \epsilon$$

for each collection of points $T = \{a = t_0 < t_1 < \cdots < t_n = b\} \supset D$ and for all *tags* τ_i with $\tau_i \in [t_{i-1}, t_i]$ (resp. $\tau_i \in (t_{i-1}, t_i)$).

The number z is called the $\mathcal{P}$-*integral* (resp. $\mathcal{P}^*$-*integral*[2]) of f with respect to g on $[a,b]$ and is denoted by $\mathcal{P}\int_a^b f\,dg$ (resp. $\mathcal{P}^*\int_a^b f\,dg$).

Suppose that g is regulated on $[a,b]$. Here we assume that $g(a-) \equiv g(a)$ and $g(b+) \equiv g(b)$. Let $T = \{a = t_0 < t_1 < \cdots < t_n = b\}$ be a collection of points, for each i we choose a tag $\tau_i \in [t_{i-1}, t_i]$. Then form the sum

$$\sum_{i=1}^{n} f(\tau_i)\,\Delta g((t_{i-1}, t_i)) + \sum_{i=0}^{n} f(t_i)\,\Delta g(t_i),$$

which is called *Young's sum* and is denoted by $(f\Delta g)_{\mathcal{Y}}(T_\tau)$. These sums were introduced by W. H. Young in his well-known paper [48]. We now define the Young integral as follows.

Definition 2.2 (T. H. Hildebrandt [13]). The function f is said to be $\mathcal{Y}$-*integrable* (resp. $\mathcal{Y}^*$-*integrable*) with respect to g on $[a,b]$ if there exists a real number z such that for each $\epsilon > 0$ there is a finite subset D of $[a,b]$ such that

$$|(f\Delta g)_{\mathcal{Y}}(T_\tau) - z| < \epsilon$$

for each collection of points $T = \{a = t_0 < t_1 < \cdots < t_n = b\} \supset D$ and for all tags τ_i with $\tau_i \in [t_{i-1}, t_i]$ (resp. $\tau_i \in (t_{i-1}, t_i)$).

The number z is called the $\mathcal{Y}$-*integral* (resp. $\mathcal{Y}^*$-*integral*) of f with respect to g on $[a,b]$ and is denoted by $\mathcal{Y}\int_a^b f\,dg$ (resp. $\mathcal{Y}^*\int_a^b f\,dg$).

Let p be an integer such that $p \geq 2$, and let $(w_1, w_2, \ldots, w_p)$ be an ordered collection of real numbers such that $w_1 + w_2 + \cdots + w_p = 1$. For a collection of points

$$T = \{a = t_0 < t_1 < \cdots < t_n = b\},$$

and for each i we choose tags

$$\tau_i = \{t_{i-1} = \tau_{1,i} < \tau_{2,i} < \cdots < \tau_{p,i} = t_i\}$$

and form the sum

$$(f\Delta g)_{(w_1,w_2,\ldots,w_p)}(T_\tau) = \sum_{i=1}^{n} \left(\sum_{j=1}^{p} w_j\, f(\tau_{j,i}) \right) \Delta g([t_{i-1}, t_i]).$$

We now define the weighted integral as follows.

[2] T. H. Hildebrandt [13] calls this integral *Dushnik's integral*. It is occasionally called the *interior integral* as well.

Definition 2.3 (F. M. Wright and J. D. Baker [47]). The function f is said to be $(w_1, w_2, \dots, w_p)$-*integrable* with respect to g on $[a,b]$ if there exists a real number z such that for each $\epsilon > 0$ there is a finite subset D of $[a,b]$ such that

$$|(f\Delta g)_{(w_1,w_2,\dots,w_p)}(T_\tau) - z| < \epsilon$$

for each collection of points $T = \{a = t_0 < t_1 < \cdots < t_n = b\} \supset D$ and for all tags

$$\tau_i = \{t_{i-1} = \tau_{1,i} < \tau_{2,i} < \cdots < \tau_{p,i} = t_i\}.$$

The number z is called the $(w_1, w_2, \dots, w_p)$-*weighted integral* of f with respect to g on $[a,b]$ and is denoted by $(w_1, w_2, \dots, w_p)\int_a^b f\,dg$.

It can easily be seen that the $(0,1,0)$-weighted integral coincides with the $\mathcal{P}^*$-integral. Furthermore, it follows from [47, Theorems 2.5, 3.1, and 3.6] that, if one of the integrand and integrator is of bounded variation on $[a,b]$ and the other is regulated on $[a,b]$, then the integrals $(1,-1,1)\int_a^b f\,dg$ and $\mathcal{Y}^*\int_a^b f\,dg$ exist and are equal.

The theory of Pollard-type integrals can be found for example in Hildebrandt's monograph [15].

Variational and gauge integrals. An arbitrary subset $\mathscr{B}$ of the Cartesian product $\mathscr{I} \times [a,b]$, where $\mathscr{I}$ denotes the set of all compact subintervals of $[a,b]$, is called a *derivation basis* (or a *basis* for short) in $[a,b]$.

Given a basis $\mathscr{B}$ in $[a,b]$, an interval $I \in \mathscr{I}$ is called a $\mathscr{B}$-*interval* if there exists a point $t \in I$ such that $(I,t) \in \mathscr{B}$.

Let E be a subset of $[a,b]$. Then, by $\mathscr{B}(E)$, we denote the set

$$\{(I,t) \in \mathscr{B} : I \subset E\}$$

and, by $\mathscr{B}[E]$, we denote the set

$$\{(I,t) \in \mathscr{B} : t \in E\}.$$

The set $\mathscr{B}(E)$ may be empty; e.g., in the case when E does not contain an interval.

Let δ be a gauge on E. Then, by $\mathscr{B}_\delta$, we denote the set

$$\{(I,t) \in \mathscr{B}[E] : I \subset (t - \delta(t), t + \delta(t))\}.$$

The set of all interval-point pairs $(I,t) \in \mathscr{I} \times [a,b]$ such that $t \in I$ is called the *full Henstock-Kurzweil* basis in $[a,b]$ and is denoted by $\mathscr{F}$.

A finite set $\pi \subset \mathscr{B}[E]$ is called a $\mathscr{B}$-*partition tagged in* E if $(I^{'}, t^{'}), (I^{''}, t^{''}) \in \pi$ and $(I^{'}, t^{'}) \neq (I^{''}, t^{''})$ imply that the intervals $I^{'}$ and $I^{''}$ are mutually nonoverlapping $\mathscr{B}$-intervals. Let I_0 be a $\mathscr{B}$-interval and π be any $\mathscr{B}$-partition tagged in I_0. If $\pi \subset \mathscr{B}(I_0)$ and $\bigcup_{(I,t)\in\pi} I = I_0$, then π is called a *$\mathscr{B}$-partition of* I_0.

We say that $\mathscr{B}$ possesses the *partitioning property* (*p-property*) if the following conditions hold:

(i) for each finite collection $I_0, I_1, \dots, I_n$ of $\mathscr{B}$-intervals with $I_1, \dots, I_n \subset I_0$ the difference $I_0 \setminus \cup_{i=1}^n \operatorname{Int}(I_i)$ can be expressed as a finite union of pairwise non-overlapping $\mathscr{B}$-intervals;

(ii) for each $\mathscr{B}$-interval I and for each gauge δ on I there exists a $\mathscr{B}_\delta$-partition of I (the so-called Cousin lemma).

