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ON $\mathcal{I}$-CAUCHY SEQUENCES

Abstract

We study $\mathcal{I}$-convergence and $\mathcal{I}$-Cauchy sequences in metric spaces where $\mathcal{I} \subset \mathcal{P}(\mathbb{N}^k)$ is an ideal containing all singletons and $k \in \{1, 2\}$.

1 Introduction.

Throughout the paper, $\mathbb{N}$ denotes the set of positive integers, $\mathcal{P}(X)$ stands for the power set of X. For a subset E of a metric space, $\mathrm{cl}E$ will denote the closure of E. The ball with center x and radius r will be written as $B(x, r)$.

Recall that a sequence $\{x_n\}_{n\in\mathbb{N}}$ of points in a metric space (X, ρ) is said to be *statistically convergent* to $x \in X$ if $d(A(\varepsilon)) = 0$ for each $\varepsilon > 0$ where $A(\varepsilon) = \{n \in \mathbb{N} : \rho(x_n, x) \geq \varepsilon\}$ and $d(E) = \lim\limits_{n\to\infty} (1/n) \cdot \mathrm{card}(\{k \in E : k \leq n\})$ is the *density* of a set $E \subset \mathbb{N}$ provided that the limit exists.

Several papers on statistical convergence have been published. See [2], [3], [5]. In [4] and [7] an interesting generalization of this notion was proposed. Namely, it is easy to check that the family $\mathcal{I}_d = \{A \subset \mathbb{N} : d(A) = 0\}$ forms an ideal of subsets of $\mathbb{N}$. Thus, one may consider an arbitrary ideal $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ (assumed non-trivial, i.e. $\emptyset \neq \mathcal{I} \neq \mathcal{P}(\mathbb{N})$) to modify the definition of statistical convergence as follows. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in (X, ρ) is called $\mathcal{I}$-*convergent* to $x \in X$ (in short $x = \mathcal{I} - \lim\limits_{n\to\infty} x_n$) if $A(\varepsilon) \in \mathcal{I}$ for each $\varepsilon > 0$. The article [4] contains many examples and properties of $\mathcal{I}$-convergence. We shall continue these studies. Our main aim is to prove that, in a complete space (X, ρ), a Cauchy-type condition (borrowed from [3]) is necessary and sufficient for the $\mathcal{I}$-convergence of a given sequence. We also give equivalent formulations of $\mathcal{I}$-Cauchy condition and obtain $\mathcal{I}$-Cauchy condition for double sequences and show some applications.

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Following [4], $\mathcal{I}$ is called *admissible* if it contains all singletons. The ideal $\mathcal{I}_{fin}$ of all finite subsets of $\mathbb{N}$ is the smallest admissible ideal in $\mathcal{P}(\mathbb{N})$. Observe that the usual convergence, in a given space (X, ρ) coincides with $\mathcal{I}_{fin}$-convergence, and that the usual convergence implies $\mathcal{I}$-convergence, for any admissible ideal $\mathcal{I}$.

2 $\mathcal{I}$-Cauchy Condition.

Let (X, ρ) be a metric space and $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be an admissible ideal. It is easy to check that the classical Cauchy condition for a sequence $\{x_n\}_{n\in\mathbb{N}}$ in (X, ρ) is equivalent to the following: for each $\varepsilon > 0$ there exists a positive integer k such that $\rho(x_n, x_k) < \varepsilon$ for all $n \geq k$. A similar idea was used by Fridy [3] in formulation of the statistical Cauchy condition for a sequence of real numbers. We can modify it to define a Cauchy-type condition associated with $\mathcal{I}$-convergence in (X, ρ). Namely, $\mathcal{I}$-*Cauchy condition* reads as follows: for each $\varepsilon > 0$ there exists a $k \in \mathbb{N}$ such that $\{n \in \mathbb{N} : \rho(x_n, x_k) \geq \varepsilon\} \in \mathcal{I}$. Note that, for $\mathcal{I}_{fin}$, this yields the usual Cauchy condition. Fridy [3] proved that statistical Cauchy condition is equivalent to the statistical convergence of a sequence of reals. However, in any metric space we have the following proposition.

