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RUNS AND INTEGRATION

Abstract

We use the convergence notion of a run to define integration processes which subsume those of Riemann-Stieltjes, Lebesgue and Henstock-Kurzweil.

1 Introduction.

Runs were introduced by Hewitt Kenyon and A. P. Morse in their paper "Runs", Pacific Journal of Mathematics, 1958 [6], as an alternative to filters [4] and nets [5]. No standard text in topology mentions them. Nonetheless, we find them to be an ideal tool for defining integration. We illustrate this viewpoint by using runs to define an integration process which subsumes Lebesgue and Riemann-Stieltjes integration. Elementary additivity properties of the integral are established under very general conditions. A variant of the main definition yields a generalization of the gauge (Henstock-Kurzweil) integral [1, 3, 12], thereby extending a description using nets (Moore-Smith limits) given in [3]. Although all of the definitions are valid in very abstract situations, they suggest viewpoints which are new even for real-valued functions on the real line. For the equivalence of runs with filters and nets we refer to [6].

For any relation R we denote by $\operatorname{dom} R$ and $\operatorname{rng} R$ respectively the domain of $R := \{x : \exists y\,(x, y) \in R\}$, and the range of $R := \{y : \exists x\,(x, y) \in R\}$. For all $x \in \operatorname{dom} R$, we denote by Rx the set $\{t : (x, t) \in R\}$. A **run** in a set Z is a relation R such that $\operatorname{rng} R \subseteq Z$, and, for all $x, y \in \operatorname{dom} R$, there exists $z \in \operatorname{dom} R$ such that $Rz \subseteq Rx \cap Ry$; i.e., such that if $(z, t) \in R$, then $(x, t), (y, t) \in R$; R' is a **subrun** of R if and only if R is a run, R' is a run, and for all $x \in \operatorname{dom} R$ there exists $y \in \operatorname{dom} R'$ such that $R'y \subseteq Rx$. Runs R and R' will be called **linked** if and only if $Rx \cap R'x' \neq \emptyset$, for all $x \in \operatorname{dom} R$

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and $x' \in \operatorname{dom} R'$. A run R in a topological space Z **converges** to $z \in Z$ if and only if for each neighborhood N of z there exists $x \in \operatorname{dom} R$ such that $Rx \subseteq N$. Clearly, if R and R' are linked runs in a Hausdorff space converging to z and z' respectively, then $z = z'$.

Let R be a relation and f a function defined on $\operatorname{rng} R$. We denote by $f \circ R$ the relation consisting of all ordered pairs $(x, f(z))$, with $(x, z) \in R$. Let R_i be a run in a set Z_i, $i = 1, \ldots, k$. We denote by $R_1 \otimes \cdots \otimes R_k$ the collection of all ordered pairs $((x_1, \ldots, x_k), (z_1, \ldots, z_k))$, in which $(x_i, z_i) \in R_i$ for each $1 \leq i \leq k$. The following properties of runs are easily checked.

If R is a run in a set Z, and f is a function from Z to a set T, then $f \circ R$ is a run in T. If R_i is a run in a set Z_i, $i = 1, \ldots, k$, then $R_1 \otimes \cdots \otimes R_k$ is a run in $Z_1 \times \cdots \times Z_k$. A run R in a topological space Z converges to $z \in Z$ if and only if every subrun of R converges to z. If, for $1 \leq i \leq k$, Z_i is a topological space and R_i is a run in Z_i converging to z_i, and f is a continuous function on the product space $Z_1 \times \cdots \times Z_k$ to a topological space T, then $f \circ (R_1 \otimes \cdots \otimes R_k)$ is a run in T converging to $f(z_1, \ldots, z_k)$.

In particular, therefore, if R_1, R_2 are runs in a commutative topological semigroup, $(Z, +)$, converging respectively to $z_i \in Z_i$, for $i = 1, 2$, then the run, $R_1 + R_2$, consisting of all ordered pairs $((x_1, x_2), s_1 + s_2)$ for which $(x_i, s_i) \in R_i$, $i = 1, 2$, converges to $z_1 + z_2$.

