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ON QUASI-UNIFORM CONVERGENCE OF SEQUENCES OF s_1-STRONGLY QUASI-CONTINUOUS FUNCTIONS ON $\mathbb{R}^m$

Abstract

A function $f : \mathbb{R}^m \to \mathbb{R}$ is called s_1-strongly quasi-continuous at a point $\mathbf{x} \in \mathbb{R}^m$ if for each real $\varepsilon > 0$ and for each set $A \ni \mathbf{x}$ belonging to the density topology, there is a nonempty open set V such that

$$\emptyset \neq A \cap V \subset f^{-1}((f(\mathbf{x}) - \varepsilon, f(\mathbf{x}) + \varepsilon)) \cap C(f),$$

where $C(f)$ denotes the set of continuity points of f. It is proved that every λ-almost everywhere continuous function $f : \mathbb{R}^m \to \mathbb{R}$ is the quasi-uniform limit of a sequence of s_1-strongly quasi-continuous functions and that each measurable function $f : \mathbb{R}^m \to \mathbb{R}$ is the quasi-uniform limit of a sequence of approximately quasi-continuous functions $f : \mathbb{R}^m \to \mathbb{R}$.

Let $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ be respectively the set of all reals, the set of all integers and the set of all positive integers, and let $\mathbb{R}^m$ be m-dimensional product space $\mathbb{R} \times \cdots \times \mathbb{R}$ with the standard metric $|\cdot|$; i.e., using the distance

$$|\mathbf{x} - \mathbf{y}| = \sqrt{\sum_{i=1}^{m} |x_i - y_i|^2}$$

between the points $\mathbf{x} = (x_1, x_2, \ldots, x_m)$, $\mathbf{y} = (y_1, y_2, \ldots, y_m) \in \mathbb{R}^m$. Moreover, let λ, (λ^*) denote Lebesgue measure, (outer Lebesgue measure) in $\mathbb{R}^m$.

For each number $n \in \mathbb{N}$ and for each system of integers $(k_1, \ldots, k_m)$ let

$$P^n_{k_1,\ldots,k_m} = \left[\frac{k_1 - 1}{2^n}, \frac{k_1}{2^n}\right) \times \cdots \times \left[\frac{k_m - 1}{2^n}, \frac{k_m}{2^n}\right).$$

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Moreover, let

$$\mathcal{P}_n = \{P^n_{k_1,\dots,k_m}; k_1,\dots,k_m \in \mathbb{Z}\} \text{ and } \mathcal{P} = \bigcup_n \mathcal{P}_n.$$

Observe that

(1) if $(k_1,\dots,k_m) \neq (l_1,\dots,l_m)$, then $P^n_{k_1,\dots,k_m} \cap P^n_{l_1,\dots,l_m} = \emptyset$;

(2) $\mathbb{R}^m = \bigcup_{k_1,\dots,k_m \in \mathbb{Z}} P^n_{k_1,\dots,k_m}$;

(3) if $n_1 > n_2$, then for each system of integers $(k_1,\dots,k_m)$ there is a unique system of integers $(l_1,\dots,l_m)$ such that $P^{n_1}_{k_1,\dots,k_m} \subset P^{n_2}_{l_1,\dots,l_m}$;

(4) for each point $\mathbf{x} \in \mathbb{R}^m$ and for each index $n \in \mathbb{N}$ there is a unique system of integers $(k_1(\mathbf{x}),\dots,k_m(\mathbf{x}))$ such that $\mathbf{x} \in P^n_{k_1(\mathbf{x}),\dots,k_m(\mathbf{x})} = P^n(\mathbf{x})$.

For a set $A \subset \mathbb{R}^m$ and a point $\mathbf{x} \in \mathbb{R}^m$ let

$$d_u(A,\mathbf{x}) = \limsup_{n\to\infty} \frac{\lambda^*(A \cap P^n(\mathbf{x}))}{\lambda(P^n(\mathbf{x}))}, \ \left(d_l(A,\mathbf{x}) = \liminf_{n\to\infty} \frac{\lambda^*(A \cap P^n(\mathbf{x}))}{\lambda(P^n(\mathbf{x}))} \right)$$

the upper, (lower) outer density of the set $A \subset \mathbb{R}$ at the point $\mathbf{x}$ (compare [1]).

A point $\mathbf{x} \in \mathbb{R}^m$ is called *a density point of a set* $A \subset \mathbb{R}^m$ if there exists a λ-measurable (i.e., measurable in the sense of Lebesgue) set $B \subset A$ such that $d_l(B,\mathbf{x}) = 1$. The family

$$\mathcal{T}_d = \{A \subset \mathbb{R}^m; A \text{ is } \lambda\text{-measurable and } d_l(A,\mathbf{x}) = 1 \text{ for } \mathbf{x} \in A\}$$

is a topology called *the density topology* ([2], [3], [12], [13]).

Moreover, let $\mathcal{T}_e$ be the Euclidean topology in $\mathbb{R}^m$ and let $C(f)$ denote the set of all points at which a real function f is continuous.

We will say that a function $f : \mathbb{R}^m \to \mathbb{R}$ is *s_1-strongly quasi-continuous at a point* $\mathbf{x}$, ($f \in Q_{s_1}(\mathbf{x})$) if for every set $A \in \mathcal{T}_d$ containing $\mathbf{x}$ and for every real $\varepsilon > 0$ there is a nonempty open set U such that

$$f^{-1}((f(\mathbf{x}) - \varepsilon, f(\mathbf{x}) + \varepsilon)) \cap C(f) \supset U \cap A \neq \emptyset, \ \ ([6],[11]).$$

Observe that if there is a nonempty open set $U \subset \mathbb{R}^m \cap C(f)$ such that $d_u(U,\ \mathbf{x}) > 0$ for $\mathbf{x} \in \mathbb{R}^m$ and the restricted function $f|_{(U \cup \{\mathbf{x}\})}$ is continuous at $\mathbf{x}$, then $f \in Q_{s_1}(\mathbf{x})$.

A sequence of functions $f_n : \mathbb{R}^m \to \mathbb{R}$, $n = 1,2,\dots$, is said to be *quasi-uniformly convergent* to a function f ([10], p.143) if it pointwise converges to f and for each real $\varepsilon > 0$ and for each index $i \in \mathbb{N}$ there is an index $p \in \mathbb{N}$ such that for each point $\mathbf{x} \in \mathbb{R}^m$

$$\min(|f_{i+1}(\mathbf{x}) - f(\mathbf{x})|, \dots, |f_{i+p}(\mathbf{x}) - f(\mathbf{x})|) < \varepsilon.$$

It is obvious (compare [6], [7], [11]) that every s_1- strongly quasi-continuous function $f:\mathbb{R}^m\to\mathbb{R}$ is λ- almost everywhere (i.e., almost everywhere with respect to λ) continuous. Since quasi-uniform convergence preserves continuity, the quasi-uniform limit of sequence of s_1-strongly quasi-continuous functions is a λ-almost everywhere continuous function.

We will prove the following.

Theorem 1. *If a function $f:\mathbb{R}^m\to\mathbb{R}$ is λ-almost everywhere continuous, then there are s_1-strongly quasi-continuous functions $g_n:\mathbb{R}^m\to\mathbb{R}$, $n=1,2,\dots,$ such that the sequence (g_n) quasi-uniformly converges to f.*

PROOF. Let cl denote the closure operation and let

$$B=\{y\in\mathbb{R};\lambda(\mathrm{cl}(f^{-1}(y))>0\}.$$

Since the function f is λ-almost everywhere continuous, the set B is countable. Without loss of the generality we can assume that $0\notin B$, because in the contrary case we may consider the function $f-a$, where $a\neq 0$ is a real. Let $L(B)$ be the linear space over the field of all rationals generated by the set B. Since the set $L(B)$ is countable, there is a positive number $c\in\mathbb{R}\setminus L(B)$. Fix $k\in\mathbb{Z}$ and $n\in\mathbb{N}$. If $\frac{k\cdot c}{2^n}\le f(\mathbf{x})<\frac{(k+1)\cdot c}{2^n}$ for $\mathbf{x}\in\mathbb{R}^m$, then we define $f_n(\mathbf{x})=\frac{k\cdot c}{2^n}$. Observe that every function f_n, $n\in\mathbb{N}$, is λ-almost everywhere continuous and if $D(f_n)$ denotes the set of all discontinuity points of f_n, then $D(f_n)$ is a closed set of λ-measure zero. Moreover, $D(f_n)\subset D(f_{n+1})$ for $n\in\mathbb{N}$ and if $\mathbf{x}\in D(f_{n+1})\setminus D(f_n)$ for some $n\in\mathbb{N}$, then for every $i>n$ the inequality $\operatorname{osc}_{f_i}(\mathbf{x})\le\frac{c}{2^{n-1}}$ holds, where $\operatorname{osc}_g(\mathbf{x})$ denote the oscillation of a function g at the point $\mathbf{x}$.

