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ON CONTINUOUS N-FUNCTIONS AND AN EXAMPLE OF MAZURKIEWICZ

Abstract

Let f and g be continuous real functions on the interval $[a, b]$, and let K denote the set of all knot points of f. Let E be a set of measure zero for which $f(E)$ has measure zero and $(f+g)(E)$ does not, and let g be differentiable at each point of E closure. We prove that K must meet E, and moreover the intersection of K with the closure of E must contain a nonvoid perfect subset. Thus in particular, the function of Mazurkiewicz is a continuous N-Function with as many knot points as there are real numbers.

In [M] Mazurkiewicz constructed a continuous N-Function F such that $F + aI$ is not an N-function if $a \neq 0$. (Here I denotes the identity function.) In the present note we carry this idea further by using knot points.

We say that the point x is a *knot* point of the continuous function f if the upper Dini derivatives of f at x (denoted $D^+f(x)$ and $D^-f(x)$) are ∞ and the lower Dini derivatives of f at x (denoted $D_+f(x)$ and $D_-f(x)$) are $-\infty$. (See also [Y, p. 168].) Perhaps the most familiar example of a knot point is 0 for the function $\sqrt{|x|}\sin\frac{1}{x}$.

We begin with three easy lemmas. Their proofs are included for the sake of completeness.

Lemma 1. *Let f and h be continuous functions on $[a, b]$ and let E be a set of measure zero such that $f(E)$ has measure zero but $h(E)$ does not. Then there exists a compact subset A of E closure (denoted E^-) such that A and $f(A)$ have measure zero but $h(A)$ does not.*

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Proof. Let U_n and V_n be open neighborhoods of E and $f(E)$ respectively such that $m(U_n) < \frac{1}{2^n}$ and $m(V_n) < \frac{1}{2^n}$, where m denotes Lebesgue outer measure. Let B_1 denote the closure of the union of finitely many components of the set $U_1 \cap f^{-1}(V_1)$ that meet E such that

$$m\big(h(E \cap B_1)\big) > \Big(1 - \frac{1}{5}\Big) m\big(h(E)\big)\,.$$

Let B_2 denote the closure of the union of finitely many components of the set $U_2 \cap f^{-1}(V_2) \cap B_1$ that meet E such that

$$m\big(h(E \cap B_2)\big) > \Big(1 - \frac{1}{5^2}\Big) m\big(h(E \cap B_1)\big)\,.$$

In general, let B_n denote the closure of the union of finitely many components of the set $U_n \cap f^{-1}(V_n) \cap B_{n-1}$ that meet E such that

$$m\big(h(E \cap B_n)\big) > \Big(1 - \frac{1}{5^n}\Big) m\big(h(E \cap B_{n-1})\big)\,.$$

Put $A = \cap_n B_n$.

Now A is the intersection of a contracting sequence of nonvoid compact sets, so A is compact. For any $a \in A$ and any index n, a lies in a component of B_n shorter than $\frac{1}{2^n}$ that contains points of E. Thus $a \in E^-$ and $A \subset E^-$. Also

$$m(A) \le m(U_n) < \frac{1}{2^n} \quad \text{and} \quad m\big(f(A)\big) \le m(V_n) < \frac{1}{2^n}$$

for each index n, so $m(A) = m\big(f(A)\big) = 0$.

It follows from the construction that $\inf_n m\big(h(E \cap B_n)\big) > 0$, so

$$m\Big(\cap_n h(B_n)\Big) > 0\,.$$

Let $b \in \cap_n h(B_n)$. Then $h^{-1}(b)$ is a compact set that meets B_n for all n. But (B_n) is a contracting sequence of compact sets, and it follows that $h^{-1}(b)$ meets $\cap_n B_n$ and $b \in h\big(\cap_n B_n\big)$. Thus $\cap_n h(B_n) = h\big(\cap_n B_n\big) = h(A)$. Finally, $m\big(h(A)\big) > 0$. □