As an illustration, the full Henstock-Kurzweil basis $\mathscr{F}$ possesses the p-property. A large amount of work concerning the concept of a derivation basis has been published by B. S. Thomson (see, for example, [42, 43, 44]).

Let $\Phi(I,t)$ and $\Psi(I,t)$ be $\mathscr{B}$-interval-point functions. If δ is a gauge on E, then

$$Var(\mathscr{B}_\delta, \Phi, E) = \sup_{\pi} \sum_{(I,t)\in\pi} |\Phi(I,t)|,$$

where π runs over $\mathscr{B}_\delta$-partitions tagged in E, is called the *δ-variation of Φ on E relative to $\mathscr{B}$. Variational measure generated by Φ, relative to $\mathscr{B}$,* is the set function

$$V(\Phi, E, \mathscr{B}) = \inf_{\delta} Var(\mathscr{B}_\delta, \Phi, E),$$

where δ runs over gauges on E.

The function Φ is said to be *variationally equivalent to Ψ on E, relative to $\mathscr{B}$,* if $V(\Phi - \Psi, E, \mathscr{B}) = 0$.

Definition 2.4 (R. Henstock [11, 12]). Suppose that the interval $[a,b]$ is a $\mathscr{B}$-interval. The function Φ is said to be *variationally integrable ($V\mathscr{B}$-integrable) in* $[a,b]$ *relative to $\mathscr{B}$* if there exists an additive $\mathscr{B}$-interval function $F(I)$, which is variationally equivalent to Φ on $[a,b]$ relative to $\mathscr{B}$. The function F is called the *variational integral of Φ in* $[a,b]$ *relative to $\mathscr{B}$*. We write $F(I) = V\mathscr{B}\int_I \Phi$ for each $\mathscr{B}$-interval I.

Definition 2.5 (R. Henstock [11, 12]). Suppose that $\mathscr{B}$ possesses the p-property. The function Φ is said to be *$\mathscr{B}$-integrable* on a $\mathscr{B}$-interval I_0 if there exists a real number z such that for each $\epsilon > 0$ there is a gauge δ on I_0 such that

$$\left| \sum_{(I,t)\in\pi} \Phi(I,t) - z \right| < \epsilon$$

for each $\mathscr{B}_\delta$-partition π of I_0. We write $z = \mathscr{B}\int_{I_0} \Phi$.

Let us now reformulate Henstock's definitions to make use of them in the theory of Riemann-Stieltjes-type integrals. Let f and g be real-valued functions defined on $[a,b]$. If the interval-point function

$$\Pi(I,t) = f(t)\,\Delta g(I)$$

is $\mathscr{F}$-integrable on $[a,b]$, then f is said to be *integrable in the Henstock-Stieltjes sense* (*$\mathcal{H}$-integrable*) with respect to g on $[a,b]$. If the function Π is variationally integrable in $[a,b]$ relative to $\mathscr{F}$, then f is said to be *variationally integrable in the Henstock-Stieltjes sense* (*$V\mathcal{H}$-integrable*) with respect to g on $[a,b]$.

McShane's derivation basis $\mathscr{M}$ in $[a,b]$ was introduced by E. J. McShane in [26] and consists of all interval-point pairs (I,t) such that $I \in \mathscr{I}$ and $t \in [a,b]$.

As for Lebesgue-Stieltjes integral[3], it can be obtained from the above definitions if we substitute McShane's derivation basis $\mathscr{M}$ for the full Henstock-Kurzweil basis $\mathscr{F}$. In [26] McShane showed that Lebesgue-Stieltjes integrable function f with respect to a function g on $[a,b]$ is $\mathscr{M}$-integrable with respect to g on $[a,b]$ and the integrals are equal. In 1986, S. Meinershagen [27] was the first to prove the converse under the additional assumption that the function g is of bounded variation. Thus, the Lebesgue-Stieltjes integral coincides with McShane's.

Monographs [5, 11, 12, 18, 21, 22, 24, 31, 33] are devoted to the theory of variational and gauge integrals.

2.2 Stieltjes-type Integrals of Banach-valued Functions.

In what follows, $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$, and " $\centerdot$ " will denote real Banach spaces and a bilinear mapping ($\centerdot : \mathfrak{X} \times \mathfrak{Y} \to \mathfrak{Z}$), which is supposed to be bounded. [4] This means that the inequality $\|x \centerdot y\| \leq \|x\| \cdot \|y\|$ is fulfilled for all $x \in \mathfrak{X}$, $y \in \mathfrak{Y}$.

We are concerned in the present paper with Stieltjes-type integrals of Banach-valued functions defined analogously to Gowurin's 1936 generalization of the Riemann-Stieltjes integral [5].

Definition 2.6 (M. Gowurin [6]). A function $f : [a,b] \to \mathfrak{X}$ is said to be *integrable in the Riemann-Stieltjes sense* with respect to a function $g : [a,b] \to \mathfrak{Y}$ on $[a,b]$ if there exists a vector $z \in \mathfrak{Z}$ such that for each $\epsilon > 0$ there is a *constant* gauge δ on $[a,b]$ such that

$$\left\| \sum_{(I,t)\in\pi} f(t) \centerdot \Delta g(I) - z \right\| < \epsilon \tag{1}$$

[3]We consider here the Lebesgue-Stieltjes integral defined in [35, Chapter VIII, § 2], which is an additive interval function.

[4]We always study the spaces $\mathfrak{X}$, $\mathfrak{Y}$ and $\mathfrak{Z}$ together with the bilinear mapping " $\centerdot$ "; in other words, a *bilinear triple* is considered and is denoted by $(\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z})$.

[5]The original Gowurin definition is reformulated in our framework.

for each $\mathscr{F}_\delta$-partition π of $[a, b]$.

The vector z is called the *Riemann-Stieltjes integral* (*$\mathcal{R}$-integral*) of f with respect to g on $[a, b]$ and is denoted by $\mathcal{R} \int_a^b f \centerdot dg$.

A classical survey on integration of Banach-valued functions can be found in [14]. More recently, the Henstock-Stieltjes integral has frequently been considered for Banach-valued functions. A series of papers devoted to the Henstock-Stieltjes integral as well as to its generalization for Banach-valued functions have been published by Š. Schwabik [36, 37, 38, 39]. Finally it should be mentioned that there is a considerable amount of work concerning applications of those integrals in equations theory (see, for example, [45]).

2.3 Integration by Parts for Stieltjes-type Integrals.

We should in the first place recall that the integration by parts formula is generally obtained for a Stieltjes-type integral by means of algebraic transformations of the integral sums. And then the respective definition of the integral, in turn, plays an important role in the proof of its validity.

A well-known fact concerning integration by parts for the Riemann-Stieltjes integral (cf. Stieltjes [41], Pollard [34], Gowurin [6]) reads as follows.

Theorem A. *Suppose that $f : [a, b] \to \mathfrak{X}$ and $g : [a, b] \to \mathfrak{Y}$ are Banach-valued functions defined on the interval $[a, b]$. Then*

$$\mathcal{R} \int_a^b f \centerdot dg + \mathcal{R} \int_a^b df \centerdot g = f(b) \centerdot g(b) - f(a) \centerdot g(a) \tag{2}$$

provided that at least one integral exists.