2 Integration.

We turn now to the definition of an integration process [11]. In what follows, E is a set, $\mathcal{H}$ and $\mathcal{N}$ are families of subsets of E such that $\mathcal{H}$ is closed under finite intersections, and $H \cap N \in \mathcal{N}$, for all $H \in \mathcal{H}$ and $N \in \mathcal{N}$, f is a function on E to a set X, $A \subseteq E$, η is a function on $\mathcal{H}$ to a set Y, and $<, >$ is a binary operation on $X \times Y$ to a commutative topological semigroup, $(Z, +)$. We shall denote $< x, y >$ by $x.y$. For any non-empty $P \subseteq \mathcal{H}$, non-empty finite $F \subseteq P$, and choice function $h : p \in P \mapsto h_p \in p$, we denote by $S(f, h, \eta, F)$ the sum $\sum_{p \in F} f(h_p).\eta(p)$.

Denote by $\mathcal{P}_{\mathcal{H}}(A)$ the family of all countable $P \subseteq \mathcal{H}$ such that $A \subseteq \bigcup P$ and $P \cap \mathcal{N} = \emptyset$. Given $P, Q \in \mathcal{P}_{\mathcal{H}}(A)$, we say that Q is **finer** than P if for each $q \in Q$ there exists $p \in P$ with $q \subseteq p$. A **truncation** on $\mathcal{P}_{\mathcal{H}}(A)$ is a function Δ on $\mathcal{P}_{\mathcal{H}}(A)$ such that $\Delta(P)$ is a finite subset of P for each $P \in \mathcal{P}_{\mathcal{H}}(A)$. A truncation Γ is **larger** than a truncation Δ if and only if $\Delta(P') \subseteq \Gamma(P')$ for all $P' \in \mathcal{P}_{\mathcal{H}}(A)$. If $P \in \mathcal{P}_{\mathcal{H}}(A)$ has pairwise intersections in $\mathcal{N}$, it will be called an $\mathcal{N}$-**mesh** in $\mathcal{H}$ covering A. Subsets A_1 and A_2 of E are **separated** by $\mathcal{H}$ with respect to $\mathcal{N}$ if and only if there exist H_1 and H_2 in $\mathcal{H}$ with $A_i \subseteq H_i$, $i = 1, 2$, such that $H_1 \cap H_2 \in \mathcal{N}$. (Notice that if $\mathcal{N}$ contains only the empty set then an $\mathcal{N}$-mesh is actually a countable, pairwise-disjoint subfamily of $\mathcal{H}$.

We introduce $\mathcal{N}$-meshes to allow covers of A which are possibly not pairwise-disjoint, as occur in Riemann integration.) For each $A \subseteq E$, let $\mathcal{P}_A$ be a subfamily of $\mathcal{P}_{\mathcal{H}}(A)$ directed by refinement. We define the run $\mathcal{F}(A, f, \eta, \mathcal{P})$ to be the collection of all ordered pairs $((P, \Delta), S(f, h, \eta, \Gamma(Q)))$, for which $P \in \mathcal{P}_A$, Δ is a truncation on $\mathcal{P}_{\mathcal{H}}(A)$, h is a choice function on Q for some $Q \in \mathcal{P}_A$ finer than P, and Γ is a truncation on $\mathcal{P}_{\mathcal{H}}(A)$ larger than Δ.

Definition 2.1. *The function f is $\mathcal{P}$-integrable over A with respect to η if and only if $\mathcal{F}(A, f, \eta, \mathcal{P})$ is a run in Z converging to some $z \in Z$.*

For motivation of the definition we refer to [11]. The idea of using an arbitrary $\mathcal{P}_A$ is taken from [7]. If f is $\mathcal{P}$-integrable over A with respect to η, and Z is Hausdorff, then we denote by $\mathcal{P}\text{-}\int_A f.d\eta$ the unique point of Z to which $\mathcal{F}(A, f, \eta, \mathcal{P})$ converges. The phrase "$\mathcal{P}\text{-}\int_A f.d\eta$ exists" will then be synonymous with "$\mathcal{F}(A, f, \eta, \mathcal{P})$ converges in Z to $\mathcal{P}\text{-}\int_A f.d\eta$". The case of non-Hausdorff Z can be treated by consideration of the standard quotient space. Our first two theorems establish additivity properties of the integral.