Lemma 2. *Let h be a continuous function on $[a,b]$. Let A be a compact set for which $m\big(h(A)\big) > 0$, and let (D_n) be a sequence of closed sets such that $m\big(h(A \cap D_n)\big) = 0$ for each n. Then there is a compact set $A_0 \subset A \setminus \cup_k D_k$ such that $m\big(h(A_0)\big) > 0$.*

PROOF. Observe that

$$\bigcup_k \Big\{ x \in A : \text{distance from } x \text{ to } D_1 \text{ is} \geq \frac{1}{k} \Big\} = A \setminus D_1 \,,$$

and each set in the union is compact. It follows that there is a compact set $P_1 \subset A \setminus D_1$ such that

$$m\big(h(P_1)\big) > \Big(1 - \frac{1}{5}\Big) m\big(h(A \setminus D_1)\big) = \Big(1 - \frac{1}{5}\Big) m\big(h(A)\big) \,.$$

In general, for each index $n > 1$, choose a compact set $P_n \subset P_{n-1} \setminus D_n$ such that

$$m\big(h(P_n)\big) > \Big(1 - \frac{1}{5^n}\Big) m\big(h(P_{n-1} \setminus D_n)\big) = \Big(1 - \frac{1}{5^n}\Big) m\big(h(P_{n-1})\big) \,.$$

It follows from the construction that $m\big(\cap_n h(P_n)\big) > 0$.

Put $A_0 = \cap_n P_n$. By an argument essentially the same as the argument in the last paragraph in the proof of Lemma 1,

$$\cap_n h(P_n) = h\big(\cap_n P_n\big) = h(A_0) \,.$$

Finally, $m\big(h(A_0)\big) > 0$, and A_0 is a compact subset of $A \setminus \cup_n D_n$. □

Lemma 3. *Let g and h be continuous functions on $[a, b]$ and let g be differentiable at each point of a set E. Then there exists a sequence of closed sets (S_n) such that for each n, g is absolutely continuous on $E \cap S_n$, h is of bounded variation on $E \cap S_n$, and every point in $E \setminus \cup_n S_n$ is a knot point of h.*

PROOF. For integers $i, j > 0$, put

$$T_{ij} = \Big\{ x : \frac{h(x+r) - h(x)}{r} \leq i \ \text{ for any } r \text{ satisfying } 0 < r \leq \frac{1}{j} \Big\} \,.$$

Then each set T_{ij} is closed by continuity, h is of bounded variation on the set $E \cap T_{ij}$, and

$$E \cap \big(\cup_{ij} T_{ij}\big) = \big\{ x \in E : D^+ h(x) < \infty \big\} \,.$$

In a similar manner we find a sequence (V_k) of closed sets such that

$$E \cap \big(\cup_k V_k\big) = \Big\{ x \in E : \text{either } D^+ h(x) < \infty \text{ or } D^- h(x) < \infty \\ \text{or } D_+ h(x) > -\infty \text{ or } D_- h(x) > -\infty \Big\} \,,$$

and h is of bounded variation on each set $E \cap V_k$. It follows that each point of $E \setminus (\cup_k V_k)$ is a knot point of h.

Likewise closed sets of the form

$$W_{ij} = \left\{ x : \left| \frac{g(x+r) - g(x)}{r} \right| \le i \ \text{ for any } r \text{ satisfying } 0 < r \le \frac{1}{j} \right\}$$

(for integers $i, j > 0$) cover E because g is differentiable on E.

Certainly g is absolutely continuous on each set $E \cap W_{ij}$. Finally, the closed sets of the form $V_k \cap W_{ij}$ suffice. □

We are now able to prove our main result.

Theorem I. *Let f and g be continuous real valued functions on $[a, b]$ and let K be the set of all knot points of f. Let $E \subset [a, b]$ be a set of measure zero such that $f(E)$ has measure zero and g is differentiable at each point of E^-. Then*

(1) the set $(f + g)(E \setminus K)$ has measure zero,

(2) if $(f + g)(E)$ does not have measure zero, then the set $K \cap E^-$ has a nonvoid perfect subset.