Step 1. Recall that the set $D(f_1)$ is closed and of λ-measure zero. For each point $\mathbf{x}\in D(f_1)$ there is a unique cube $P^1(\mathbf{x})\in\mathcal{P}_1$ such that $\mathbf{x}\in P^1(\mathbf{x})$. Observe that the diameter (with respect to the standard metric in $\mathbb{R}^m$), $\operatorname{diam}(P^1(\mathbf{x}))\le\frac{\sqrt{m}}{2}$. For a such cube $P^1(\mathbf{x})$ there is a finite family of cubes

$$Q_{1,1,\mathbf{x}},\ Q_{2,1,\mathbf{x}},\dots,Q_{i(1,1,\mathbf{x}),1,\mathbf{x}}\in\mathcal{P},$$

whose closures are pairwise disjoint and contained in $\operatorname{int}(P^1(\mathbf{x}))\setminus D(f_1)$ (int denotes the interior operation) and such that

$$\frac{\lambda(\bigcup_{i=1}^{i(1,1,\mathbf{x})}Q_{i,1,\mathbf{x}})}{\lambda(P^1(\mathbf{x}))}>\frac{1}{2}.$$

Moreover, we assume that if for $\mathbf{x}$, $\mathbf{y}\in D(f_1)$ the cubes $P^1(\mathbf{x})$ and $P^1(\mathbf{y})$ are the same, then $i(1,1,\mathbf{x})=i(1,1,\mathbf{y})$ and $Q_{i,1,\mathbf{x}}=Q_{i,1,\mathbf{y}}$ for $i\le i(1,1,\mathbf{x})$. Let

$$S_1^1=\bigcup_{\mathbf{x}\in D(f_1)}\ \bigcup_{i\le i(1,1,\mathbf{x})}Q_{i,1,\mathbf{x}}.$$

Observe that

$$\mathrm{cl}(S_1^1) \setminus D(f_1) = \bigcup_{\mathbf{x}\in D(f_1)} \bigcup_{i\le i(1,1,\mathbf{x})} \mathrm{cl}(Q_{i,1,\mathbf{x}})$$

and the family $\{Q_{i,1,\mathbf{x}}; i \le i(1,1,\mathbf{x})$ and $\mathbf{x} \in D(f_1)\}$ is $\mathcal{P}$-locally finite; i.e., for each point $\mathbf{y} \in \mathbb{R}^m$ there is an index $l \in \mathbb{N}$ such that the family of triples $(i,1,\mathbf{x})$, where $\mathbf{x} \in D(f_1)$, for which $Q_{i,1,\mathbf{x}} \cap P^l(\mathbf{y}) \neq \emptyset$ is finite.

Now, for each point $\mathbf{x} \in D(f_1)$ there is the first positive integer $s(1,2,\mathbf{x})$ such that $\mathrm{diam}(P^{s(1,2,\mathbf{x})}(\mathbf{x})) < \frac{1}{2^2}$ and

$$\mathbf{x} \in P^{s(1,2,\mathbf{x})}(\mathbf{x}) \subset P^1(\mathbf{x}) \setminus \mathrm{cl}(S_1^1).$$

For a such integer $s(1,2,\mathbf{x})$ there is a finite family of cubes

$$Q_{1,s(1,2,\mathbf{x})},\ Q_{2,s(1,2,\mathbf{x})},\ \dots,\ Q_{i(1,s(1,2,\mathbf{x})),s(1,2,\mathbf{x})} \in \mathcal{P},$$

whose closures are pairwise disjoint and contained in $\mathrm{int}(P^{s(1,2,\mathbf{x})}(\mathbf{x})) \setminus D(f_1)$ and such that

$$\frac{\lambda(\bigcup_{i=1}^{i(1,s(1,2,\mathbf{x}))} Q_{i,s(1,2,\mathbf{x})})}{\lambda(P^{s(1,2,\mathbf{x})}(\mathbf{x}))} > 1 - \frac{1}{2^2}.$$

Assume that if for $\mathbf{x},\ \mathbf{y} \in D(f_1)$ the point $\mathbf{y} \in P^{s(1,2,\mathbf{x})}(\mathbf{x})$, then $P^{s(1,2,\mathbf{x})}(\mathbf{x}) = P^{s(1,2,\mathbf{y})}(\mathbf{y})$, $i(1,s(1,2,\mathbf{x})) = i(1,s(1,2,\mathbf{y}))$ and $Q_{i,s(1,2,\mathbf{x})} = Q_{i,s(1,2,\mathbf{y})}$ for $i \le i(1,s(1,2,\mathbf{x}))$.

Let

$$S_2^1 = \bigcup_{\mathbf{x}\in D(f_1)} \bigcup_{i\le i(1,s(1,2,\mathbf{x}))} Q_{i,s(1,2,\mathbf{x})}.$$

Observe that

$$\mathrm{cl}(S_2^1) \setminus D(f_1) = \bigcup_{\mathbf{x}\in D(f_1)} \bigcup_{i\le i(1,s(1,2,\mathbf{x}))} \mathrm{cl}(Q_{i,s(1,2,\mathbf{x})})$$

and the family $\{Q_{i,s(1,2,\mathbf{x})}; i \le i(1,s(1,2,\mathbf{x}))$ and $\mathbf{x} \in D(f_1)\}$ is $\mathcal{P}$-locally finite.

Generally, for $j > 2$, we proceed analogously and for each point $\mathbf{x} \in D(f_1)$ we find the first positive integer $s(1,j,\mathbf{x})$ such that $\mathrm{diam}(P^{s(1,j,\mathbf{x})}(\mathbf{x})) < \frac{1}{2^j}$ and

$$\mathbf{x} \in P^{s(1,j,\mathbf{x})}(\mathbf{x}) \subset P^{s(1,j-1,\mathbf{x})}(\mathbf{x}) \setminus \mathrm{cl}(S_{j-1}^1).$$

For such an integer $s(1,j,\mathbf{x})$ there is a finite family of cubes

$$Q_{1,s(1,j,\mathbf{x})},\ Q_{2,s(1,j,\mathbf{x})},\ \dots,\ Q_{i(1,s(1,j,\mathbf{x})),s(1,j,\mathbf{x})} \in \mathcal{P}$$

whose closures are pairwise disjoint and contained in $\operatorname{int}(P^{s(1,j,\mathbf{x})}(\mathbf{x})) \setminus D(f_1)$ and such that

$$\frac{\lambda(\bigcup_{i=1}^{i(1,s(1,j,\mathbf{x}))} Q_{i,s(1,j,\mathbf{x})})}{\lambda(P^{s(1,j,\mathbf{x})}(\mathbf{x}))} > 1 - \frac{1}{2^j}.$$

Moreover, we assume that if for $\mathbf{x},\ \mathbf{y} \in D(f_1)$ the point $\mathbf{y} \in P^{s(1,j,\mathbf{x})}(\mathbf{x})$, then $P^{s(1,j,\mathbf{x})}(\mathbf{x}) = P^{s(1,j,\mathbf{y})}(\mathbf{y})$, $i(1,s(1,j,\mathbf{x})) = i(1,s(1,j,\mathbf{y}))$ and $Q_{i,s(1,j,\mathbf{x})} = Q_{i,s(1,j,\mathbf{y})}$ for $i \le i(1,s(1,j,\mathbf{x}))$. Let

$$S_j^1 = \bigcup_{\mathbf{x}\in D(f_1)} \ \bigcup_{i\le i(1,s(1,j,\mathbf{x}))} Q_{i,s(1,j,\mathbf{x})}.$$

Then

$$\operatorname{cl}(S_j^1) \setminus D(f_1) = \bigcup_{\mathbf{x}\in D(f_1)} \ \bigcup_{i\le i(1,s(1,j,\mathbf{x}))} \operatorname{cl}(Q_{i,s(1,j,\mathbf{x})})$$

and the family $\{Q_{i,s(1,j,\mathbf{x})}; i \le i(1,s(1,j,\mathbf{x}))$ and $\mathbf{x} \in D(f_1)\}$ is $\mathcal{P}$-locally finite.

Now, let N_l, $l \in \mathbb{Z}$, be pairwise disjoint infinite subsets of positive integers such that $\mathbb{N} = \bigcup_{l\in\mathbb{Z}} N_l$. Observe that for each index $l \in \mathbb{Z}$ and for each point $\mathbf{x} \in D(f_1)$ the upper density

$$d_u\left(\bigcup_{j\in N_l} \operatorname{int}(S_j^1),\ \mathbf{x}\right) = 1.$$

Let

$$g_1(\mathbf{x}) = \begin{cases} \frac{k\cdot c}{2} & \text{if } \mathbf{x} \in S_j^1,\ j \in N_{2k-1},\ k \in \mathbb{Z} \\ f_1(\mathbf{x}) & \text{otherwise on } \mathbb{R}^m \end{cases}$$

and let

$$g_2(\mathbf{x}) = \begin{cases} \frac{k\cdot c}{2} & \text{if } \mathbf{x} \in S_j^1,\ j \in N_{2k},\ k \in \mathbb{Z} \\ f_1(\mathbf{x}) & \text{otherwise on } \mathbb{R}^m. \end{cases}$$

The functions g_1, g_2 are s_1-strongly quasi-continuous at each point $\mathbf{x}$. Indeed, if $\mathbf{x} \in D(f_1)$, then there is an integer k with $f_1(\mathbf{x}) = \frac{k\cdot c}{2}$. Since $\mathbf{x} \in D(f_1)$, for each positive integer $j \in N_{2k-1}$ there is a cube $P^{1,s(1,j,\mathbf{x})}(\mathbf{x}) \ni \mathbf{x}$. But

$$\frac{\lambda(S_j^1 \cap P^{1,s(1,j,\mathbf{x})}(\mathbf{x}))}{\lambda(P^{1,s(1,j,\mathbf{x})}(\mathbf{x}))} > 1 - \frac{1}{2^j}\ ,$$

$g_1(\mathbf{x}) = f_1(\mathbf{x})$ and $\operatorname{int}(S_j^1) \cap P^{1,s(1,j,\mathbf{x})}(\mathbf{x}) \subset C(g_1)$, so

$$d_u\left(\operatorname{int}\left((g_1)^{-1}(\frac{k\cdot c}{2})\right),\ \mathbf{x}\right) = 1$$

and consequently g_1 is s_1-strongly quasi-continuous at $\mathbf{x}$. If $\mathbf{x} \in \mathbb{R}^m \setminus D(f_1)$, then from the construction of g_1 follows that $g_1 \in Q_{s_1}(\mathbf{x})$.