Thus, formula (2) is the "internal" property of the Riemann-Stieltjes integral; in other words, it does not depend on the integrands taken separately. And normally we must impose additional conditions on the integrands for other integrals of the Stieltjes type.

Integration by parts for Pollard-type integrals. T. H. Hildebrandt discussed certain integrals of the Pollard type and also integration by parts for the $\mathcal{Y}^*$-integral (see Definition 2.2) in [13].

Theorem B (T. H. Hildebrandt, 1938). *Let f and g be real-valued functions of bounded variation on the interval $[a, b]$. Then the integrals $\mathcal{Y}^* \int_a^b f\, dg$ and $\mathcal{Y}^* \int_a^b g\, df$ exist and the following formula holds:*

$$\begin{aligned} \mathcal{Y}^* \int_a^b f\, dg + \mathcal{Y}^* \int_a^b g\, df =& f(b)\, g(b) - f(a)\, g(a) - \\ & \sum_{a \le t \le b} [\Delta^+ f(t)\, \Delta^+ g(t) - \Delta^- f(t)\, \Delta^- g(t)]. \end{aligned} \tag{3}$$

Note that the $\mathcal{Y}^*$-integral under the hypotheses of Theorem B is "absolute" because it is equivalent to the Lebesgue-Stieltjes integral. Theorem B includes also the famous theorems obtained by W. H. Young [48, pp. 136–137] and S. Saks [35, Chapter III, § 14]. A generalization of Theorem B for Banach-valued functions under similar hypotheses can be found in [39, Theorem 13].

In [25] J. S. MacNerney considered a special case of Gowurin's construction, when the spaces $\mathfrak{X}$, $\mathfrak{Y}$ and $\mathfrak{Z}$ coincide with a Banach algebra and the bilinear mapping " $\centerdot$ " is the algebra multiplication. In this special case MacNerney established Theorem C on integration by parts.

Theorem C (J. S. MacNerney, 1963). *If f and g are regulated Banach algebra valued-functions, and one of f and g is of bounded variation on $[a, b]$. Then the integrals $\mathcal{Y}^* \int_a^b f \centerdot dg$ and $\mathcal{P}^* \int_a^b df \centerdot g$ exist and the following formula holds:*

$$\mathcal{Y}^* \int_a^b f \centerdot dg + \mathcal{P}^* \int_a^b df \centerdot g = f(b) \centerdot g(b) - f(a) \centerdot g(a). \tag{4}$$

It is interesting that there are no additional terms in the right side of formula (4) unlike in the right side of formula (3). The property of bounded variation plays an important role in MacNerney's proof.

In [47], F. M. Wright and J. D. Baker investigated the question of integration by parts for (w_1, w_2, w_3)-weighted integrals (see Definition 2.3).

Theorem D (F. M. Wright and J. D. Baker, 1969). *Suppose that f and g are real-valued functions defined on $[a, b]$, g is of bounded variation and f is bounded on $[a, b]$. Let*

$$S^+ = \{t \in [a, b) : g(t+) \neq g(t)\}, \text{ and } S^- = \{t \in (a, b] : g(t-) \neq g(t)\}.$$

If the integral $(w_1, w_2, w_3) \int_a^b f\, dg$ exists, $f(t+)$ exists for $t \in S^+$, and $f(t-)$ exists for $t \in S^-$, then the integrals $(w_1, w_2, w_3) \int_a^b g\, df$, $(w_3, w_2, w_1) \int_a^b g\, df$, and $(1 - w_1, -w_2, 1 - w_3) \int_a^b g\, df$ exist and the following formulae hold:

$$(w_1, w_2, w_3) \int_a^b f\, dg + (w_1, w_2, w_3) \int_a^b g\, df = f(b)\, g(b) - f(a)\, g(a) - \\ (2\, w_1 - 1) \sum_{t \in S^+} \Delta^+ f(t)\, \Delta^+ g(t) + (2\, w_3 - 1) \sum_{t \in S^-} \Delta^- f(t)\, \Delta^- g(t);$$

$$(w_1, w_2, w_3) \int_a^b f\, dg + (w_3, w_2, w_1) \int_a^b g\, df = f(b)\, g(b) - f(a)\, g(a) + \\ w_2 \left\{ \sum_{t \in S^+} \Delta^+ f(t)\, \Delta^+ g(t) - \sum_{t \in S^-} \Delta^- f(t)\, \Delta^- g(t) \right\};$$

$$(w_1, w_2, w_3) \int_a^b f\, dg + (1 - w_1, -w_2, 1 - w_3) \int_a^b g\, df = f(b)\, g(b) - f(a)\, g(a).$$

Theorem B follows from Theorem D when f is of bounded variation and $w_2 = -1$, $w_1 = w_3 = 1$. In addition, Theorem C follows from Theorem D in the special case when the Banach algebra coincides with $\mathbb{R}$. The generalization of Theorem C for the case when one of the integrands is of bounded variation on $[a, b]$ and the other is bounded on $[a, b]$ is given in [47] as well.

Theorem E (F. M. Wright and J. D. Baker, 1969). *Suppose that f and g are real-valued functions defined on $[a, b]$, g is of bounded variation and f is bounded on $[a, b]$. Then, the integral $\mathcal{Y}^* \int_a^b f\, dg$ exists if and only if the integral $\mathcal{P}^* \int_a^b g\, df$ exists, and (4) holds.*

Thus, all the above-mentioned results on integration by parts involving the integrals of the Pollard type keep the bounded variation condition for at least one of the integrands. However, E. R. Love [23] has shown that this condition can be omitted for *regular* real-valued functions. So, a formula for integration by parts might be extended over a wide range of functions of unbounded variation.

A vector-valued function f is said to be *regular* at a point τ if the limits $f(\tau-)$ and $f(\tau+)$ exist and

$$f(\tau) = \frac{f(\tau-) + f(\tau+)}{2}.$$

Theorem F (E. R. Love, 1998). *Let f and g be regulated real-valued functions defined on $[a, b]$ that are regular at each point of the open interval (a, b). Then*

$$\begin{aligned} \mathcal{Y}^* \int_a^b f\, dg + \mathcal{Y}^* \int_a^b g\, df = & f(b)\, g(b) - f(a)\, g(a) \\ & + \Delta^- f(b)\, \Delta^- g(b) - \Delta^+ f(a)\, \Delta^+ g(a) \end{aligned} \tag{5}$$

provided that at least one integral exists.

It can easily be seen that (5) is simpler than (3) since all the intermediate terms in the right side of formula (3) vanish due to the regularity of the integrands. Using Love's approach to integration by parts, the author obtained Theorem G in [29].

Theorem G. *Suppose that $f : [a, b] \to \mathfrak{X}$ and $g : [a, b] \to \mathfrak{Y}$ are Banach-valued functions and g is regulated on $[a, b]$. If the integral $\mathcal{P}^* \int_a^b df \centerdot g$ exists, then the integral $\mathcal{Y}^* \int_a^b f \centerdot dg$ exists and (4) holds.*

Moreover, the conditions of Theorem G are non-symmetrical; that is, if we replace the existence condition of the $\mathcal{P}^*$-integral with that of the $\mathcal{Y}^*$-integral, then the other integral in (4) may not exist (see [30, Example 3.3.1]).