Proposition 1. *If a sequence of points in X is $\mathcal{I}$-convergent in X then it fulfills $\mathcal{I}$-Cauchy condition.*

PROOF. Let $\mathcal{I} - \lim_{n\to\infty} x_n = x$ and $\varepsilon > 0$. Thus $A(\varepsilon/2) = \{n \in \mathbb{N} : \rho(x_n, x) \geq \varepsilon/2\} \in \mathcal{I}$. Pick an $k \in \mathbb{N}$ such that $k \notin A(\varepsilon/2)$. Hence $\{n \in \mathbb{N} : \rho(x_n, x_k) \geq \varepsilon\} \subset \{n \in \mathbb{N} : \rho(x_n, x) \geq \varepsilon/2 \text{ or } \rho(x, x_k) \geq \varepsilon/2\} = A(\varepsilon/2) \in \mathcal{I}$. □

In the next theorem we shall show that the equivalence of $\mathcal{I}$-convergence and $\mathcal{I}$-Cauchy condition is true for complete metric spaces. Moreover, we shall give a sufficient condition for a metric space to be complete, by the use of $\mathcal{I}$-convergence of $\mathcal{I}$-Cauchy sequences. The proofs of Proposition 1 and of part (1) in Theorem 2 mimic the arguments of Fridy [3].

Theorem 2. *(1). If (X, ρ) is a complete space then every $\mathcal{I}$-Cauchy sequence in X is $\mathcal{I}$-convergent in X.*

(2). If every $\mathcal{I}$-Cauchy sequence in X is $\mathcal{I}$-convergent in X then X is complete.

PROOF. (1). Let $\{x_n\}_{n\in\mathbb{N}}$ be an $\mathcal{I}$-Cauchy sequence in a complete space (X, ρ). Consider $\varepsilon_m = 1/2^m$, $m \in \mathbb{N}$, and, according to $\mathcal{I}$-Cauchy condition, pick numbers $k(m) \in \mathbb{N}$, $m \in \mathbb{N}$, such that $A_m = \{n \in \mathbb{N} : \rho(x_n, x_{k(m)}) \geq \varepsilon_m/2\} \in \mathcal{I}$ for all $m \in \mathbb{N}$. Define inductively $B_1 = \mathrm{cl}B(x_{k(1)}, \varepsilon_1)$, $B_{m+1} = B_m \cap \mathrm{cl}B(x_{k(m+1)}, \varepsilon_{m+1})$, $m \in \mathbb{N}$. Let us prove that $B_m \neq \emptyset$ for each $m \in \mathbb{N}$.

Indeed, we have $A_1 \in \mathcal{I}$ and $x_n \in B_1$ for all $n \notin A_1$. Assume that $m \in \mathbb{N}$ and $C \in \mathcal{I}$ is a set such that $x_n \in B_m$ for each $n \notin C$. We have $A_{m+1} \in \mathcal{I}$ and $x_n \in \mathrm{cl}B(x_{k(m+1)}, \varepsilon_{m+1})$ for each $n \notin A_{m+1}$. Thus $C \cup A_{m+1} \in \mathcal{I}$ and $x_n \in B_{m+1}$ for all $n \notin C \cup A_{m+1}$. Since additionally $B_{m+1} \subset B_m$ for all $m \in \mathbb{N}$, and the diameter of B_m tends to 0, there is an $x \in X$ such that $\bigcap_{m\in\mathbb{N}} B_m = \{x\}$, by the Cantor theorem for complete spaces. It suffices to show that $\mathcal{I} - \lim_{n\to\infty} x_n = x$. Let $\varepsilon > 0$ and pick an $m \in \mathbb{N}$ such that $\varepsilon_m < \varepsilon/2$. We have

$$A(\varepsilon) \subset \{n \in \mathbb{N} : \rho(x_n, x_{k(m)}) + \rho(x_{k(m)}, x) \geq \varepsilon\}.$$