Analogously we can show that $g_2 \in Q_{s_1}(\mathbf{x})$ for each point $\mathbf{x} \in \mathbb{R}^m$. Moreover,

$$|f_1 - f| < \frac{c}{2} \text{ and } \min(|g_1 - f_1|, |g_2 - f_1|) = 0, \text{ so}$$

$$\min(|g_1 - f|, |g_2 - f|) \leq \min(|g_1 - f_1| + |f_1 - f|, |g_2 - f_1| + |f_1 - f|) < \frac{c}{2}.$$

Step 2. For a nonempty closed set $H \subset \mathbb{R}^m$ and for a real $\eta > 0$, we put

$$\mathcal{O}(H, \eta) = \bigcup_{\mathbf{x} \in H} K(\mathbf{x}, \eta), \text{ where } K(\mathbf{x}, \eta) = \{\mathbf{u} \in \mathbb{R}^m; |\mathbf{u} - \mathbf{x}| < \eta\}.$$

The set $D(f_2)$ is closed and of λ-measure zero. Let

$$D_2^1 = (D(f_2) \setminus D(f_1)) \cap \left(\bigcup_{j \in \mathbb{N}} S_j^1\right), \text{ and } D_2^2 = (D(f_2) \setminus D(f_1)) \setminus D_2^1.$$

If for an index $p_0 \in \mathbb{N}$ and a cube $Q_{i,s(1,p_0,\mathbf{y})}$, where $\mathbf{y} \in D(f_1)$ and $i \leq i(1, s(1, p_0, \mathbf{y}))$, the set

$$D_{i,s(1,p_0,\mathbf{y})} = D_2^1 \cap Q_{i,s(1,p_0,\mathbf{y})} \neq \emptyset,$$

then we find an open (in $Q_{i,s(1,p_0,\mathbf{y})}$) set $U(Q_{i,s(1,p_0,\mathbf{y})}) \subset Q_{i,s(1,p_0,\mathbf{y})}$ containing $D_{i,s(1,p_0,\mathbf{y})}$ such that

$$\frac{\lambda\left(\bigcup_{i \leq i(1,s(1,p_0,\mathbf{y}))} U(Q_{i,s(1,p_0,\mathbf{y})})\right)}{\lambda\left(S_{p_0}^1\right)} < \frac{1}{2^{p_0}}$$

and $d_u(U(Q_{i,s(1,p_0,\mathbf{y})}), \mathbf{x}) = 0$ for $\mathbf{x} \in \operatorname{Fr}(Q_{i,s(1,p_0,\mathbf{y})})$, where $\operatorname{Fr}(H)$ denotes the boundary of the set H. If $D_{i,s(1,p_0,\mathbf{y})} = \emptyset$, then we take $U(Q_{i,s(1,p_0,\mathbf{y})}) = \emptyset$. Thus, for every $p \in \mathbb{N}$ such that $S_p^1 \cap D_2^1 \neq \emptyset$ and for $\mathbf{x} \in \operatorname{Fr}(Q_{i,s(1,p,\mathbf{y})})$,

$$\lambda\left(\bigcup_{i \leq i(1,s(1,p,\mathbf{y}))} U(Q_{i,s(1,p,\mathbf{y})})\right) < \frac{1}{2^p} \cdot \lambda\left(S_p^1\right), \tag{$*$}$$

where $U(Q_{i,s(1,p,\mathbf{y})}) \supset D_{i,s(1,p,\mathbf{y})}$ and $d_u(U(Q_{i,s(1,p,\mathbf{y})}), \mathbf{x}) = 0$.

Similarly, as in the first step, for each point $\mathbf{x} \in D(f_2) \setminus D(f_1)$ there is the first positive integer $s(2, 1, \mathbf{x})$ such that:

- if $\mathbf{x} \in D_{i,s(1,p,\mathbf{y})}$ for $i \leq i(1, s(1, p, \mathbf{y}))$, $p \in \mathbb{N}$ and $\mathbf{y} \in D(f_1)$, then

$$\mathbf{x} \in P^{s(2,1,\mathbf{x})}(\mathbf{x}) \subset U(Q_{i,s(1,p,\mathbf{y})}) \cap \mathcal{O}\left(D_2^1, \frac{1}{2^4}\right);$$

- if $\mathbf{x} \in D_2^2$, then

$$\mathbf{x} \in P^{s(2,1,\mathbf{x})}(\mathbf{x}) \ \subset \ \mathcal{O}\left(D_2^2,\ \frac{1}{2^4}\right) \setminus \bigcup_{j\in\mathbb{N}} \mathrm{cl}(S_j^1),$$

- $\mathrm{diam}(P^{s(2,1,\mathbf{x})}(\mathbf{x})) < \frac{1}{2^4}$ and f_1 is constant on $P^{s(2,1,\mathbf{x})}(\mathbf{x})$.

For a such positive integer $s(2,1,x)$ there is a finite family of cubes

$$Q_{1,s(2,1,\mathbf{x})},\ Q_{2,s(2,1,\mathbf{x})},\ \ldots,\ Q_{i(1,s(2,1,\mathbf{x})),s(2,1,\mathbf{x})} \in \mathcal{P},$$

whose closures are pairwise disjoint and contained in $\mathrm{int}(P^{s(2,1,\mathbf{x})}(\mathbf{x})) \setminus D(f_2)$ and such that

$$\frac{\lambda(\bigcup_{i=1}^{i(1,s(2,1,\mathbf{x}))} Q_{i,s(2,1,\mathbf{x})})}{\lambda(P^{s(2,1,\mathbf{x})}(\mathbf{x}))} > \frac{1}{2}.$$

Moreover, we assume that if for $\mathbf{x},\ \mathbf{y} \in D(f_2)\setminus D(f_1)$ the point $\mathbf{y} \in P^{s(2,1,\mathbf{x})}(\mathbf{x})$, then $P^{s(2,1,\mathbf{x})}(\mathbf{x}) = P^{s(2,1,\mathbf{y})}(\mathbf{y})$, $i(1,s(2,1,\mathbf{x})) = i(1,s(2,1,\mathbf{y}))$ and $Q_{i,s(2,1,\mathbf{x})} = Q_{i,s(2,1,\mathbf{y})}$ for $i \le i(1,s(2,1,\mathbf{x}))$. Let

$$S_1^{2,1} = \bigcup_{\mathbf{x}\in D_2^1} \ \bigcup_{i\le i(1,s(2,1,\mathbf{x}))} Q_{i,s(2,1,\mathbf{x})},\ S_1^{2,2} = \bigcup_{\mathbf{x}\in D_2^2} \ \bigcup_{i\le i(1,s(2,1,\mathbf{x}))} Q_{i,s(2,1,\mathbf{x})},$$

$$\text{and } S_1^2 = S_1^{2,1} \cup S_1^{2,2} = \bigcup_{\mathbf{x}\in(D(f_2)\setminus D(f_1))} \ \bigcup_{i\le i(1,s(2,1,\mathbf{x}))} Q_{i,s(2,1,\mathbf{x})}.$$

Obviously

$$\mathrm{cl}(S_1^2) \setminus D(f_2) = \bigcup_{\mathbf{x}\in(D(f_2)\setminus D(f_1))} \ \bigcup_{i\le i(1,s(2,1,\mathbf{x}))} \mathrm{cl}(Q_{i,s(2,1,\mathbf{x})})$$

and the family $\{Q_{i,s(2,1,\mathbf{x})}; i \le i(1,s(2,1,\mathbf{x}))$ and $\mathbf{x} \in D(f_2) \setminus D(f_1)\}$ is $\mathcal{P}$-locally finite.

Now, for each point $\mathbf{x} \in D(f_2) \setminus D(f_1)$ there is the first positive integer $s(2,2,\mathbf{x})$ such that $\mathrm{diam}(P^{s(2,2,\mathbf{x})}(\mathbf{x})) < \frac{1}{2^2} \cdot \mathrm{diam}(P^{s(2,1,\mathbf{x})}(\mathbf{x}))$ and

$$\mathbf{x} \in P^{s(2,2,\mathbf{x})}(\mathbf{x}) \subset P^{s(2,1,\mathbf{x})}(\mathbf{x}) \setminus \mathrm{cl}(S_1^2).$$

For a such integer $s(2,2,\mathbf{x})$ there is a finite family of cubes

$$Q_{1,s(2,2,\mathbf{x})},\ Q_{2,s(2,2,\mathbf{x})},\ \ldots,\ Q_{i(1,s(2,2,\mathbf{x})),s(2,2,\mathbf{x})} \in \mathcal{P},$$

whose closures are pairwise disjoint and contained in $\mathrm{int}(P^{s(2,2,\mathbf{x})}(\mathbf{x})) \setminus D(f_2)$ and such that

$$\frac{\lambda(\bigcup_{i=1}^{i(1,s(2,2,\mathbf{x}))} Q_{i,s(2,2,\mathbf{x})})}{\lambda(P^{s(2,2,\mathbf{x})}(\mathbf{x}))} > 1 - \frac{1}{2^2}.$$