Integration by parts for variational and gauge integrals. Typically the VBG^**-property* plays the role of the bounded variation property when the Henstock-Stieltjes integral is considered. The classical definitions of the VB^*- and VBG^*-functions on a set can be found in Saks [35, Chapter VII, § 7].

A. J. Ward in [46] obtained Theorem H and Theorem I, which give necessary and sufficient conditions for integration by parts in the case of the $\mathcal{W}$-integral. Ward employed Perron's approach to integration in his proofs.

Theorem H (A. J. Ward, 1936). *Suppose that f and g are real-valued functions defined on $[a,b]$, g is bounded and VBG^* on $[a,b]$, f is bounded on $[a,b]$ and also continuous on $[a,b]$ except at the points of a set N such that $|g(N)| = 0$ and g is continuous at each point of N. Then*

$$\mathcal{W}\int_a^b f\,dg + \mathcal{W}\int_a^b g\,df = f(b)\,g(b) - f(a)\,g(a) \tag{6}$$

provided that at least one integral exists.

Further Ward noted: "It may be remarked that the conditions of the theorem, while far from being necessary, are not so artificial as they might at first appear." Also it is possible to prove Theorem I. However, no proof of this theorem was published.

Theorem I (A. J. Ward, 1936). *Suppose that f and g are real-valued functions defined on $[a,b]$, g is VBG^* on $[a,b]$, and, for $t \in [a,b]$,*

$$\mathcal{W}\int_a^t f\,dg + \mathcal{W}\int_a^t g\,df = f(t)\,g(t) - f(a)\,g(a).$$

Then f is continuous on $[a,b]$ except at the points of a set N such that $|g(N)| = 0$.

P. S. Bullen gave a very short proof of the integration by parts formula for the Perron integral (see [1]). In addition, his extensive survey on integration by parts for Perron-type integrals can be found in [2].

In his expository paper [10] R. Henstock pointed out that, although the family of variational and gauge integrals does not contain, for example, the $\mathcal{P}^*$-integral, it is wide enough to contain many of known integrals of the Stieltjes type[6]. In particular in [10] Henstock established Theorem J, which provides a

[6]It should be noted at this point that a Riemann-type definition of an integral that includes both the $\mathcal{Y}^*$- and $\mathcal{H}$-integrals has been given in [20].

set of necessary and sufficient conditions for integration by parts in the special variational integral case.

Consider the *end-point tagged Henstock-Kurzweil* basis

$$\mathscr{E} = \{(I,t) \in \mathscr{I} \times [a,b] : I \in \mathscr{I},\, t \in \partial I\}.$$

It is not hard to prove that the $V\mathscr{E}$-integral coincides with the $\mathcal{H}$-integral.

Theorem J (R. Henstock, 1973). *Suppose that f and g are real-valued functions defined on $[a,b]$. If the formula*

$$V\mathscr{E}\int_{[a,t]} f\,\Delta g + V\mathscr{E}\int_{[a,t]} g\,\Delta f = f(t)\,g(t) - f(a)\,g(a) \tag{7}$$

holds for each $t \in (a,b]$, then

$$V(\Delta f \Delta g, [a,b], \mathscr{E}) = 0. \tag{8}$$

Conversely, if (8) *holds and one of the integrals in* (7) *exists when $t = b$, then the other one exists and* (7) *holds for each $t \in (a,b]$.*

Thus Theorem J gives a common approach to integration by parts for integrals that are covered by the Henstock-Stieltjes integral. Nevertheless, the conditions of Theorem J are quite difficult to check for concrete functions f and g. Henstock also obtained Ward's Theorem H as a consequence of Theorem J and gave several examples. These examples show that the condition (8) is independent of the existence condition for the integrals in (7).

Finally we must mention W. F. Pfeffer's paper [32]. He found there that integration by parts theorems for the Henstock-Stieltjes integral, which is "non-absolute", ought not to be concerned with absolutely integrable functions exclusively. Pfeffer's theorem on integration by parts reads as follows.

Theorem K (W. F. Pfeffer, 1983). *Suppose that f, g, and α are real-valued functions defined on $[a,b]$ where α is of bounded variation and write*

$$F(t) = \mathcal{H}\int_a^t f\,d\alpha \text{ and } G(t) = \mathcal{H}\int_a^t g\,d\alpha$$

where it is assumed that these functions exist and are continuous on $[a,b]$. Then

$$\mathcal{H}\int_a^b F\,dG + \mathcal{H}\int_a^b G\,dF = F(b)\,G(b) \tag{9}$$

provided that at least one integral exists.

Note that under these hypotheses *both* F and G must be VBG^* on $[a,b]$. Pfeffer made use of the *Saks-Henstock lemma*, equivalently, of the variational approach to the integral in his proof.

However, A. P. Solodov has shown that the gauge integral *properly* contains the variational integral when any infinite dimensional Banach space is considered (see [40]). Henceforth we will present a set of necessary and sufficient conditions for integration by parts in the case of Banach-valued gauge integrals. Our approach is fully based on Henstock's definition of the Riemann type integral and a *weak variational condition*.

3 Integration by Parts for Banach-Valued Gauge Integrals.

3.1 Preliminaries.

This section will give definitions and basic properties of the gauge integrals and of the functions of *weakly bounded variation* with respect to $(\mathfrak{X},\mathfrak{Y},\mathfrak{Z})$. In particular, an *interior integral* is considered relative to a special kind of bases. Using constructions, due to I. M. Gelfand [3, 4] and M. Gowurin [6], we introduce new classes wVB, $wVBG$, wVB^*, and $wVBG^*$, which generalize the well-known classes VB, VBG, VB^*, and VBG^*, respectively in the case of Banach-valued functions. We take a different approach to a definition of the wVB^*-property on a set in comparison with [35, Chapter VII, § 7]. This approach was developed in [21, 22] according to problems related to the theory of gauge integrals.

Let $\mathscr{B}$ be a basis in $[a,b]$. We say that a $\mathscr{B}$-partition π of a $\mathscr{B}$-interval I is an *interior tagged $\mathscr{B}$-partition* (*$\mathscr{B}^*$-partition*) if the following conditions hold:

(i) $(I_0,t)\in\pi$ and $I_0\subset \operatorname{Int}(I)$ imply $t\in\operatorname{Int}(I_0)$;

(ii) $(I_0,t)\in\pi$ and $I_0\cap\partial I\neq\emptyset$ imply $t\in I_0\cap\partial I$.

We say that $\mathscr{B}$ possesses the *interior partitioning property* (*p^*-property*) if the following conditions hold:

(i) for each finite collection $I_0,\ I_1,\dots,I_n$ of $\mathscr{B}$-intervals with $I_1,\dots,I_n\subset I_0$ the difference $I_0\setminus\cup_{i=1}^n\operatorname{Int}(I_i)$ can be expressed as a finite union of pairwise non-overlapping $\mathscr{B}$-intervals;

(ii) $(I_0,t)\in\mathscr{B}$, $(I_1,t)\in\mathscr{B}$, and $I_0\cap I_1=\{t\}$ imply $(I_0\cup I_1,t)\in\mathscr{B}$;

(iii) $(I,t)\in\mathscr{B},\ t\in\operatorname{Int}(I)$ imply $((-\infty,t]\cap I,t)\in\mathscr{B}$ and $(I\cap[t,+\infty),t)\in\mathscr{B}$;

(iv) for each $\mathscr{B}$-interval I and for each gauge δ on I there exists $\mathscr{B}^*_\delta$-partition of I.