Theorem 2.1. *Let X be a commutative semigroup, with $<,>$ being additive in its first argument, and let f_1, f_2 be X-valued functions on E. If $\mathcal{P}\text{-}\int_A f_1.d\eta$, $\mathcal{P}\text{-}\int_A f_2.d\eta$ both exist, then $\mathcal{P}\text{-}\int_A (f_1 + f_2).d\eta$ exists, and*

$$\mathcal{P}\text{-}\int_A (f_1 + f_2).d\eta = \mathcal{P}\text{-}\int_A f_1.d\eta + \mathcal{P}\text{-}\int_A f_2.d\eta$$

Proof. Let f_1, f_2 be X-valued functions on E which are integrable over A. Then, by earlier remarks, $\mathcal{F}(A, f_1, \eta, \mathcal{P}) + \mathcal{F}(A, f_2, \eta, \mathcal{P})$ is a run converging to $\mathcal{P}\text{-}\int_A f_1.d\eta + \mathcal{P}\text{-}\int_A f_2.d\eta$. We shall now show that it has $\mathcal{F}(A, f_1 + f_2, \eta, \mathcal{P})$ as a subrun. We shall write $(P, \Delta) \prec (Q, \Gamma)$ if and only if $P, Q \in \mathcal{P}_{\mathcal{H}}(A)$ with Q finer than P, and Δ, Γ are truncations on $\mathcal{P}_{\mathcal{H}}(A)$ with Γ larger than Δ. Let $\mathcal{D}$ consist of all ordered pairs (P, Δ), where $P \in \mathcal{P}_A$ and Δ is a truncation on $\mathcal{P}_{\mathcal{H}}(A)$. Then $\mathcal{D}$ is directed by $\prec$. Given any $(P_i, \Delta_i) \in \mathcal{D}$, $i = 1, 2$, choose $(P', \Delta') \in \mathcal{D}$ such that $(P_i, \Delta_i) \prec (P', \Delta')$, $i = 1, 2$. Suppose $((P', \Delta'), S(f_1 + f_2, h, \eta, \Gamma(Q))) \in \mathcal{F}(A, f_1 + f_2, \eta, \mathcal{P})$. Since $\prec$ is transitive, $((P_i, \Delta_i), S(f_i, h, \eta, \Gamma(Q))) \in \mathcal{F}(A, f_i, \eta, \mathcal{P})$. Further, by the additivity of $<,>$ in its first argument,

$$S(f_1 + f_2, h, \eta, \Gamma(Q)) = S(f_1, h, \eta, \Gamma(Q)) + S(f_2, h, \eta, \Gamma(Q)),$$

and therefore,

$$(((P_1, \Delta_1), (P_2, \Delta_2)), S(f_1 + f_2, h, \eta, \Gamma(Q))) \in \mathcal{F}(A, f_1, \eta, \mathcal{P}) + \mathcal{F}(A, f_2, \eta, \mathcal{P}).$$

Thus, $\mathcal{F}(A, f_1 + f_2, \eta, \mathcal{P})$ is a subrun of $\mathcal{F}(A, f_1, \eta, \mathcal{P}) + \mathcal{F}(A, f_2, \eta, \mathcal{P})$. □

Similarly, we can show the linearity of the integral when X is a vector space over the real or complex numbers, Z is a topological vector space over the same field of scalars, and $<,>$ is linear in its first argument. Note that $\mathcal{P}_A$ may consist of all countable, pairwise-disjoint subfamilies of $\mathcal{H}$ which cover A, or of all finite, pairwise-disjoint subfamilies of $\mathcal{H}$ which cover A. Indeed, we may take $\mathcal{P}_A$ to be the family of all finite $\mathcal{N}$-meshes in $\mathcal{H}$ covering A.

We can show that the integral is an additive set-function, if the families $\mathcal{P}_A$ satisfy the following conditions:

- $\mathcal{P}_A \subseteq \mathcal{P}_{\mathcal{H}}(A)$ is directed by refinement, and $\{H\} \in \mathcal{P}_A$, for all $A \subseteq H \in \mathcal{H}$;
- if $P \in \mathcal{P}_{A\cup B}$ then there exists $P' \in \mathcal{P}_A$ finer than $\{p \in P : p \cap A \neq \emptyset\}$;
- if $Q_i \in \mathcal{P}_{A_i}$, and $p_1 \cap p_2 \in \mathcal{N}$ for all $p_i \in Q_i$, $i = 1, 2$, then $Q_1 \cup Q_2 \in \mathcal{P}_{A_1\cup A_2}$.