Moreover, we assume that if for $\mathbf{x},\ \mathbf{y} \in D(f_2)\backslash D(f_1)$ the point $\mathbf{y} \in P^{s(2,2,\mathbf{x})}(\mathbf{x})$, then $P^{s(2,2,\mathbf{x})}(\mathbf{x}) = P^{s(2,2,\mathbf{y})}(\mathbf{y}),\ i(1,s(2,2,\mathbf{x})) = i(1,s(2,2,\mathbf{y}))$ and $Q_{i,s(2,2,\mathbf{x})} = Q_{i,s(2,2,\mathbf{y})}$ for $i \le i(1,s(2,2,\mathbf{x}))$. Let

$$S_2^{2,1} = \bigcup_{\mathbf{x}\in D_2^1} \bigcup_{i\le i(1,s(2,2,\mathbf{x}))} Q_{i,s(2,2,\mathbf{x})},\ S_2^{2,2} = \bigcup_{\mathbf{x}\in D_2^2} \bigcup_{i\le i(1,s(2,2,\mathbf{x}))} Q_{i,s(2,2,\mathbf{x})},$$

$$\text{and } S_2^2 = S_2^{2,1} \cup S_2^{2,2} = \bigcup_{\mathbf{x}\in(D(f_2)\backslash D(f_1))} \bigcup_{i\le i(1,s(2,2,\mathbf{x}))} Q_{i,s(2,2,\mathbf{x})}.$$

Then

$$\mathrm{cl}(S_2^2) \setminus D(f_2) = \bigcup_{\mathbf{x}\in(D(f_2)\backslash D(f_1))} \bigcup_{i\le i(1,s(2,2,\mathbf{x}))} \mathrm{cl}(Q_{i,s(2,2,\mathbf{x})}).$$

and the family $\{Q_{i,s(2,2,\mathbf{x})}; i \le i(1,s(2,2,\mathbf{x}))$ and $\mathbf{x} \in D(f_2) \setminus D(f_1)\}$ is $\mathcal{P}$-locally finite.

Generally, for $j > 2$ and for each point $\mathbf{x} \in D(f_2)\backslash D(f_1)$ let $s(2,j,\mathbf{x})$ be the smallest positive integer such that $\mathrm{diam}(P^{s(2,j,\mathbf{x})}(\mathbf{x})) < \frac{1}{2^j}\cdot\mathrm{diam}(P^{s(2,j-1,\mathbf{x})}(\mathbf{x}))$ and

$$\mathbf{x} \in P^{s(2,j,\mathbf{x})}(\mathbf{x}) \subset P^{s(2,j-1,\mathbf{x})}(\mathbf{x}) \setminus \mathrm{cl}(S_{j-1}^2).$$

For a such integer $s(2,j,\mathbf{x})$ there is a finite family of cubes

$$Q_{1,s(2,j,\mathbf{x})},\ Q_{2,s(2,j,\mathbf{x})},\ \ldots,\ Q_{i(1,s(2,j,\mathbf{x})),s(2,j,\mathbf{x})} \in \mathcal{P},$$

whose closures are pairwise disjoint and contained in $\mathrm{int}(P^{s(2,j,\mathbf{x})}(\mathbf{x})) \setminus D(f_2)$ and such that

$$\frac{\lambda(\bigcup_{i=1}^{i(1,s(2,j,\mathbf{x}))} Q_{i,s(2,j,\mathbf{x})})}{\lambda(P^{s(2,j,\mathbf{x})}(\mathbf{x}))} > 1 - \frac{1}{2^j}.$$

Moreover, we assume that if for $\mathbf{x},\ \mathbf{y} \in D(f_2)\backslash D(f_1)$ the point $\mathbf{y} \in P^{s(2,j,\mathbf{x})}(\mathbf{x})$, then $P^{s(2,j,\mathbf{x})}(\mathbf{x}) = P^{s(2,j,\mathbf{y})}(\mathbf{y}),\ i(1,s(2,j,\mathbf{x})) = i(1,s(2,j,\mathbf{y}))$ and $Q_{i,s(2,j,\mathbf{x})} = Q_{i,s(2,j,\mathbf{y})}$ for $i \le i(1,s(2,j,\mathbf{x}))$. Let

$$S_j^{2,1} = \bigcup_{\mathbf{x}\in D_2^1} \bigcup_{i\le i(1,s(2,j,\mathbf{x}))} Q_{i,s(2,j,\mathbf{x})},\ S_j^{2,2} = \bigcup_{\mathbf{x}\in D_2^2} \bigcup_{i\le i(1,s(2,j,\mathbf{x}))} Q_{i,s(2,j,\mathbf{x})},$$

$$\text{and } S_j^2 = S_j^{2,1} \cup S_j^{2,2} = \bigcup_{\mathbf{x}\in(D(f_2)\backslash D(f_1))} \bigcup_{i\le i(1,s(2,j,\mathbf{x}))} Q_{i,s(2,j,\mathbf{x})}.$$

Then

$$\mathrm{cl}(S_j^2) \setminus D(f_2) = \bigcup_{\mathbf{x}\in(D(f_2)\backslash D(f_1))} \bigcup_{i\le i(1,s(2,j,\mathbf{x}))} \mathrm{cl}(Q_{i,s(2,j,\mathbf{x})}).$$

and the family $\{Q_{i,s(2,j,\mathbf{x})}; i \leq i(1,s(2,j,\mathbf{x}))$ and $\mathbf{x} \in D(f_2) \setminus D(f_1)\}$ is $\mathcal{P}$-locally finite. Note too, since $\left(\bigcup_{j\in\mathbb{N}} S_j^{2,1}\right) \cap S_p^1 \subset \bigcup_{i\leq i(1,s(1,p,\mathbf{y}))} U(Q_{i,s(1,p,\mathbf{y})})$ for every $p \in \mathbb{N}$ and $\mathbf{y} \in D(f_1)$, by $(*)$ we have

$$\lambda\left((\bigcup_{j\in\mathbb{N}} S_j^{2,1}) \cap S_p^1\right) < \frac{1}{2^p} \cdot \lambda(S_p^1). \qquad (**)$$

Let $N_{k,t},\ k \in \mathbb{Z},\ t \in \mathbb{N}$, be pairwise disjoint infinite subsets of positive integers such that for all $k \in \mathbb{Z}$, $N_k = \bigcup_{t\in\mathbb{N}} N_{k,t}$. Observe that for all integers k and each point $\mathbf{x} \in D(f_2) \setminus D(f_1)$ the upper density

$$d_u\left(\bigcup_{j\in N_{k,t}} \operatorname{int}(S_j^2),\ \mathbf{x}\right) = 1.$$

Recall that

$$\bigcup_{l\in\mathbb{N}} S_l^{2,1} \subset \bigcup_{j\in\mathbb{N}} S_j^1 \text{ and } \bigcup_{l\in\mathbb{N}} S_l^{2,2} \subset \mathbb{R}^m \setminus \bigcup_{j\in\mathbb{N}} S_j^1.$$

Moreover, there is an index $j_2 \in \mathbb{N}$ such that

$$S_j^1 \subset \mathcal{O}\left(D(f_1), \frac{1}{2^4}\right) \text{ for } j > j_2.$$

Next, for $k \in \mathbb{Z}$ we define the functions $g_3,\ g_4 : \mathbb{R}^m \to \mathbb{R}$ by

$$g_3(\mathbf{x}) = \begin{cases} f_2(\mathbf{x}) & \text{if } \mathbf{x} \in D(f_2) \\ g_1(\mathbf{x}) & \text{if } \mathbf{x} \in S_j^1 \setminus \bigcup_{l\in\mathbb{N}} S_l^{2,1},\ (j \in N_{2k-1,1}) \wedge (j > j_2) \\ g_1(\mathbf{x}) + \frac{c}{2^2} & \text{if } \mathbf{x} \in S_j^1 \setminus \bigcup_{l\in\mathbb{N}} S_l^{2,1},\ (j \in N_{2k-1,2}) \wedge (j > j_2) \\ f_1(\mathbf{x}) & \text{if } \mathbf{x} \in \bigcup_{l\in N_{2k-1,1}} \left(S_l^{2,1} \cup S_l^{2,2}\right) \\ f_1(\mathbf{x}) + \frac{c}{2^2} & \text{if } \mathbf{x} \in \bigcup_{l\in N_{2k-1,2}} \left(S_l^{2,1} \cup S_l^{2,2}\right) \\ f_2(\mathbf{x}) & \text{otherwise on } \mathbb{R}^m \end{cases}$$

and let

$$g_4(\mathbf{x}) = \begin{cases} f_2(\mathbf{x}) & \text{if } \mathbf{x} \in D(f_2) \\ g_2(\mathbf{x}) & \text{if } \mathbf{x} \in S_j^1 \setminus \bigcup_{l\in\mathbb{N}} S_l^{2,1},\ (j \in N_{2k,1}) \wedge (j > j_2) \\ g_2(\mathbf{x}) + \frac{c}{2^2} & \text{if } \mathbf{x} \in S_j^1 \setminus \bigcup_{l\in\mathbb{N}} S_l^{2,1},\ (j \in N_{2k,2}) \wedge (j > j_2) \\ f_1(\mathbf{x}) & \text{if } \mathbf{x} \in \bigcup_{l\in N_{2k,1}} \left(S_l^{2,1} \cup S_l^{2,2}\right) \\ f_1(\mathbf{x}) + \frac{c}{2^2} & \text{if } \mathbf{x} \in \bigcup_{l\in N_{2k,2}} \left(S_l^{2,1} \cup S_l^{2,2}\right) \\ f_2(\mathbf{x}) & \text{otherwise on } \mathbb{R}^m. \end{cases}$$