Lemma 1. *The full Henstock-Kurzweil basis $\mathscr{F}$ possesses the p^*-property.*

Proof. The proof can be found for example in [19, Lemma 1.2]. □

Definition 3.1. Suppose that $\mathscr{B}$ possesses the p-property (resp. p^*-property). A function $f : [a, b] \to \mathfrak{X}$ is said to be *$\mathscr{B}$-integrable* (resp. *$\mathscr{B}^*$-integrable*) with respect to a function $g : [a, b] \to \mathfrak{Y}$ on a $\mathscr{B}$-interval I_0 if there exists a vector $z \in \mathfrak{Z}$ such that for each $\epsilon > 0$ there is a gauge δ on I_0 such that (1) holds for each $\mathscr{B}_\delta$-partition (resp. $\mathscr{B}^*_\delta$-partition) π of I_0.

The vector z is called the *$\mathscr{B}$-integral* (resp. *$\mathscr{B}^*$-integral*) of f with respect to g on I_0 and is denoted by $\mathscr{B}\int_{I_0} f \centerdot dg$ (resp. $\mathscr{B}^* \int_{I_0} f \centerdot dg$).

Remark 1. It can be easily proved by standard methods that the $\mathscr{B}$-integral and the $\mathscr{B}^*$-integral are linear with respect to both the integrand and integrator as well as are additive $\mathscr{B}$-interval functions. Also, the so-called *weak Saks-Henstock lemma* [7] is valid for the $\mathscr{B}$-integral.

The $\mathscr{F}$-integral and the $\mathscr{F}^*$-integral are called the *Henstock-Stieltjes integral ($\mathcal{H}$-integral)* and the *interior Henstock-Stieltjes integral ($\mathcal{H}^*$-integral)*, respectively. It seems that the notion of the $\mathcal{H}^*$-integral for real-valued functions was first proposed by Š. Schwabik in [37]. It follows from [36, Example 2.1] and [37, Proposition 2] that the $\mathscr{F}^*$-integral is essentially wider than the $\mathscr{F}$-integral. The proofs of all basic properties of the $\mathscr{F}$-integral for Banach-valued functions can be found in [38]. We need only Lemma 2 in what follows.

Lemma 2. *Suppose that $f : [a, b] \to \mathfrak{X}$ and $g : [a, b] \to \mathfrak{Y}$ are Banach-valued functions such that g is locally bounded at a point c and f is $\mathscr{F}$-integrable with respect to g on $[a, b]$. Then the indefinite $\mathscr{F}$-integral of f with respect to g is locally bounded at c.*

Proof. It follows from [38, Theorem 19] that

$$\lim_{\substack{t\to c\\ t\in[a,b]}} \left\{ \mathscr{F}\int_a^t f \centerdot dg + f(c) \centerdot [g(c) - g(t)] \right\} = \mathscr{F}\int_a^c f \centerdot dg.$$

Hence

$$\limsup_{\substack{t\to c\\ t\in[a,b]}} \left\| \mathscr{F}\int_a^t f \centerdot dg \right\| \leq \left\| \mathscr{F}\int_a^c f \centerdot dg \right\| + \|f(c)\| \cdot \limsup_{\substack{t\to c\\ t\in[a,b]}} \|g(c) - g(t)\| < +\infty.$$

This completes our proof of Lemma 2. □

[7] This version of the Saks-Henstock lemma, in contrast to the ordinary version, is fulfilled for Banach-valued gauge integrals (see [40, Lemma 1.2]).

Definition 3.2. A function $g : [a,b] \to \mathfrak{Y}$ is of *weakly bounded variation*, (wVB) on a set $E \subset [a,b]$ if there exists a positive number M such that

$$\left\| \sum_{k=1}^{K} x_k \centerdot \Delta g(I_k) \right\| \le M \cdot \max_{1\le k\le K} \|x_k\| \tag{10}$$

for each finite collection $\{x_k\}_{k=1}^{K} \subset \mathfrak{X}$ and for each finite collection $\{I_k\}_{k=1}^{K}$ of pairwise non-overlapping compact intervals with $\partial I_k \subset E$.

The lower bound of those M is denoted by $\mathbf{W}(g,E)$ and is called the *w-weak variation of g on E.*

Definition 3.3. A function $g : [a,b] \to \mathfrak{Y}$ is of *weakly bounded variation in the restricted sense*, (wVB^*) on a set $E \subset [a,b]$ if there exists a positive number M such that (10) holds for each finite collection $\{x_k\}_{k=1}^{K} \subset \mathfrak{X}$ and for each finite collection $\{I_k\}_{k=1}^{K}$ of pairwise non-overlapping compact intervals with $I_k \subset [a,b]$ and $\partial I_k \cap E \neq \emptyset$.

The lower bound of those M is denoted by $\mathbf{W}^*(g,E)$ and is called the *w-strong variation of g on E.*

We remark that if the spaces $\mathfrak{X}$, $\mathfrak{Y}$ and $\mathfrak{Z}$ coincide with $\mathbb{R}$ and the bilinear mapping " $\centerdot$ " is ordinary multiplication, then it is possible to substitute

$$\sum_{k=1}^{K} |\Delta g(I_k)| \le M$$

for (10) in Definitions 3.2 and 3.3. In this case Lemma 3 is very important for our aims. The proof of this assertion can be found in [22, Lemma 5.3.8].

Lemma 3. *Let f be a real-valued function defined on $[a,b]$ and $E \subset [a,b]$. Then, f is wVB^* on E if and only if f is VB^* on E and is bounded on $[a,b]$.*

Definition 3.4. A function $g : [a,b] \to \mathfrak{Y}$ is of *generalized weakly bounded variation ($wVBG$)* on a set $E \subset [a,b]$ if E can be written as a countable union of sets on each of which g is wVB. The function g is of *generalized weakly bounded variation in the restricted sense ($wVBG^*$)* on E if E can be written as a countable union of sets on each of which g is wVB^*.

Lemma 4. *Let $g : [a,b] \to \mathfrak{Y}$ be $wVBG^*$ (resp. $wVBG$) on a set $E \subset [a,b]$. Then E can be written as a countable disjoint union of sets on each of which g is wVB^* (resp. wVB).*

Proof. The proof is clear. $\square$

Example 1. This example is borrowed from [6]. Let g be the function from $[0,1]$ to $L^\infty[0,1]$ such that $g(t) = \mathbf{1}_{[0,t]}$ for all $t \in [0,1]$ and let the bilinear mapping " $\centerdot$ " from $\mathbb{R} \times L^\infty[0,1]$ to $L^\infty[0,1]$ be scalar multiplication. Then it can easily be checked that g is wVB^* on $[0,1]$, moreover $\mathbf{W}^*(g,[0,1]) = 1$. Yet $t' \neq t''$ implies $\|g(t') - g(t'')\|_\infty = 1$. Therefore g is not of bounded variation on $[0,1]$. Note that g is discontinuous at each point of $[0,1]$.

3.2 Necessary Conditions for Integration by Parts.

This section will give necessary conditions for integration by parts in the case of the $\mathcal{H}$-integral. The main result, Theorem L, is analogous to Theorem I and Theorem J. This theorem was proved by the author in [28].

We say that vector-valued functions f and g satisfy the *conditions of the first kind* at a point τ if they have neither simultaneous left-hand discontinuity nor simultaneous right-hand discontinuity at τ.