But $\rho(x_{k(m)}, x) \leq \varepsilon_m < \varepsilon/2$ since $x \in B_m$. Therefore

$$A(\varepsilon) \subset \{n \in \mathbb{N} : \rho(x_n, x_{k(m)}) + \varepsilon/2 \geq \varepsilon\} = \{n \in \mathbb{N} : \rho(x_n, x_{k(m)}) \geq \varepsilon/2\}$$
$$\subset \{n \in \mathbb{N} : \rho(x_n, x_{k(m)}) > \varepsilon_m\} \subset A_m \in \mathcal{I}.$$

(2). Let $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in (X, ρ). Since $\mathcal{I}$ is admissible, $\{x_n\}_{n\in\mathbb{N}}$ is an $\mathcal{I}$-Cauchy sequence. Thus, by assumption, we have $\mathcal{I} - \lim_{n\to\infty} x_n = x$ for some $x \in X$. Put $k_0 = 0$ and for $\varepsilon = 1/n, n \in \mathbb{N}$, pick inductively $k_n \in \mathbb{N} \setminus (\{0, \ldots, k_{n-1}\} \cup A(\varepsilon_n))$. Thus $\rho(x_{k_n}, x) < 1/n$ for every n which implies that $\lim_{n\to\infty} x_{k_n} = x$. Consequently, $\lim_{n\to\infty} x_n = x$. □

Note that $\mathcal{I}$-Cauchy sequences lack some natural properties of Cauchy sequences. For instance, a subsequence of an $\mathcal{I}$-Cauchy sequence can be not $\mathcal{I}$-Cauchy which is shown in the following example inspired by [4, Prop. 3.1(ii)].

Example 3. Assume that a metric space (X, ρ) contains at least two distinct points x and y. Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be an admissible ideal such that there exists a partition of $\mathbb{N}$ into pairwise disjoint infinite sets such that $A \in \mathcal{I}$ and $B \notin \mathcal{I}$, $C \notin \mathcal{I}$. Let $A = \{m_n : n \in \mathbb{N}\}$, $B \cup C = \{k_n : n \in \mathbb{N}\}$ with m_n and k_n strictly increasing. Define $\{x_n\}_{n\in\mathbb{N}}$ as follows. Put $x_{k_n} = x$ for all $n \in \mathbb{N}$. Let

$$x_{m_n} = \begin{cases} x & \text{if} \quad n \in A \cup B \\ y & \text{if} \quad n \in C \end{cases}.$$

Observe that $\mathcal{I} - \lim_{n\to\infty} x_n = x$, thus $\{x_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-Cauchy, by Theorem 2. However, the subsequence $\{x_{m_n}\}_{n\in\mathbb{N}}$ is not $\mathcal{I}$-Cauchy (consider $\varepsilon = \rho(x, y)$.)

The statements of Proposition 1 and Theorem 2 (1) were mentioned in [7]. The authors of [7] use however filters rather than ideals. Their $\mathcal{I}$-Cauchy condition is formulated in a different but equivalent form. Now, we shall prove this equivalence and we add one more equivalent condition.

For $\varepsilon > 0$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$ of points in (X, ρ), we denote $E_k(\varepsilon) = \{n \in \mathbb{N} : \rho(x_n, x_k) \geq \varepsilon\}$, $k \in \mathbb{N}$.