The function g_3 and g_4 are s_1-strongly quasi-continuous. Indeed,

(-) if $\mathbf{x} \in D(f_2)$, then there exists an index $k \in \mathbb{Z}$ such that $g_3(\mathbf{x}) = f_2(\mathbf{x}) = \frac{k \cdot c}{4}$. Then, for some $k_0 \in \mathbb{Z}$, we have two cases

$$g_3(\mathbf{x}) = \frac{2k_0 \cdot c}{4} = \frac{k_0 \cdot c}{2} = f_1(\mathbf{x}) \text{ or } g_3(\mathbf{x}) = \frac{(2k_0+1) \cdot c}{4} = \frac{k_0 \cdot c}{2} + \frac{c}{4}.$$

Suppose that $\mathbf{x} \in D(f_1) \subset D(f_2)$. If $k = 2k_0$, then for each index $j \in N_{2k-1,1}$ there is a cube $P^{s(1,j,\mathbf{x})}(\mathbf{x}) \ni \mathbf{x}$ such that

$$\frac{\lambda\left(S_j^1 \cap P^{s(1,j,\mathbf{x})}(\mathbf{x})\right)}{\lambda(P^{s(1,j,\mathbf{x})}(\mathbf{x}))} > 1 - \frac{1}{2^j}.$$

Moreover, if $j > j_2$ is such that $S_j^1 \cap D_2^1 \neq \emptyset$, then by the formula $(**)$ we have

$$\begin{aligned}
&\frac{\lambda\left((S_j^1 \setminus \bigcup_{l\in\mathbb{N}} S_l^{2,1}) \cap P^{s(1,j,\mathbf{x})}(\mathbf{x})\right)}{\lambda(P^{s(1,j,\mathbf{x})}(\mathbf{x}))} \\
=&\frac{\lambda\left(\left(S_j^1 \setminus ((\bigcup_{l\in\mathbb{N}} S_l^{2,1}) \cap S_j^1)\right) \cap P^{s(1,j,\mathbf{x})}(\mathbf{x})\right)}{\lambda(P^{s(1,j,\mathbf{x})}(\mathbf{x}))} \\
=&\frac{\lambda\left(S_j^1 \cap P^{s(1,j,\mathbf{x})}(\mathbf{x})\right)}{\lambda(P^{s(1,j,\mathbf{x})}(\mathbf{x}))} - \frac{\lambda\left(\left((\bigcup_{l\in\mathbb{N}} S_l^{2,1}) \cap S_j^1\right) \cap P^{s(1,j,\mathbf{x})}(\mathbf{x})\right)}{\lambda(P^{s(1,j,\mathbf{x})}(\mathbf{x}))} \\
>&\frac{\lambda\left(S_j^1 \cap P^{s(1,j,\mathbf{x})}(\mathbf{x})\right)}{\lambda(P^{s(1,j,\mathbf{x})}(\mathbf{x}))} - \frac{\frac{1}{2^j} \cdot \lambda\left(S_j^1 \cap P^{s(1,j,\mathbf{x})}(\mathbf{x})\right)}{\lambda(P^{s(1,j,\mathbf{x})}(\mathbf{x}))} \\
>&\left(1 - \frac{1}{2^j}\right) - \frac{1}{2^j} \cdot \left(1 - \frac{1}{2^j}\right) > 1 - \frac{1}{2^{j-1}}.
\end{aligned}$$

For all $\mathbf{y} \in \left(S_j^1 \setminus \bigcup_{l\in\mathbb{N}} S_l^{2,1}\right) \cap P^{s(1,j,\mathbf{x})}(\mathbf{x})$, by definition, $g_3(\mathbf{y}) = g_1(\mathbf{y}) = \frac{k_0 \cdot c}{2}$. Thus

$$d_u\left(\operatorname{int}\left((g_3)^{-1}(\frac{k_0 \cdot c}{2})\right),\ \mathbf{x}\right) = 1.$$

Similarly, if $k = 2k_0 + 1$, then for each index $j \in N_{2k-1,2}$ there is a cube $P^{s(1,j,\mathbf{x})}(\mathbf{x}) \ni \mathbf{x}$ such that if $j > j_2$ and $S_j^1 \cap D_2^1 \neq \emptyset$, then

$$\frac{\lambda\left((S_j^1 \setminus \bigcup_{l\in\mathbb{N}} S_l^{2,1}) \cap P^{s(1,j,\mathbf{x})}(\mathbf{x})\right)}{\lambda(P^{s(1,j,\mathbf{x})}(\mathbf{x}))} \ > \ 1 - \frac{1}{2^{j-1}}.$$

Then, for all $\mathbf{y} \in \left(S_j^1 \setminus \bigcup_{l\in\mathbb{N}} S_l^{2,1}\right) \cap P^{s(1,j,\mathbf{x})}(\mathbf{x})$, by definition, $g_3(\mathbf{y}) = g_1(\mathbf{y}) + \frac{c}{4} = \frac{k_0 \cdot c}{2} + \frac{c}{4} = \frac{(2k_0+1) \cdot c}{4}$. Thus

$$d_u\left(\operatorname{int}\left((g_3)^{-1}(\frac{(2k_0+1) \cdot c}{4})\right),\ \mathbf{x}\right) = 1$$

and consequently $g_3 \in Q_{s_1}(\mathbf{x})$ for each $\mathbf{x} \in D(f_1)$.

Suppose that $\mathbf{x} \in D(f_2) \setminus D(f_1)$. If $k = 2k_0$, then for each index $l \in N_{2k-1,1}$ there is a cube $P^{s(2,l,\mathbf{x})}(\mathbf{x}) \ni \mathbf{x}$ such that

$$\frac{\lambda\left(S_l^{2,1} \cap P^{s(2,l,\mathbf{x})}(\mathbf{x})\right)}{\lambda(P^{s(n,l,\mathbf{x})}(\mathbf{x}))} > 1 - \frac{1}{2^l} \text{ or } \frac{\lambda\left(S_l^{2,2} \cap P^{s(2,l,\mathbf{x})}(\mathbf{x})\right)}{\lambda(P^{s(n,l,\mathbf{x})}(\mathbf{x}))} > 1 - \frac{1}{2^l}$$

because for all $l \in N_{2k-1,1}$ we have $S_l^{2,1} \cap S_l^{2,2} = \emptyset$. Thus, by definition, for all $\mathbf{y} \in (S_l^{2,1} \cup S_l^{2,2}) \cap P^{s(2,l,\mathbf{x})}(\mathbf{x})$ we have $g_3(\mathbf{y}) = f_1(\mathbf{y}) = \frac{k_0 \cdot c}{2}$ and

$$d_u\left(\text{int}\left((g_3)^{-1}(\frac{k_0 \cdot c}{2})\right),\ \mathbf{x}\right) = 1$$

Similarly, if $k = 2k_0 + 1$, then for each index $l \in N_{2k-1,2}$ there is a cube $P^{s(2,l,\mathbf{x})}(\mathbf{x}) \ni \mathbf{x}$ such that

$$\frac{\lambda\left(S_l^{2,1} \cap P^{s(2,l,\mathbf{x})}(\mathbf{x})\right)}{\lambda(P^{s(n,l,\mathbf{x})}(\mathbf{x}))} > 1 - \frac{1}{2^l} \text{ or } \frac{\lambda\left(S_l^{2,2} \cap P^{s(2,l,\mathbf{x})}(\mathbf{x})\right)}{\lambda(P^{s(n,l,\mathbf{x})}(\mathbf{x}))} > 1 - \frac{1}{2^l}$$

because for all $l \in N_{2k-1,2}$ we have $S_l^{2,1} \cap S_l^{2,2} = \emptyset$. Observe that for all $\mathbf{y} \in (S_l^{2,1} \cup S_l^{2,2}) \cap P^{s(2,l,\mathbf{x})}(\mathbf{x})$, by definition , $g_3(\mathbf{y}) = f_1(\mathbf{y}) + \frac{c}{4} = \frac{k_0 \cdot c}{2} + \frac{c}{4} = \frac{(2k_0+1)\cdot c}{4}$. Thus

$$d_u\left(\text{int}\left((g_3)^{-1}(\frac{(2k_0+1) \cdot c}{4})\right),\ \mathbf{x}\right) = 1$$

and consequently $g_3 \in Q_{s_1}(\mathbf{x})$ for each $\mathbf{x} \in D(f_2) \setminus D(f_1)$.

(-) if $\mathbf{x} \in \mathbb{R}^m \setminus D(f_2)$, then by the construction of g_3, we have that $g_3 \in Q_{s_1}(\mathbf{x})$.