Theorem 2.2. *Let A_1, A_2 be subsets of E which are separated by $\mathcal{H}$ with respect to $\mathcal{N}$. If the integrals on both sides exist then*

$$\mathcal{P}\text{-}\int_{A_1} f.d\eta + \mathcal{P}\text{-}\int_{A_2} f.d\eta = \mathcal{P}\text{-}\int_{A_1\cup A_2} f.d\eta.$$

PROOF. For $i = 1, 2$, let $A_i \subseteq H_i \in \mathcal{H}$ with $H_1 \cap H_2 \in \mathcal{N}$. Let $P_i \in \mathcal{P}_{A_i}$, Δ_i be a truncation on $\mathcal{P}_{\mathcal{H}}(A_i)$, $P \in \mathcal{P}_{A_1\cup A_2}$, and Δ be a truncation on $\mathcal{P}_{\mathcal{H}}(A_1 \cup A_2)$. Choose $Q_i \in \mathcal{P}_{A_i}$ finer than $\{H_i\}$, P_i, and $\{p \in P : p \cap A_i \neq \emptyset\}$, and denote $Q_1 \cup Q_2$ by Q. Then $Q \in \mathcal{P}_{A_1\cup A_2}$. Let Γ_i be a truncation on $\mathcal{P}_{\mathcal{H}}(A_i)$ such that Γ_i is larger than Δ_i, and $\Gamma_1(Q_1) \cup \Gamma_2(Q_2) \supseteq \Delta(Q_1 \cup Q_2)$. Let h_i be a choice function on Q_i, and h be the choice function on Q given by $h(q) = h_i(q)$, if $q \in Q_i$. Let Γ be a truncation on $\mathcal{P}_{\mathcal{H}}(A_1 \cup A_2)$ such that $\Gamma(Q) = \Gamma_1(Q_1) \cup \Gamma_2(Q_2)$, and $\Gamma(P) = \Delta(P)$, if $P \neq Q_1 \cup Q_2$. Then,

$$\begin{aligned}
&S(f, h_1, \eta, \Gamma_1(Q_1)) + S(f, h_2, \eta, \Gamma_2(Q_2)) = S(f, h, \eta, \Gamma(Q)),\\
&((P_i, \Delta_i), S(f, h_i, \eta, \Gamma_i(Q_i))) \in \mathcal{F}(A_i, f, \eta, \mathcal{P}_{A_i}), \text{ and}\\
&((P, \Delta), S(f, h, \eta, \Gamma(Q))) \in \mathcal{F}(A_1 \cup A_2, f, \eta, \mathcal{P}_{A_1\cup A_2}).
\end{aligned}$$

Thus $\mathcal{F}(A_1 \cup A_2, f, \eta, \mathcal{P}_{A_1\cup A_2})$, $\mathcal{F}(A_1, f, \eta, \mathcal{P}_{A_1}) + \mathcal{F}(A_2, f, \eta, \mathcal{P}_{A_2})$ are linked runs, and their limits are therefore equal. $\square$

For the classical definition of the Riemann-Stieltjes integral [9], p.122, and [2], p.298, we identify E, X, Y and Z with the real line, and $<,>$ with the binary operation of multiplication on the real line. Let $\mathcal{H}$ be the family of all non-empty, closed subintervals, $[s, t]$, of the real line, $\eta([s, t]) = g(t) - g(s)$ for some monotone increasing function g, $\mathcal{N}$ consist of the empty set together with all one-point subsets of the real line, $A = [a, b]$, and $\mathcal{P}_A$ be the family of all finite $\mathcal{N}$-meshes $P \subseteq \mathcal{H}$ with $\bigcup P = [a, b]$. In this case the use of truncations is superfluous, and, for a bounded real-valued function f on $[a, b]$, the Riemann-Stieltjes integral may be defined as the limit, when it exists, of the run whose ordered pairs are of the form, $(P, S(f, h, \eta, Q))$, where $P \in \mathcal{P}_A$ and h is a choice function on Q for some $Q \in \mathcal{P}_A$ which is a refinement of P ([9], thms. 6.6, 6.7, and [2], thms. 5.8, 5.9), or, equivalently, as the limit of the run whose ordered pairs are of the form $(\delta, S(f, h, \eta, Q))$, where $\delta > 0$, $Q \in \mathcal{P}_A$, with the length of q less than δ for all $q \in Q$, and h is a choice function on Q, [2], Theorem 5.10, p.309.