Theorem L. *Suppose that the bilinear mapping " $\centerdot$ " is such that $x \centerdot y = 0$ if and only if $x = 0$ or $y = 0$. Let $f : [a,b] \to \mathfrak{X}$ and $g : [a,b] \to \mathfrak{Y}$ be functions possessing all the unilateral limits at a point τ. In order that*

$$\mathscr{F}\int_I f \centerdot dg + \mathscr{F}\int_I df \centerdot g = \Delta(f \centerdot g)(I)$$

holds for each compact interval I it is necessary that f and g satisfy the conditions of the first kind at τ.

3.3 Sufficient Conditions for Integration by Parts.

This section will give sufficient conditions for integration by parts in the case of the $\mathcal{H}$-integral and in the case of the $\mathcal{H}^*$-integral. Henstock's methods (see [10]) are employed for generalizing our previous results obtained in [28].

First of all, we prove several lemmas. Recall that a vector-valued function has a *removable discontinuity* at a point τ if its unilateral limits at τ are equal and it is not continuous at τ.

Lemma 5. *Suppose that $\pi = \{([t_{i-1}, t_i], \tau_i)\}_{i=1}^n$ is a set of interval-point pairs such that $[a,b] = \cup_{i=1}^n [t_{i-1}, t_i]$ and also $\tau_i \in [a,b]$, and $f : [a,b] \to \mathfrak{X}$, $g : [a,b] \to \mathfrak{Y}$ being given. Then the following equation holds:*

$$\begin{aligned}\Delta(f,g;\pi) \equiv & f(b) \centerdot g(b) - f(a) \centerdot g(a) - \sum_{i=1}^n f(\tau_i) \centerdot \Delta g([t_{i-1}, t_i]) \\ & - \sum_{i=1}^n \Delta f([t_{i-1}, t_i]) \centerdot g(\tau_i) = \sum_{i=1}^n \Delta_i(f,g;\pi),\end{aligned}$$

where $\Delta_i(f, g; \pi)$ *can be expressed as*

1. $\Delta f([\tau_i, t_i]) \centerdot \Delta g([t_{i-1}, t_i]) - \Delta f([t_{i-1}, t_i]) \centerdot \Delta g([t_{i-1}, \tau_i])$;
2. $\Delta f([t_{i-1}, t_i]) \centerdot \Delta g([\tau_i, t_i]) - \Delta f([t_{i-1}, \tau_i]) \centerdot \Delta g([t_{i-1}, t_i])$;
3. $\Delta f([\tau_i, t_i]) \centerdot \Delta g([\tau_i, t_i]) - \Delta f([t_{i-1}, \tau_i]) \centerdot \Delta g([t_{i-1}, \tau_i])$;
4. *if* f *and* g *have removable discontinuities at* τ_i, *then*
$$\begin{aligned}&\Delta f([\tau_i, t_i]) \centerdot \{\Delta g([t_{i-1}, \tau_i)) + \Delta g((\tau_i, t_i])\}\\ &- \{\Delta f([t_{i-1}, \tau_i)) + \Delta f((\tau_i, t_i])\} \centerdot \Delta g([t_{i-1}, \tau_i]);\end{aligned}$$
5. *if* f *and* g *are regular at* τ_i, *then*
$$\begin{aligned}&\left\{\frac{\Delta f((\tau_i, t_i]) - \Delta f([t_{i-1}, \tau_i))}{2}\right\} \centerdot \Delta g([t_{i-1}, t_i])\\ &+ \Delta f([t_{i-1}, t_i]) \centerdot \left\{\frac{\Delta g((\tau_i, t_i]) - \Delta g([t_{i-1}, \tau_i))}{2}\right\}.\end{aligned}$$

PROOF. The lemma can be proved by direct calculations. □

Lemma 6 (The weak variational condition, cf. Theorem J, Kurzweil [17], Schwabik [39]). *Let* $\mathscr{B}$ *be a basis in* $[a, b]$ *possessing the p-property (resp.* p^**-property), and let* $f : [a, b] \to \mathfrak{X}$ *and* $g : [a, b] \to \mathfrak{Y}$ *be functions such that, under the same notations as above,*

$$\inf_{\delta} \sup_{\pi} \|\Delta(f, g; \pi)\| = 0, \tag{11}$$

where δ *and* π *run over gauges on* $[a, b]$ *and* $\mathscr{B}_\delta$*-partitions (resp.* $\mathscr{B}^*_\delta$*-partitions) of* $[a, b]$, *respectively. Then*

$$\int_a^b f \centerdot dg + \int_a^b df \centerdot g = f(b) \centerdot g(b) - f(a) \centerdot g(a) \tag{12}$$

holds for $\mathscr{B}$*-integrals (resp.* $\mathscr{B}^*$*-integrals) provided that at least one integral exists.*

PROOF. Assume without loss of generality that the $\mathscr{B}$-integral (resp. $\mathscr{B}^*$-integral) $\int_a^b df \centerdot g$ exists. Denote by z the integral $\int_a^b df \centerdot g$, by B the vector $f(b) \centerdot g(b)$ and by A the vector $f(a) \centerdot g(a)$.

Let π be an arbitrary $\mathscr{B}$-partition (resp. $\mathscr{B}^*$-partition) of $[a,b]$. Then we have

$$\begin{aligned}&\Big\|(B-A)-z-\sum_{(I,t)\in\pi} f(t)\centerdot\Delta g(I)\Big\|\\ &=\Big\|(B-A)-\sum_{(I,t)\in\pi}[f(t)\centerdot\Delta g(I)+\Delta f(I)\centerdot g(t)]\\ &\quad+\sum_{(I,t)\in\pi}\Delta f(I)\centerdot g(t)-z\Big\| = \Big\|\Delta(f,g;\pi)+\sum_{(I,t)\in\pi}\Delta f(I)\centerdot g(t)-z\Big\|\\ &\le \|\Delta(f,g;\pi)\|+\Big\|\sum_{(I,t)\in\pi}\Delta f(I)\centerdot g(t)-z\Big\|.\end{aligned}$$

Using (11) and the definition of the $\mathscr{B}$-integral (resp. $\mathscr{B}^*$-integral), we get

$$\inf_{\delta}\sup_{\pi}\Big\|(B-A)-z-\sum_{(I,t)\in\pi} f(t)\centerdot\Delta g(I)\Big\|=0.$$

Thus we obtain (12) for $\mathscr{B}$-integrals (resp. $\mathscr{B}^*$-integrals). □

We say that vector-valued functions f and g satisfy the *conditions of the second kind* at a point τ if at least one of the following conditions holds:

(i) they satisfy the conditions of the first kind at τ;

(ii) they are regular at τ;

(iii) they have removable discontinuities at τ.

Further, we say that vector-valued functions f and g satisfy the conditions of the first (resp. second) kind in an I if they satisfy those at each point of I.

Lemma 7. *Let $f:[a,b]\to\mathfrak{X}$ and $g:[a,b]\to\mathfrak{Y}$ be $wVBG^*$-functions on $[a,b]$. If f and g satisfy the conditions of the first (resp. second) kind in $[a,b]$, then* (11) *holds for $\mathscr{F}$-partitions (resp. $\mathscr{F}^*$-partitions) of* $[a,b]$.