Proposition 4. *For a sequence $\{x_n\}_{n\in\mathbb{N}}$ of points in (X, ρ), the following conditions are equivalent:*

(1). $\{x_n\}_{n\in\mathbb{N}}$ *is an $\mathcal{I}$-Cauchy sequence,*

(2). *(cf. [7])* $(\forall\varepsilon > 0)(\exists D \in \mathcal{I})(\forall m, n \notin D) \quad \rho(x_m, x_n) < \varepsilon$,

(3). $(\forall\varepsilon > 0) \quad \{k \in \mathbb{N} : E_k(\varepsilon) \notin \mathcal{I}\} \in \mathcal{I}$.

PROOF. (1) $\Rightarrow$ (2). Let $\varepsilon > 0$. Put $D = E_k(\varepsilon/2)$ where $k \in \mathbb{N}$ is chosen for $\varepsilon/2$ in the $\mathcal{I}$-Cauchy condition for $\{x_n\}_{n\in\mathbb{N}}$. Thus $D \in \mathcal{I}$ and for any $m, n \notin D$ we have $\rho(x_n, x_k) < \varepsilon/2$ and $\rho(x_m, x_k) < \varepsilon/2$. Hence $\rho(x_n, x_m) < \varepsilon$ by the triangle inequality.
(2) $\Rightarrow$ (3). Let $\varepsilon > 0$ and let D be chosen as in (2). We shall show that $\{k \in \mathbb{N} : E_k(\varepsilon) \notin \mathcal{I}\} \subset D$. Let $k \in \mathbb{N}$ be such that $E_k(\varepsilon) \notin \mathcal{I}$. Suppose that $k \notin D$. Pick an $n \in E_k(\varepsilon) \setminus D$. Thus $\rho(x_n, x_k) \geq \varepsilon$ by the definition of $E_k(\varepsilon)$. But $n, k \notin D$ implies $\rho(x_n, x_k) < \varepsilon$ by (2), contradiction.
(3) $\Rightarrow$ (1). ¿From (3) we have $(\forall\varepsilon > 0) \quad \{k \in \mathbb{N} : E_k(\varepsilon) \in \mathcal{I}\} \neq \emptyset$ which yields (1). □

3 Double Sequences.

In [1], the notion of $\mathcal{I}$-convergence was extended to the case when $\mathcal{I}$ is an ideal of subsets of $\mathbb{N}^2$ and one considers a double sequence $\{x_{mn}\}_{m,\, n\in\mathbb{N}}$ of points in (X, ρ). (The further generalization deals with multi-indexed sequences and with ideals in $\mathcal{P}(\mathbb{N}^k)$ for $k \in \mathbb{N}$.) Namely, we say that $\{x_{mn}\}_{m,\, n\in\mathbb{N}}$ is $\mathcal{I}$-*convergent* to $x \in X$ (in short $\mathcal{I} - \lim x_{mn} = x$) if $\{(m, n) \in \mathbb{N}^2 : \rho(x_{mn}, x) \geq \varepsilon\} \in \mathcal{I}$ for each $\varepsilon > 0$. Again an ideal $\mathcal{I} \subset \mathcal{P}(\mathbb{N}^2)$ is called admissible if it is non-trivial and contains all singletons.

Proposition 5. *Let $\{x_{mn}\}_{m,\, n\in\mathbb{N}}$ be a sequence of points in a complete metric space (X, ρ) and let $\mathcal{I} \subset \mathcal{P}(\mathbb{N}^2)$ be an admissible ideal. The following conditions are equivalent:*

(1). $\{x_{mn}\}_{m,\, n\in\mathbb{N}}$ *is an $\mathcal{I}$-convergent sequence,*

(2). $(\forall\varepsilon > 0)(\exists (M, N) \in \mathbb{N}^2) \quad \{(m, n) \in \mathbb{N}^2 : \rho(x_{mn}, x_{MN}) \geq \varepsilon\} \in \mathcal{I}$.