Modifying the latter approach, we can use runs to define a generalized gauge (Henstock-Kurzweil) integral [1]. Let $\mathcal{U}$ be a uniformity on E [4]. A **gauge** on E is a function on E to $\mathcal{U}$. Given $Q \in \mathcal{P}_{\mathcal{H}}(A)$, a choice function h on Q and a gauge δ on E, we shall say that (Q, h) is δ-**fine** if and only if $(t, h_q) \in \delta(h_q)$ for all $q \in Q$ and $t \in q$. Now let $\mathcal{P}_A$ be a subfamily of $\mathcal{P}_{\mathcal{H}}(A)$ such that for each gauge δ on E there exists a $Q \in \mathcal{P}_A$ and a choice function h on Q such that (Q, h) is δ-fine. (For the gauge integral of a real-valued function on a closed, bounded interval of the real line, validity of the latter condition is guaranteed by Cousin's theorem [1].) Denote by $\mathcal{K}(A, f, \eta, \mathcal{P}, \mathcal{U})$ the collection of all ordered pairs, $((\delta, \Delta), S(f, h, \eta, \Gamma(Q)))$, in which δ is a gauge on E, Δ is a truncation of $\mathcal{P}_{\mathcal{H}}(A)$, Γ is a truncation of $\mathcal{P}_{\mathcal{H}}(A)$ larger than Δ, $Q \in \mathcal{P}_A$ and h is a choice function on Q such that (Q, h) is δ-fine.

Definition 2.2. *The function f is* **gauge-integrable** *over A with respect to $\mathcal{P}$ if and only if $\mathcal{K}(A, f, \eta, \mathcal{P}, \mathcal{U})$ is a run in Z converging to some $z \in Z$.*

The generalized gauge integral of an X-valued function f on E will be denoted by $\mathcal{G}\text{-}\int_A f.d\eta$. Through a straightforward modification of its proof, Theorem 2.1 extends to this integral.

Theorem 2.3. *Let X be a semigroup, and $<,>$ be additive in its first argument. If $\mathcal{G}\text{-}\int_A f_1.d\eta$, $\mathcal{G}\text{-}\int_A f_2.d\eta$ both exist, then $\mathcal{G}\text{-}\int_A (f_1 + f_2).d\eta$ exists, and*

$$\mathcal{G}\text{-}\int_A (f_1 + f_2).d\eta = \mathcal{G}\text{-}\int_A f_1.d\eta + \mathcal{G}\text{-}\int_A f_2.d\eta$$

A similar extension of Theorem 2.2 can be established if the families $\mathcal{P}_A$ satisfy the following conditions:

- for all $A \subseteq H \in \mathcal{H}$, and gauge δ on E, there exist $Q \in \mathcal{P}_A$, and a choice function h on Q, such that $q \subseteq H$ for all $q \in Q$, and (Q, h) is δ-fine.
- if $Q_i \in \mathcal{P}_{A_i}$, and $p_1 \cap p_2 \in \mathcal{N}$ for all $p_i \in Q_i$, $i = 1, 2$, then $Q_1 \cup Q_2 \in \mathcal{P}_{A_1 \cup A_2}$.