So, $g_3 \in Q_{s_1}(\mathbf{x})$ for each $\mathbf{x} \in \mathbb{R}^m$. Analogously we can show that g_4 is s_1-strongly quasi-continuous at each point of its domain. Observe too, that $g_3(\mathbf{x}) = g_4(\mathbf{x}) = f_2(\mathbf{x})$ for all

$$\mathbf{x} \notin \mathcal{O}\left(D(f_2),\ \frac{1}{2^4}\right) = \mathcal{O}\left(D(f_1),\ \frac{1}{2^4}\right) \cup \mathcal{O}\left(D(f_2) \setminus D(f_1),\ \frac{1}{2^4}\right).$$

Moreover, since $|f_2 - f| < \frac{c}{4}$ and $\min(|g_3 - f_2|, |g_4 - f_2|) = 0$,

$$\min(|g_3 - f|, |g_4 - f|) = \min(|g_3 - f_2| + |f_2 - f|, |g_4 - f_2| + |f_2 - f|) < \frac{c}{4}.$$

Step 3. The set $D(f_3)$ is closed and of λ-measure zero. Let

$$D_3^1 = (D(f_3) \setminus D(f_2)) \cap \left(\bigcup_{r=1}^{2} \bigcup_{j \in \mathbb{N}} S_j^r\right) \text{ and let } D_3^2 = (D(f_3) \setminus D(f_2)) \setminus D_3^1.$$

If for an index $r_0 \in \{1,\ 2\}$ and an index $p_0 \in \mathbb{N}$ and a cube $Q_{i,s(r_0,p_0,\mathbf{y})}$, where $\mathbf{y} \in D(f_2)$ and $i \le i(1, s(r_0, p_0, \mathbf{y}))$, the set

$$D_{i,s(r_0,p_0,\mathbf{y})} = D_3^1 \cap Q_{i,s(r_0,p_0,\mathbf{y})} \neq \emptyset,$$

then we find an open (in $Q_{i,s(r_0,p_0,\mathbf{y})}$) set $U(Q_{i,s(r_0,p_0,\mathbf{y})}) \subset Q_{i,s(r_0,p_0,\mathbf{y})}$ containing $D_{i,s(r_0,p_0,\mathbf{y})}$ such that

$$\frac{\lambda\left(\bigcup_{i\le i(1,s(r_0,p_0,\mathbf{y}))} U(Q_{i,s(r_0,p_0,\mathbf{y})})\right)}{\lambda\left(S_{p_0}^{r_0}\right)} < \frac{1}{2^{p_0}}$$

and $d_u(U(Q_{i,s(r_0,p_0,\mathbf{y})}),\ \mathbf{x}) = 0$ for $\mathbf{x} \in \mathrm{Fr}(Q_{i,s(r_0,p_0,\mathbf{y})})$. If $D_{i,s(r_0,p_0,\mathbf{y})} = \emptyset$, we take $U(Q_{i,s(r_0,p_0,\mathbf{y})} = \emptyset$. Thus, for every $p \in \mathbb{N}$ and $r \in \{1,\ 2\}$ such that $S_p^r \cap D_3^1 \neq \emptyset$ and for $\mathbf{x} \in \mathrm{Fr}(Q_{i,s(r,p,\mathbf{y})})$ we have

$$\lambda\left(\bigcup_{i\le i(1,s(r,p,\mathbf{y}))} U(Q_{i,s(r,p,\mathbf{y})})\right) < \frac{1}{2^p}\cdot\lambda\left(S_p^r\right),$$

where $U(Q_{i,s(r,p,\mathbf{y})}) \supset D_{i,s(1,p,\mathbf{y})}$ and $d_u(U(Q_{i,s(r,p,\mathbf{y})}),\ \mathbf{x}) = 0$.

Similarly, as in the second step, for each point $\mathbf{x} \in D(f_3) \setminus D(f_2)$ there is the first positive integer $s(3, 1, \mathbf{x})$ such that:

- if $\mathbf{x} \in D_{i,s(r,p,\mathbf{y})}$ for $p \in \mathbb{N},\ r \in \{1,\ 2\},\ i \le i(1, s(r, p, \mathbf{y}))$ and $\mathbf{y} \in D(f_2)$, then
$$\mathbf{x} \in P^{s(3,1,\mathbf{x})}(\mathbf{x}) \subset U\left(Q_{i,s(r,p,\mathbf{y})}\right) \cap \mathcal{O}(D_3^1,\ \frac{1}{2^9})\ ;$$
- if $\mathbf{x} \in D_3^2$, then
$$\mathbf{x} \in P^{s(3,1,\mathbf{x})}(\mathbf{x}) \subset \mathcal{O}\left(D_3^2,\ \frac{1}{2^9}\right) \setminus \bigcup_{r=1}^{2}\bigcup_{j\in\mathbb{N}} \mathrm{cl}(S_j^r)\ ;$$
- $\mathrm{diam}(P^{s(3,1,\mathbf{x})}(\mathbf{x}) < \frac{1}{2^9}$ and f_2 is constant on $P^{s(3,1,\mathbf{x})}(\mathbf{x})$.

For a such positive integer $s(3, 1, x)$ there is a finite family of cubes

$$Q_{1,s(3,1,\mathbf{x})},\ Q_{2,s(3,1,\mathbf{x})},\ \ldots,\ Q_{i(1,s(3,1,\mathbf{x})),s(3,1,\mathbf{x})} \in \mathcal{P},$$

whose closures are pairwise disjoint and contained in $\mathrm{int}(P^{s(3,1,\mathbf{x})}(\mathbf{x})) \setminus D(f_3)$ and such that

$$\frac{\lambda(\bigcup_{i=1}^{i(1,s(3,1,\mathbf{x}))} Q_{i,s(3,1,\mathbf{x})})}{\lambda(P^{s(3,1,\mathbf{x})}(\mathbf{x}))} > \frac{1}{2}.$$

Moreover, we assume that if for $\mathbf{x},\ \mathbf{y} \in D(f_3)\setminus D(f_2)$ the point $\mathbf{y} \in P^{s(3,1,\mathbf{x})}(\mathbf{x})$, then $P^{s(3,1,\mathbf{x})}(\mathbf{x}) = P^{s(3,1,\mathbf{y})}(\mathbf{y})$, $i(1,s(3,1,\mathbf{x})) = i(1,s(3,1,\mathbf{y}))$ and $Q_{i,s(3,1,\mathbf{x})} = Q_{i,s(3,1,\mathbf{y})}$ for $i \leq i(1,s(3,1,\mathbf{x}))$. Let

$$S_1^{3,1} = \bigcup_{\mathbf{x}\in D_3^1} \bigcup_{i\leq i(1,s(3,1,\mathbf{x}))} Q_{i,s(3,1,\mathbf{x})},\quad S_1^{3,2} = \bigcup_{\mathbf{x}\in D_3^2} \bigcup_{i\leq i(1,s(3,1,\mathbf{x}))} Q_{i,s(3,1,\mathbf{x})} \text{ and}$$

$$S_1^3 = S_1^{3,1} \cup S_1^{3,2} = \bigcup_{\mathbf{x}\in(D(f_3)\setminus D(f_2))} \bigcup_{i\leq i(1,s(3,1,\mathbf{x})} Q_{i,s(3,1,\mathbf{x})}.$$

Obviously

$$\mathrm{cl}(S_1^3) \setminus D(f_3) = \bigcup_{\mathbf{x}\in(D(f_3)\setminus D(f_2))} \bigcup_{i\leq i(1,s(3,1,\mathbf{x}))} \mathrm{cl}(Q_{i,s(3,1,\mathbf{x})})$$

and the family $\{Q_{i,s(3,1,\mathbf{x})}; i \leq i(1,s(3,1,\mathbf{x}))$ and $\mathbf{x} \in D(f_3) \setminus D(f_2)\}$ is $\mathcal{P}$-locally finite.

Now, for each point $\mathbf{x} \in D(f_3)\setminus D(f_2)$ let $s(3,2,\mathbf{x})$ be the smallest positive integer such that $\mathrm{diam}(P^{s(3,2,\mathbf{x})}(\mathbf{x})) < \frac{1}{2^2} \cdot \mathrm{diam}(P^{s(3,1,\mathbf{x})}(\mathbf{x}))$ and

$$\mathbf{x} \in P^{s(3,2,\mathbf{x})}(\mathbf{x}) \subset P^{s(3,1,\mathbf{x})}(\mathbf{x}) \setminus \mathrm{cl}(S_1^3).$$

For a such integer $s(3,2,\mathbf{x})$ there is a finite family of cubes

$$Q_{1,s(3,2,\mathbf{x})},\ Q_{2,s(3,2,\mathbf{x})},\ \ldots,\ Q_{i(1,s(3,2,\mathbf{x})),s(3,2,\mathbf{x})} \in \mathcal{P},$$

whose closures are pairwise disjoint and contained in $\mathrm{int}(P^{s(3,2,\mathbf{x})}(\mathbf{x})) \setminus D(f_3)$ and such that

$$\frac{\lambda(\bigcup_{i=1}^{i(1,s(3,2,\mathbf{x}))} Q_{i,s(3,2,\mathbf{x})})}{\lambda(P^{s(3,2,\mathbf{x})}(\mathbf{x}))} > 1 - \frac{1}{2^2}.$$

Moreover, we assume that if for $\mathbf{x},\ \mathbf{y} \in D(f_3)\setminus D(f_3)$ the point $\mathbf{y} \in P^{s(3,2,\mathbf{x})}(\mathbf{x})$, then $P^{s(3,2,\mathbf{x})}(\mathbf{x}) = P^{s(3,2,\mathbf{y})}(\mathbf{y})$, $i(1,s(3,2,\mathbf{x})) = i(1,s(3,2,\mathbf{y}))$ and $Q_{i,s(3,2,\mathbf{x})} = Q_{i,s(3,2,\mathbf{y})}$ for $i \leq i(1,s(3,2,\mathbf{x}))$. Let

$$S_2^{3,1} = \bigcup_{\mathbf{x}\in D_3^1} \bigcup_{i\leq i(1,s(3,2,\mathbf{x}))} Q_{i,s(3,2,\mathbf{x})},\quad S_2^{3,2} = \bigcup_{\mathbf{x}\in D_3^2} \bigcup_{i\leq i(1,s(3,2,\mathbf{x}))} Q_{i,s(3,2,\mathbf{x})} \text{ and}$$

$$S_2^3 = S_2^{3,1} \cup S_2^{3,2} = \bigcup_{\mathbf{x}\in(D(f_3)\setminus D(f_2))} \bigcup_{i\leq i(1,s(3,2,\mathbf{x}))} Q_{i,s(3,2,\mathbf{x})}.$$

Observe that

$$\mathrm{cl}(S_2^3) \setminus D(f_3) = \bigcup_{\mathbf{x}\in(D(f_3)\setminus D(f_2))} \bigcup_{i\leq i(1,s(3,2,\mathbf{x}))} \mathrm{cl}(Q_{i,s(3,2,\mathbf{x})})$$

and the family $\{Q_{i,s(3,2,\mathbf{x})}; i \leq i(1, s(3,2,\mathbf{x}))$ and $\mathbf{x} \in D(f_3) \setminus D(f_2)\}$ is $\mathcal{P}$-locally finite.