PROOF. To be definite, we will prove the assertion of our lemma for $\mathscr{F}^*$-partitions.

It follows from Lemma 4 that the interval $[a,b]$ can be divided into sets $\{P_k\}_{k=1}^{\infty}$ and into sets $\{Q_l\}_{l=1}^{\infty}$ so that f is wVB^* on each of P_k and g is wVB^* on each of Q_l.

To each point $\tau \in [a,b]$ assign three numbers $k(\tau)$, $l(\tau)$, and $M(\tau)$ so that $\tau \in P_{k(\tau)}$, $\tau \in Q_{l(\tau)}$, and

$$M(\tau) = \max\{\mathbf{W}^*(f, P_{k(\tau)}), \mathbf{W}^*(g, Q_{l(\tau)})\}.$$

Fix a positive number ϵ. Define a gauge δ at τ as follows.

1. If $f(\tau) = f(\tau+)$ or $g(\tau) = g(\tau+)$, we can choose $\delta(\tau)$ so that

$$2^{l(\tau)+1}\, M(\tau) \cdot \|\Delta f([\tau, t])\| < \epsilon \text{ or } 2^{k(\tau)+1}\, M(\tau) \cdot \|\Delta g([\tau, t])\| < \epsilon$$

   holds for each $t \in [\tau, \tau + \delta(\tau))$.

2. If $f(\tau-) = f(\tau)$ or $g(\tau-) = g(\tau)$, we can choose $\delta(\tau)$ so that

$$2^{l(\tau)+1}\, M(\tau) \cdot \|\Delta f([t, \tau])\| < \epsilon \text{ or } 2^{k(\tau)+1}\, M(\tau) \cdot \|\Delta g([t, \tau])\| < \epsilon$$

   holds for each $t \in (\tau - \delta(\tau), \tau]$.

3. If both f and g are regular at τ or have removable discontinuities at τ, we can choose $\delta(\tau)$ so that

$$2^{l(\tau)+1}\, M(\tau) \cdot \|\Delta f((\tau, t])\| < \epsilon \text{ and } 2^{k(\tau)+1}\, M(\tau) \cdot \|\Delta g((\tau, t])\| < \epsilon$$

   hold for each $t \in (\tau, \tau + \delta(\tau))$ as well as

$$2^{l(\tau)+1}\, M(\tau) \cdot \|\Delta f([t, \tau))\| < \epsilon \text{ and } 2^{k(\tau)+1}\, M(\tau) \cdot \|\Delta g([t, \tau))\| < \epsilon$$

   for hold each $t \in (\tau - \delta(\tau), \tau)$.

Consider an arbitrary $\mathscr{F}^*_\delta$-partition $\pi = \{([t_{i-1}, t_i], \tau_i)\}_{i=1}^n$ of $[a,b]$. Denote, by K, the maximum $\max\limits_{1\le i\le n}\{k(\tau_i)\}$ and, by L, the maximum $\max\limits_{1\le i\le n}\{l(\tau_i)\}$.

We now introduce the sets of integers S_0, S_1, $S_\pm$, $S_\mp$, S_2, and S_3 as follows.

$S_0 = \{i : f \text{ or } g \text{ is continuous at } \tau_i\}; S_1 = \{i : f \text{ is continuous at } \tau_i\};$
$S_\pm = \{i : f(\tau_i) = f(\tau_i+) \text{ and } g(\tau_i-) = g(\tau_i)\} \setminus S_0;$
$S_\mp = \{i : f(\tau_i-) = f(\tau_i) \text{ and } g(\tau_i) = g(\tau_i+)\} \setminus S_0;$
$S_2 = \{i : f \text{ and } g \text{ are regular at } \tau_i\} \setminus S_0;$
$S_3 = \{i : f \text{ and } g \text{ have removable discontinuities at } \tau_i\}.$

Using item 3 of Lemma 5, we get

$$\begin{aligned}\Big\|\sum_{i\in S_\pm}\Delta_i(f,g;\pi)\Big\| &\le \sum_{l=1}^{L}\Big\|\sum_{S_\pm\ni i:l(\tau_i)=l}\Delta f([\tau_i,t_i])\centerdot\Delta g([\tau_i,t_i])\Big\|\\ &\quad+\sum_{k=1}^{K}\Big\|\sum_{\substack{S_\pm\ni i:\\ k(\tau_i)=k}}\Delta f([t_{i-1},\tau_i])\centerdot\Delta g([t_{i-1},\tau_i])\Big\|\\ &\le \sum_{l=1}^{L}\epsilon\,2^{-l-1}+\sum_{k=1}^{K}\epsilon\,2^{-k-1}<\epsilon.\end{aligned}$$

In addition, we get the estimate $\big\|\sum_{i\in S_\mp}\Delta_i(f,g;\pi)\big\|<\epsilon$ in a similar manner. Further,

$$\begin{aligned}\Big\|\sum_{i\in S_1}\Delta_i(f,g;\pi)\Big\| &\le \sum_{l=1}^{L}\Big\|\sum_{S_1\ni i:l(\tau_i)=l}\Delta f([\tau_i,t_i])\centerdot\Delta g([\tau_i,t_i])\Big\|\\ &\quad+\sum_{l=1}^{L}\Big\|\sum_{S_1\ni i:l(\tau_i)=l}\Delta f([t_{i-1},\tau_i])\centerdot\Delta g([t_{i-1},\tau_i])\Big\|\\ &\le \sum_{l=1}^{L}\{\epsilon\,2^{-l-1}+\epsilon\,2^{-l-1}\}<\epsilon\end{aligned}$$

and in a similar manner we get $\big\|\sum_{i\in S_0\setminus S_1}\Delta_i(f,g;\pi)\big\|<\epsilon$.

Using item 4 of Lemma 5, we get

$$\begin{aligned}\Big\|\sum_{i\in S_3}\Delta_i(f,g;\pi)\Big\| &\le \sum_{k=1}^{K}\Big\|\sum_{\substack{S_3\ni i:\\ k(\tau_i)=k}}\Delta f([\tau_i,t_i])\centerdot\{\Delta g([t_{i-1},\tau_i))+\Delta g((\tau_i,t_i])\}\Big\|\\ &\quad+\sum_{l=1}^{L}\Big\|\sum_{S_3\ni i:l(\tau_i)=l}\{\Delta f([t_{i-1},\tau_i))+\Delta f((\tau_i,t_i])\}\centerdot\Delta g([t_{i-1},\tau_i])\Big\|\\ &\le \sum_{k=1}^{K}\epsilon\,2^{-k}+\sum_{l=1}^{L}\epsilon\,2^{-l}<2\,\epsilon.\end{aligned}$$

Using item 5 of Lemma 5, we get

$$\Big\|\sum_{i\in S_2}\Delta_i(f,g;\pi)\Big\| \le \sum_{l=1}^{L}\Big\|\sum_{\substack{S_2\ni i:\\ l(\tau_i)=l}}\Big\{\frac{\Delta f((\tau_i,t_i])-\Delta f([t_{i-1},\tau_i))}{2}\Big\}\cdot\Delta g([t_{i-1},t_i])\Big\|$$

$$+\sum_{k=1}^{K}\Big\|\sum_{\substack{S_2\ni i:\\ k(\tau_i)=k}}\Delta f([t_{i-1},t_i])\cdot\Big\{\frac{\Delta g((\tau_i,t_i])-\Delta g([t_{i-1},\tau_i))}{2}\Big\}\Big\|$$

$$\le \sum_{l=1}^{L}\epsilon\, 2^{-l-1}+\sum_{k=1}^{K}\epsilon\, 2^{-k-1}<\epsilon.$$

Thus, we obtain the estimate $\|\Delta(f,g;\pi)\|<7\,\epsilon$ for each $\mathscr{F}^*_\delta$-partition π of $[a,b]$ and, so, our lemma is proved. □

Combining Lemmas 6 and 7, we obtain the following theorem.