If moreover, $\mathcal{I}$ contains all sets of the form $\{n\} \times \mathbb{N}$, $\mathbb{N} \times \{n\}$ (for $n \in \mathbb{N}$), each of the above conditions is equivalent to:

(3). $(\forall\varepsilon > 0)(\forall l \in \mathbb{N})(\exists M, N \geq l) \quad \{(m, n) \in \mathbb{N}^2 : \rho(x_{mn}, x_{MN}) \geq \varepsilon\} \in \mathcal{I}$.

Proof. To show (1) $\Leftrightarrow$ (2), fix a bijection $\varphi : \mathbb{N}^2 \to \mathbb{N}$ and put $\mathcal{J} = \{\varphi(A) : A \in \mathcal{I}\}$. For an $x \in X$ we have $\mathcal{I} - \lim x_{mn} = x \Leftrightarrow \mathcal{J} - \lim_{k\to\infty} x_{\varphi^{-1}(k)} = x$. By Proposition 1 and Theorem 2 part (1) this last condition is equivalent to $(\forall \varepsilon > 0)(\exists j \in \mathbb{N}) \quad \{k \in \mathbb{N} : \rho(x_{\varphi^{-1}(k)}, x_{\varphi^{-1}(j)}) \geq \varepsilon\} \in \mathcal{J}$ which in turn is equivalent to (2) when we put $(M, N) = \varphi^{-1}(j)$. Now, assume that $\mathcal{I}$ contains all sets of the form $\{n\} \times \mathbb{N}$, $\mathbb{N} \times \{n\}$ (for $n \in \mathbb{N}$). It is obvious that (3) $\Rightarrow$ (2). Let us show implication (1) $\Rightarrow$ (3). Assume that $\mathcal{I} - \lim x_{mn} = x$ and fix $\varepsilon > 0$ and $l \in \mathbb{N}$. Thus $\{(m,n) \in \mathbb{N}^2 : \rho(x_{mn}, x) \geq \varepsilon/2\} \in \mathcal{I}$. Since $(\mathbb{N} \times \{1, \ldots, l-1\}) \cup (\{1, \ldots, l-1\} \times \mathbb{N}) \in \mathcal{I}$, we can pick $(M, N) \in \{l, l+1, \ldots\} \times \{l, l+1, \ldots\}$ with $\rho(x_{MN}, x) < \varepsilon/2$. Now, we have $\{(m,n) \in \mathbb{N}^2 : \rho(x_{mn}, x_{MN}) \geq \varepsilon\} \subset \{(m,n) \in \mathbb{N}^2 : \rho(x_{mn}, x) \geq \varepsilon/2$ or $\rho(x_{MN}, x) \geq \varepsilon/2\} = \{(m,n) \in \mathbb{N}^2 : \rho(x_{mn}, x) \geq \varepsilon/2\} \in \mathcal{I}$ as desired. $\square$

Remark. $\mathcal{I}$-Cauchy condition in the form (3) was proved by Móricz [6] in the particular case when $\mathcal{I}$ consists of all sets $S \subset \mathbb{N}^2$ with two-dimensional density $d_2(S)$ equal to 0 where $d_2(S) = \lim_{m,n\to\infty} (1/(mn)) \cdot \operatorname{card}(S \cap (\{1, \ldots, m\} \times \{1, \ldots, n\}))$ and $\lim_{m,n\to\infty} z_{mn} = z$ is meant in the Pringsheim's sense, that is $(\forall \varepsilon > 0)(\exists l \in \mathbb{N})(\forall m, n \geq l) \quad |z_{mn} - z| < \varepsilon$. It is easy to check that the ideal $\mathcal{I}$ defined in such a manner contains all sets of the form $\{n\} \times \mathbb{N}$, $\mathbb{N} \times \{n\}$ (for $n \in \mathbb{N}$).

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Added in Proof. Recently another proof of Theorem 2(1) has been published in the paper B. K. Lahiri, Pratulananda Das, *Further results on I-limit superior and I-limit inferior*, Mathematical Communications, **8** (2003), 151–156.

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