Theorem 2.4. *Let A_1, A_2 be subsets of E which are separated by $\mathcal{H}$ with respect to $\mathcal{N}$. If the integrals on both sides exist then*

$$\mathcal{G}\text{-}\int_{A_1} f.d\eta + \mathcal{G}\text{-}\int_{A_2} f.d\eta = \mathcal{G}\text{-}\int_{A_1\cup A_2} f.d\eta.$$

We note that the above definition of a generalized gauge integral is applicable to any subset of a uniform space, and allows countable covers of the domain of integration. However, when E, X, Y and Z are identified with the real line, under the uniformity generated by the standard Euclidean metric, and $<,>$ with multiplication, A is a closed subinterval of E with non-empty interior, $\mathcal{H}$ is the family of all closed subintervals $[s,t]$ of E having non-empty interior, with $\eta([s,t]) = g(t) - g(s)$ for some monotone increasing function g on E, $\mathcal{N}$ consists of the empty set together with all one-point subsets of the real line, and $\mathcal{P}_A$ consists of all finite $\mathcal{N}$-meshes $P \subseteq \mathcal{H}$ with $\bigcup P = A$, then the above yields a definition of the gauge integral of a real-valued function f on A [1]. Our definition using runs therefore improves on that given in [3] using nets.

We can give a unified definition of the integration processes defined above. For each $A \subseteq E$, let $\mathcal{M}_A$ be a run in the family of all ordered pairs (P,g) for which $P \in \mathcal{P}_{\mathcal{H}}(A)$ and g is a choice function on P. Let $\mathcal{J}(A,f,\eta,\mathcal{M})$ be the run in Z consisting of all ordered pairs $((x,\Delta), S(f,h,\eta,\Gamma(Q)))$ for which $x \in \operatorname{dom}\mathcal{M}_A$, Δ is a truncation on $\mathcal{P}_{\mathcal{H}}(A)$, $(x,(Q,h)) \in \mathcal{M}_A$ and Γ is a truncation on $\mathcal{P}_{\mathcal{H}}(A)$ larger than Δ.

Definition 2.3. *The function f is* $\mathcal{M}$-**integrable** *over A if and only if the run $\mathcal{J}(A,f,\eta,\mathcal{M})$ converges to some $z \in Z$.*

This generalized integral of an X-valued function f on E will be denoted by $\mathcal{M}\text{-}\int_A f.d\eta$. It is easily shown that it is additive.

Theorem 2.5. *Let X be a semigroup, and $<,>$ be additive in its first argument. If $\mathcal{M}\text{-}\int_A f_1.d\eta$, $\mathcal{M}\text{-}\int_A f_2.d\eta$ both exist, then $\mathcal{M}\text{-}\int_A (f_1+f_2).d\eta$ exists, and*

$$\mathcal{M}\text{-}\int_A (f_1+f_2).d\eta = \mathcal{M}\text{-}\int_A f_1.d\eta + \mathcal{M}\text{-}\int_A f_2.d\eta$$

Its additivity as a set function can be proved under the following assumptions:

- for all $A \subseteq H \in \mathcal{H}$, and $x \in \operatorname{dom}\mathcal{M}_A$, there exists Q, h such that $(x,(Q,h)) \in \mathcal{M}_A$, and $q \subseteq H$ for all $q \in Q$;
- for all $y \in \operatorname{dom}\mathcal{M}_{A_1\cup A_2}$, and $i = 1,2$, there exist $x_i \in \operatorname{dom}\mathcal{M}_{A_i}$, such that if $(x_i,(Q_i,h_i)) \in \mathcal{M}_{A_i}$, $p_1 \cap p_2 \in \mathcal{N}$ for all $p_i \in Q_i$, and $h(q) = h_i(q)$ whenever $q \in Q_i$, then $(y,(Q_1\cup Q_2,h)) \in \mathcal{M}_{A_1\cup A_2}$.

Theorem 2.6. *Let A_1, A_2 be subsets of E which are separated by $\mathcal{H}$ with respect to $\mathcal{N}$. If the integrals on both sides exist then*

$$\mathcal{M}\text{-}\int_{A_1} f.d\eta + \mathcal{M}\text{-}\int_{A_2} f.d\eta = \mathcal{M}\text{-}\int_{A_1 \cup A_2} f.d\eta,$$

By appropriate specialization of $\mathcal{M}_A$ we obtain the integrals defined previously, and their additivity properties.

3 Set Functions.

We close with applications of runs to the generation of set functions by inner or outer approximation, showing that runs may be of interest outside of integration theory. Given a Z-valued function ν on a family, $\mathcal{H}$, of subsets of E, we say that ν is **additive** if and only if $\nu(A \cup B) = \nu(A) + \nu(B)$, for all disjoint A and B in $\mathcal{H}$.