Generally, for $j > 2$ and for each point $\mathbf{x} \in D(f_3) \setminus D(f_2)$ let $s(3, j, \mathbf{x})$ be the smallest positive integer such that $\operatorname{diam}(P^{s(3,j,\mathbf{x})}(\mathbf{x})) < \frac{1}{2^j} \cdot \operatorname{diam}(P^{s(3,j-1,\mathbf{x})}(\mathbf{x}))$ and

$$\mathbf{x} \in P^{s(3,j,\mathbf{x})}(\mathbf{x}) \subset P^{s(3,j-1,\mathbf{x})}(\mathbf{x}) \setminus \operatorname{cl}(S^3_{j-1}).$$

For a such integer $s(3, j, \mathbf{x})$ there is a finite family of cubes

$$Q_{1,s(3,j,\mathbf{x})},\ Q_{2,s(3,j,\mathbf{x})},\ \ldots,\ Q_{i(1,s(3,j,\mathbf{x})),s(3,j,\mathbf{x})} \in \mathcal{P},$$

whose closures are pairwise disjoint and contained in $\operatorname{int}(P^{s(3,j,\mathbf{x})}(\mathbf{x})) \setminus D(f_3)$ and such that

$$\frac{\lambda(\bigcup_{i=1}^{i(1,s(3,j,\mathbf{x}))} Q_{i,s(3,j,\mathbf{x})})}{\lambda(P^{s(3,j,\mathbf{x})}(\mathbf{x}))} > 1 - \frac{1}{2^j}.$$

Moreover, we assume that if for $\mathbf{x},\ \mathbf{y} \in D(f_3) \setminus D(f_3)$ the point $\mathbf{y} \in P^{s(3,j,\mathbf{x})}(\mathbf{x})$, then $P^{s(3,j,\mathbf{x})}(\mathbf{x}) = P^{s(3,j,\mathbf{y})}(\mathbf{y})$, $i(1, s(3,j,\mathbf{x})) = i(1, s(3,j,\mathbf{y}))$ and $Q_{i,s(3,j,\mathbf{x})} = Q_{i,s(3,j,\mathbf{y})}$ for $i \leq i(1, s(3,j,\mathbf{x}))$. Let

$$S^{3,1}_j = \bigcup_{\mathbf{x} \in D^1_3} \bigcup_{i \leq i(1,s(3,j,\mathbf{x}))} Q_{i,s(3,j,\mathbf{x})},\ S^{3,2}_j = \bigcup_{\mathbf{x} \in D^2_3} \bigcup_{i \leq i(1,s(3,j,\mathbf{x}))} Q_{i,s(3,j,\mathbf{x})} \text{ and}$$

$$S^3_j = S^{3,1}_j \cup S^{3,2}_j = \bigcup_{\mathbf{x} \in (D(f_3) \setminus D(f_2))} \bigcup_{i \leq i(1,s(3,j,\mathbf{x}))} Q_{i,s(3,j,\mathbf{x})}.$$

Observe that

$$\operatorname{cl}(S^3_j) \setminus D(f_3) = \bigcup_{\mathbf{x} \in (D(f_3) \setminus D(f_2))} \bigcup_{i \leq i(1,s(3,j,\mathbf{x}))} \operatorname{cl}(Q_{i,s(3,j,\mathbf{x})})$$

and the family $\{Q_{i,s(3,j,\mathbf{x})}; i \leq i(1, s(3,j,\mathbf{x}))$ and $\mathbf{x} \in D(f_3) \setminus D(f_2)\}$ is $\mathcal{P}$-locally finite.

Let $N_{k,t,l}$, $k \in \mathbb{Z}$ and $t, l \in \mathbb{N}$; be pairwise disjoint infinite subsets of positive integers such that for all $k \in \mathbb{Z}$ and $t \in \mathbb{N}$, $N_{k,t} = \bigcup_{l \in \mathbb{N}} N_{k,t,l}$. Observe, that for each point $\mathbf{x} \in D(f_3) \setminus D(f_2)$ and $k \in \mathbb{Z},\ t \in \mathbb{N}$ the upper density

$$d_u\left(\bigcup_{j \in N_{k,t,l}} \operatorname{int}(S^3_j),\ \mathbf{x}\right) = 1.$$

Put

- $N_{2k-1,3} = N_{2k-1,1,1} \cup N_{2k-1,2,1}$ and $N_{2k-1,4} = N_{2k-1,1,2} \cup N_{2k-1,2,2}$;

- $N_{2k,3} = N_{2k,1,1} \cup N_{2k,2,1}$ and $N_{2k,4} = N_{2k,1,2} \cup N_{2k,2,2}$.

Recall, too, that

- $\bigcup_{p\in\mathbb{N}} S_p^{3,1} \subset \bigcup_{r=1}^2 \bigcup_{l\in\mathbb{N}} S_l^r$, $\bigcup_{p\in\mathbb{N}} S_p^{3,2} \subset \mathbb{R}^m \setminus \bigcup_{r=1}^2 \bigcup_{l\in\mathbb{N}} cl\,(S_l^r)$ and
- $\bigcup_{p\in\mathbb{N}} S_p^3 = \bigcup_{p\in\mathbb{N}} S_p^{3,1} \cup \bigcup_{p\in\mathbb{N}} S_p^{3,2}$.

There are indexes $j_3,\ l_3 \in \mathbb{N}$ such that

$$S_j^1 \subset \mathcal{O}\left(D(f_1),\ \frac{1}{2^9}\right) \text{ for } j > j_3 \text{ and } S_l^2 \subset \mathcal{O}\left(D(f_2)\setminus D(f_1),\ \frac{1}{2^9}\right) \text{ for } l > l_3.$$

Next, for $k \in \mathbb{Z}$ we define the functions $g_5,\ g_6 : \mathbb{R}^m \to \mathbb{R}$ by

$$g_5(\mathbf{x})\begin{cases} f_3(\mathbf{x}) & \text{if } \mathbf{x} \in D(f_3) \\ g_3(\mathbf{x}) & \text{if } \mathbf{x} \in S_j^1 \setminus \left(\bigcup_{p\in\mathbb{N}} S_p^{3,1} \cup \bigcup_{l\in\mathbb{N}} S_l^{2,1}\right),\ (j \in N_{2k-1,3}) \wedge (j > j_3) \\ g_3(\mathbf{x}) + \frac{c}{2^3} & \text{if } \mathbf{x} \in S_j^1 \setminus \left(\bigcup_{p\in\mathbb{N}} S_p^{3,1} \cup \bigcup_{l\in\mathbb{N}} S_l^{2,1}\right),\ (j \in N_{2k-1,4}) \wedge (j > j_3) \\ g_3(\mathbf{x}) & \text{if } \mathbf{x} \in S_l^2 \setminus \bigcup_{p\in\mathbb{N}} S_p^{3,1},\ (l \in N_{2k-1,3}) \wedge (l > l_3) \\ g_3(\mathbf{x}) + \frac{c}{2^3} & \text{if } \mathbf{x} \in S_l^2 \setminus \bigcup_{p\in\mathbb{N}} S_p^{3,1},\ (l \in N_{2k-1,4}) \wedge (l > l_3) \\ f_2(\mathbf{x}) & \text{if } \mathbf{x} \in \bigcup_{p\in N_{2k-1,1}} S_p^3 \\ f_2(\mathbf{x}) + \frac{c}{2^3} & \text{if } \mathbf{x} \in \bigcup_{p\in N_{2k-1,2}} S_p^3 \\ f_3(\mathbf{x}) & \text{otherwise on } \mathbb{R}^m \end{cases}$$

and

$$g_6(\mathbf{x}) = \begin{cases} f_3(\mathbf{x}) & \text{if } \mathbf{x} \in D(f_3) \\ g_3(\mathbf{x}) & \text{if } \mathbf{x} \in S_j^1 \setminus \left(\bigcup_{p\in\mathbb{N}} S_p^{3,1} \cup \bigcup_{l\in\mathbb{N}} S_l^{2,1}\right),\ (j \in N_{2k,3}) \wedge (j > j_3) \\ g_3(\mathbf{x}) + \frac{c}{2^3} & \text{if } \mathbf{x} \in S_j^1 \setminus \left(\bigcup_{p\in\mathbb{N}} S_p^{3,1} \cup \bigcup_{l\in\mathbb{N}} S_l^{2,1}\right),\ (j \in N_{2k,4}) \wedge (j > j_3) \\ g_3(\mathbf{x}) & \text{if } \mathbf{x} \in S_l^2 \setminus \bigcup_{p\in\mathbb{N}} S_p^{3,1},\ (l \in N_{2k,3}) \wedge (l > l_3) \\ g_3(\mathbf{x}) + \frac{c}{2^3} & \text{if } \mathbf{x} \in S_l^2 \setminus \bigcup_{p\in\mathbb{N}} S_p^{3,1},\ (l \in N_{2k,4}) \wedge (l > l_3) \\ f_2(\mathbf{x}) & \text{if } \mathbf{x} \in \bigcup_{p\in N_{2k,1}} S_p^3 \\ f_2(\mathbf{x}) + \frac{c}{2^3} & \text{if } \mathbf{x} \in \bigcup_{p\in N_{2k,2}} S_p^3 \\ f_3(\mathbf{x}) & \text{otherwise on } \mathbb{R}^m. \end{cases}$$