Theorem 1. *Let $f:[a,b]\to\mathfrak{X}$ and $g:[a,b]\to\mathfrak{Y}$ be $wVBG^*$-functions on $[a,b]$. If f and g satisfy the conditions of the first (resp. second) kind in $[a,b]$, then* (12) *holds for $\mathscr{F}$-integrals (resp. $\mathscr{F}^*$-integrals) provided that at least one integral in the left side of* (12) *exists.*

We will now prove a generalization of Pfeffer's Theorem K.

Corollary 1. *The assertion of Theorem K holds provided that α is bounded and VBG^* and at least one of the functions F and G is continuous.*

Proof. It follows from [46, p. 592, Theorem 9] that F and G are VBG^* on $[a,b]$. The application of Lemmas 2 and 3 and Theorem 1 yields our corollary. □

Remark 2. Theorem G and Theorem 1 are independent of one another. It can be demonstrated in the following way. For the functions $f=\mathbf{1}_{\mathbb{Q}}$ and $g=\mathbf{1}_{\{1\}}$ on $[0,1]$ none of the integrals $\mathcal{H}^*\int_0^1 f\,dg$ and $\mathcal{H}^*\int_0^1 g\,df$ exist, the same being true for the integral $\mathcal{P}^*\int_0^1 f\,dg$. However, $\mathcal{Y}^*\int_0^1 f\,dg=1, \mathcal{P}^*\int_0^1 g\,df=0$ and $f(1)\,g(1)-f(0)\,g(0)=1$. This agrees completely with Theorem G.

Remark 3. It is not possible to replace the $wVBG^*$-property, even for one of the integrands in Theorem 1, with the $wVBG$-property. Consider for example the functions $f=\mathbf{1}_{\mathbb{Q}}$, $g(t)=t$. It is well known that $\mathcal{H}\int_0^1\mathbf{1}_{\mathbb{Q}}\,dt=0$. However, the integral $\mathcal{H}^*\int_0^1 t\,d\mathbf{1}_{\mathbb{Q}}$ does not exist. Finally note that f is $wVBG$ on $[0,1]$, while it is not $wVBG^*$ on $[0,1]$.

Remark 4. A proof patterned after our proof of Theorem 1 shows that (12) holds for $\mathscr{F}$-integrals provided that at least one integral in the left side of formula (12) exists as well as one of the integrands is continuous on $[a,b]$ and the other is $wVBG^*$ on $[a,b]$.

Next we give three formulae for integration by parts when the integrands have discontinuities of the first kind at a or at b.

Corollary 2. *Let $f:[a,b]\to\mathfrak{X}$ and $g:[a,b]\to\mathfrak{Y}$ be $wVBG^*$-functions on $[a,b]$ possessing right-hand limits at a. If f and g satisfy the conditions of the first (resp. second) kind in $(a,b]$, then*

$$\int_a^b f\centerdot dg+\int_a^b df\centerdot g=f(b)\centerdot g(b)-f(a)\centerdot g(a)-\Delta^+f(a)\centerdot\Delta^+g(a) \tag{13}$$

holds for $\mathscr{F}$-integrals (resp. $\mathscr{F}^*$-integrals) provided that at least one integral exists.

PROOF. It can easily be checked that

$$\begin{aligned}\int_a^b\left\{\Delta^+f(a)\mathbf{1}_{(a,b]}\right\}\centerdot dg&=\Delta^+f(a)\centerdot\Delta g((a,b]),\\ \int_a^b d\left\{\Delta^+f(a)\mathbf{1}_{(a,b]}\right\}\centerdot g&=\Delta^+f(a)\centerdot g(a).\end{aligned}$$

Using Theorem 1, we get

$$\begin{aligned}&\int_a^b\left\{f-\Delta^+f(a)\mathbf{1}_{(a,b]}\right\}\centerdot dg+\int_a^b d\left\{f-\Delta^+f(a)\mathbf{1}_{(a,b]}\right\}\centerdot g\\ &=\left\{f(b)-\Delta^+f(a)\right\}\centerdot g(b)-f(a)\centerdot g(a).\end{aligned}$$

Hence,

$$\begin{aligned}\int_a^b f\centerdot dg+\int_a^b df\centerdot g=&\left\{f(b)-\Delta^+f(a)\right\}\centerdot g(b)-f(a)\centerdot g(a)\\ &+\Delta^+f(a)\centerdot\Delta g((a,b])+\Delta^+f(a)\centerdot g(a)\\ =&f(b)\centerdot g(b)-f(a)\centerdot g(a)-\Delta^+f(a)\centerdot\Delta^+g(a).\end{aligned}$$

This completes our proof of Corollary 2. □

Corollaries 3 and 4 can be proved in a similar way.

Corollary 3. *Let* $f : [a, b] \to \mathfrak{X}$ *and* $g : [a, b] \to \mathfrak{Y}$ *be* $wVBG^*$*-functions on* $[a, b]$ *possessing the left-hand limits at* b*. If* f *and* g *satisfy the conditions of the first (resp. second) kind in* $[a, b)$*, then*

$$\int_a^b f \centerdot dg + \int_a^b df \centerdot g = f(b) \centerdot g(b) - f(a) \centerdot g(a) + \Delta^- f(b) \centerdot \Delta^- g(b) \qquad (14)$$

holds for $\mathscr{F}$*-integrals (resp.* $\mathscr{F}^*$*-integrals) provided that at least one integral exists.*

Corollary 4. *Let* $f : [a, b] \to \mathfrak{X}$ *and* $g : [a, b] \to \mathfrak{Y}$ *be* $wVBG^*$*-functions on* $[a, b]$ *possessing all the unilateral limits at* a *and at* b*. If* f *and* g *satisfy the conditions of the first (resp. second) kind in* (a, b)*, then*

$$\begin{aligned} \int_a^b f \centerdot dg + \int_a^b df \centerdot g =& f(b) \centerdot g(b) - f(a) \centerdot g(a) \\ &+ \Delta^- f(b) \centerdot \Delta^- g(b) - \Delta^+ f(a) \centerdot \Delta^+ g(a) \end{aligned}$$

holds for $\mathscr{F}$*-integrals (resp.* $\mathscr{F}^*$*-integrals) provided that at least one integral exists.*

4 Further Problems.

The above discussion reveals certain questions related to integration by parts for Stieltjes-type integrals. Interesting problems seem to be the following:

- Obtain integration-by-parts type formulae analogous to those in Theorem D under the hypothesis that the condition of bounded variation is omitted or is replaced with a weaker condition.
- Find pairs of integrals of the Stieltjes type that might be contained in integration-by-parts type formulae.

As for Banach-valued functions of weakly bounded variation, it would be interesting to find some structural properties of those functions and to determine how these properties depend on structural properties of $(\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z})$.

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