Let $(\mathcal{K}, \mathcal{G})$ be a pair of families of subsets of E, such that $\mathcal{K}$ is closed under finite unions, $\mathcal{G}$ is closed under finite unions and finite intersections, and for all $K \in \mathcal{K}$ and $G \in \mathcal{G}$: (1) $K \setminus G \in \mathcal{K}$, $G \setminus K \in \mathcal{G}$, (2) there exist $K' \in \mathcal{K}$ and $G' \in \mathcal{G}$ with $K' \subseteq G$ and $G' \subseteq K$, and (3) if $K \subseteq G$ then there exist $K' \in \mathcal{K}$ and $G' \in \mathcal{G}$ with $K \subseteq G' \subseteq K' \subseteq G$ [8, 10].

Let κ and γ be Z-valued functions, respectively on $\mathcal{K}$ and $\mathcal{G}$. For each $G \in \mathcal{G}$, let $\mathcal{K}^-(G, \kappa)$ be the run consisting of all ordered pairs $(K, \kappa(K'))$, for which $K, K' \in \mathcal{K}$ and $K \subseteq K' \subseteq G$. For each $A \subseteq E$ let $\mathcal{G}^+(A, \gamma)$ be the run consisting of all ordered pairs $(G, \gamma(G'))$, for which $G, G' \in \mathcal{G}$, with $A \subseteq G' \subseteq G$.

Theorem 3.1. *Let κ be additive, and $G_1, G_2 \in \mathcal{G}$ be disjoint. If $\mathcal{K}^-(G_1, \kappa)$ and $\mathcal{K}^-(G_2, \kappa)$ converge to z_1, z_2 respectively, then $\mathcal{K}^-(G_1 \cup G_2, \kappa)$ converges to $z_1 + z_2$.*

PROOF. Let $G_1, G_2 \in \mathcal{G}$ be disjoint, and $K_1, K_2 \in \mathcal{K}$ with $K_i \subseteq G_i \in \mathcal{G}$, $i = 1, 2$. For each $K' \in \mathcal{K}$ such that $K_1 \cup K_2 \subseteq K' \subseteq G_1 \cup G_2$, let $K'_1 = K' \setminus G_2$, $K'_2 = K' \setminus G_1$. Then $\kappa(K') = \kappa({K'}_1) + \kappa({K'}_2)$, and $(K_i, \kappa(K'_i)) \in \mathcal{K}^-(G_i, \kappa)$, $i = 1, 2$. Thus $\mathcal{K}^-(G_1 \cup G_2, \kappa)$ is a subrun of $\mathcal{K}^-(G_1, \kappa) + \mathcal{K}^-(G_2, \kappa)$. □

Theorem 3.2. *Let γ be additive, and $K_1, K_2 \in \mathcal{K}$ be disjoint. If $\mathcal{G}^+(K_1, \gamma)$ and $\mathcal{G}^+(K_2, \gamma)$ converge to z_1, z_2 respectively, then $\mathcal{G}^+(K_1 \cup K_2, \gamma)$ converges to $z_1 + z_2$.*

PROOF. $\mathcal{G}^+(K_1 \cup K_2, \gamma)$ is a subrun of $\mathcal{G}^+(K_1, \gamma) + \mathcal{G}^+(K_2, \gamma)$. (The proof uses the following separation property of $(\mathcal{K}, \mathcal{G})$. *For all disjoint $K_1, K_2 \in \mathcal{K}$, there exist disjoint $G_1, G_2 \in \mathcal{G}$ with $K_i \subseteq G_i$, $i = 1, 2$.*) □

Further conditions are required to guarantee the existence of the limits appearing in the theorems above [1, 2, 11], or to guarantee that the set functions defined by inner or outer approximation are extensions σ-additive on a σ-algebra containing the original domain of definition.

We do not doubt the usefulness of filters and nets for describing convergence. However we do think that there is still a place for runs, especially in the theory of integration, where they correspond to the process under consideration more efficiently than filters or nets. (See, [6], p.813, last paragraph.)

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