As in the second step we can verify that $g_5,\ g_6 \in Q_{s_1}(\mathbf{x})$ for each $\mathbf{x} \in \mathbb{R}^m$. Observe too, that $g_5(\mathbf{x}) = g_6(\mathbf{x}) = f_3(\mathbf{x})$ for all

$$\mathbf{x} \notin \mathcal{O}\left(D(f_3),\ \frac{1}{2^9}\right) = \bigcup_{i=0}^{2} \mathcal{O}\left(D(f_{i+1})\setminus D(f_i),\ \frac{1}{2^9}\right),\ \text{where } D(f_0) = \emptyset.$$

Moreover, since $|f_3 - f| < \frac{c}{2^3}$ and $\min(|g_5 - f_3|, |g_6 - f_3|) = 0$,

$$\min(|g_5 - f|, |g_6 - f|) = \min(|g_5 - f_3| + |f_3 - f|, |g_6 - f_3| + |f_3 - f|) < \frac{c}{2^3}.$$

Generally, for $n > 3$, as in step 3, we define functions $g_{2n-1},\ g_{2n} : \mathbb{R}^m \to \mathbb{R}$ such that $g_{2n-1},\ g_{2n} \in Q_{s_1}(\mathbf{x})$ for each point $\mathbf{x} \in \mathbb{R}^m$, $g_{2n-1}(\mathbf{x}) - g_{2n}(\mathbf{x}) = f_n(\mathbf{x})$ for all

$$\mathbf{x} \notin \mathcal{O}\left(D(f_n),\ \frac{1}{2^{n^2}}\right) = \bigcup_{i=0}^{n-1} \mathcal{O}\left(D(f_{i+1}) \setminus D(f_i),\ \frac{1}{2^{n^2}}\right) \text{ where } D(f_0) = \emptyset,$$

and $\min(|g_{2n-1} - f_n|, |g_{2n} - f_n|) = 0$. We will prove that the sequence (g_n) quasi-uniformly converges to f. First we shall show that the sequence (g_n) converges pointwise to f. Suppose that $\mathbf{x} \in \bigcup_{n=1}^{\infty} D(f_n) = \bigcup_{n=2}^{\infty} \bigcup_{i=0}^{n-1} (D(f_{i+1}) \setminus D(f_i)$ where $D(f_0) = \emptyset$. Then there is an index $M \in \mathbb{N}$ such that $g_{2n-1}(\mathbf{x}) = g_{2n}(\mathbf{x}) = f_n(\mathbf{x})$ for $n > M$ and consequently

$$\lim_{n\to\infty} g_{2n-1}(\mathbf{x}) = \lim_{n\to\infty} g_{2n}(\mathbf{x}) = \lim_{n\to\infty} f_n(\mathbf{x}) = f(\mathbf{x}).$$

Now suppose that $\mathbf{x} \notin \bigcup_{n=1}^{\infty} D(f_n)$. Fix a real $\varepsilon > 0$. There is an index $T \in \mathbb{N}$ such that $\dfrac{c}{2^{T-2}} < \varepsilon$. Since $\mathbf{x} \notin D(f_T)$, there is a real $\eta > 0$ with $\mathbf{x} \notin \mathcal{O}(D(f_T),\ \eta)$. Let $M > T$ be a positive integer such that $\frac{1}{2^M} < \eta$. Then, for all $n > M > T$ we have $\mathbf{x} \notin \mathcal{O}\left(D(f_T),\ \frac{1}{2^M}\right)$ and consequently

$$\max(|g_{2n-1}(\mathbf{x}) - f_n(\mathbf{x})|, |g_{2n}(\mathbf{x}) - f_n(\mathbf{x})|) < \frac{c}{2^M}.$$

Since for all $n > M$ we obtain

$$\max(|g_{2n-1}(\mathbf{x}) - f(\mathbf{x})|, |g_{2n}(\mathbf{x}) - f(\mathbf{x})|) < \frac{c}{2^M} + \frac{c}{2^n} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon,$$

the sequence (g_n) converges pointwise to f. It is obvious that $\min(|g_{2n-1} - f|, |g_{2n} - f|) < \varepsilon$ for all $n > M$ and the proof is completed. $\square$

Recall that a function $f : \mathbb{R}^m \to \mathbb{R}$ is *approximately quasi-continuous at a point* $\mathbf{x} \in \mathbb{R}^m$, $(f \in Q_{ap}(\mathbf{x}))$ if for each real $\varepsilon > 0$ and each set $U \in \mathcal{T}_d$ containing $\mathbf{x}$ there is a nonempty set $V \subset U$ belonging to $\mathcal{T}_d$ with $f(V) \subset (f(\mathbf{x}) - \varepsilon, f(\mathbf{x}) + \varepsilon)$ ([4]).

In [4] it is proved that each λ-measurable function $f : \mathbb{R}^m \to \mathbb{R}$ is the limit of a pointwise convergent sequence of approximately quasi-continuous functions. We will prove the following assertion.

Theorem 2. *If $f : \mathbb{R}^m \to \mathbb{R}$ is a λ-measurable function, then there is a sequence of approximately quasi-continuous functions $g_n : \mathbb{R}^m \to \mathbb{R}$ which quasi-uniformly converges to f.*

PROOF. Since f is λ-measurable, the set

$$D_{ap}(f) = \{\mathbf{x} \in \mathbb{R}^m; f \text{ is not approximately continuous at } \mathbf{x}\}$$

is of λ-measure zero. There exists an G_δ-set $A \supset D_{ap}(f)$ of λ-measure zero.

Let (G_n) be a decreasing sequence of open sets $G_1 \supset G_2 \supset \dots$ such that $A = \bigcap_{n=1}^{\infty} G_n$. Fix an index $n \in \mathbb{N}$. From Lemma 3 in [4] there is a sequence of pairwise disjoint measurable sets $A_{n,k} \subset G_n \setminus A$ such that

- $\bigcup_{k=0}^{\infty} A_{n,k} = G_n \setminus A$;
- $d_u(A_{n,k},\ \mathbf{x}) > 0$ for each $\mathbf{x} \in A \cup A_{n,k}$ and each $k \geq 0,$ and
- $d_u((\mathbb{R}^m \setminus G_n) \cup A_{n,0},\ \mathbf{x}) > 0$ for each $\mathbf{x} \in \mathbb{R}^m \setminus G_n$.

Let (w_k) be a sequence of all rationals such that $w_i \neq w_j$ for $i \neq j$ and let

$$g_{2n-1}(\mathbf{x}) = \begin{cases} w_k & \text{for } \mathbf{x} \in A_{n,2k},\ k = 1, 2, \dots \\ f(\mathbf{x}) & \text{otherwise on } \mathbb{R}^m \end{cases}$$

and

$$g_{2n}(\mathbf{x}) = \begin{cases} w_k & \text{for } \mathbf{x} \in A_{n,2k-1},\ k = 1, 2, \dots \\ f(\mathbf{x}) & \text{otherwise on } \mathbb{R}^m. \end{cases}$$

Evidently the functions $g_n,\ (n \in \mathbb{N})$ are approximately quasi-continuous. Since $A = \bigcap_n G_n$ and $G_n \supset G_{n+1}$ for $n \geq 1$, we have $f = \lim_{n\to\infty} g_n$. Moreover, since $\min(|g_{2n-1} - f|, |g_{2n} - f|) = 0$ for every $n \in \mathbb{N}$, the sequence (g_n) quasi-uniformly converges to f and the proof is completed.

References

[1] A. M. Bruckner, *Differentiation of real functions*, Lectures Notes in Math., 659 (1978), Springer–Verlag.

[2] A. M. Bruckner, *Differentiation of the integrals*, Amer. Math. Monthly, **78** (1978), 1–54.

[3] C. Goffman, C. J. Neugebauer, T. Nishiura, *Density topology and approximate continuity*, Duke Math. J., **28** (1961), 497–506.

[4] Z. Grande, T. Natkaniec, E. Strońska, *Algebraic structures generated by d-quasicontinuous functions*, Bull. Polish Acad. Sci., Math., **35 No. 11-12** (1987), 717–723.

[5] Z. Grande, *On quasi-uniform convergence of a sequence of s.q.c. functions*, Math. Slovaca, **48** (1998), 507–511.

[6] Z. Grande., *On some special notions of approximate quasicontinuity*, Real Analysis Exch., **24** (1998-99), 171–183.

[7] Z. Grande, E. Strońska E., *On some properties connected with continuity and measurability*, Tatra Mont. Math. Publ., **19** (2000), 117– 125.

[8] T. Neubrunn, *Quasi–continuity*, Real Anal. Exch., **14, No. 2** (1988-89), 259–306.

[9] S. Saks, *Theory of the integral*, Monografie Matematyczne, Warszawa, (1937).

[10] R. Sikorski, *Real functions*, PWN, Warszawa, (1958), vol.I.

[11] E. Strońska, *On some special notions of approximate quasicontinuity on* $\mathbb{R}^m$, Demonstratio Math., (2004/2005), (to appear).

[12] F. D. Tall, *The density topology*, Pacific J. Math., **62** (1976), 275–284.

[13] W. Wilczyński, *Density topologies,* Edited by Pap E. *Handbook of measure theory,* Elsevier Science B.V., Amsterdam (2002), ch.15.