(It follows that Mazurkiewicz' function F is a continuous N-Function with as many knot points as there are real numbers. Note that in Theorem I the hypothesis imposed on f is independent of the choice of g.)

Proof. By Lemma 3, there exists a sequence of closed sets (S_n) such that for each n, g is absolutely continuous on $E \cap S_n$ and f is of bounded variation on $E \cap S_n$, and each point of $E \setminus \cup_n S_n$ is a knot point of f. For (1) it suffices to prove that $(f + g)(E \cap S_n)$ has measure zero for each n.

We proceed by contradiction. Let N be an index for which $(f+g)(E \cap S_N)$ does not have measure zero. By Lemma 1, there is a compact subset A of $(E \cap S_N)^-$ such that A and $f(A)$ have measure zero but $(f + g)(A)$ does not. Now f is of bounded variation on $E \cap S_N$ and A is a subset of $(E \cap S_N)^-$. It follows that f is of bounded variation on A; likewise g is absolutely continuous on $E \cap S_N$ and on A. But f is a continuous N-function on A because $f(A)$ has measure zero. It follows from [S, (6.7) chapter VII] that f is an absolutely continuous function on A. Then $f + g$ is absolutely continuous on A. Again by [S, (6.7) chapter VII], $(f + g)(A)$ has measure zero, contrary to the choice of A. This contradiction proves (1).

To prove (2) we assume that $(f + g)(E)$ does not have measure zero. By Lemma 1, there is a compact subset B of E^- such that B and $f(B)$ have measure zero but $(f + g)(B)$ does not. By Lemma 3, there exists a sequence of closed sets (T_n) such that for each n, g is absolutely continuous on $B \cap T_n$, and $f + g$ is of bounded variation on $B \cap T_n$, and such that each point of

$B \setminus \cup_n T_n$ is a knot point of the functions $f + g$ and f. From an argument in the preceding paragraph we see $(f+g)(B \cap T_n)$ has measure zero for each n. Hence $(f+g)(B \setminus \cup_n T_n)$ does not have measure zero. By Lemma 2, there is a compact subset X of $B \setminus \cup_n T_n$ such that $(f+g)(X)$ does not have measure zero. Then X must be uncountable, so X contains a nonvoid perfect subset Y. Finally,

$$Y \subset X \subset B \setminus \big(\cup_n T_n\big) \subset K \text{ and } Y \subset B \subset E^- .$$

This proves (2). □

The following corollaries are immediate.

Corollary 1. *Let f be a continuous N-function and let g be a differentiable function on $[a,b]$. Let K be the set of all knot points of f. Then $f+g$ is an N-function on the set $[a,b] \setminus K$.*

Corollary 2. *In Corollary 1, let K have no nonvoid perfect subset. Then $f+g$ is an N-function on $[a,b]$.*

Corollary 3. *Let p be a continuous function that is not an N-function on $[a,b]$, let K be the set of all knot points of p, and let $m\big(p(K)\big) = 0$. Let g be a differentiable function on $[a,b]$. Then $p - g$ is not an N-function on $[a,b]$.*

To see this, put $f = p - g$ in the proof of Theorem I. We leave the argument.

We conclude with one further observation. Let L be the set of all N-functions f on $[a,b]$ such that $f+h$ is an N-function for every N-function h on $[a,b]$. Then L is closed under addition; for if f_1 and f_2 lie in L, then for any N-function h, $f_2 + h$ and

$$(f_1 + f_2) + h = f_1 + (f_2 + h)$$

are N-functions on $[a,b]$. Obviously if f lies in L and if c is any real number, then cf lies in L. Thus L can be regarded as a linear space that contains all the constant functions. However L does not contain Mazurkiewicz' function F or the identity function I.

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