Real Analysis Exchange
Vol. 30(1), 2004/2005, pp. 43–58

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ITERATED REDUCED CLUSTER FUNCTIONS

Abstract

Given a multifunction F between topological spaces X and Y, the reduced cluster function $C^r(F;\cdot) : X \to 2^Y$ of F is defined by $C^r(F;x) = \bigcap \mathrm{cl}(F(U \setminus \{x\}))$, U running through the neighborhood system of x. By transfinite recursion, one defines iterated reduced cluster functions $C^{r,\alpha}(F;\cdot)$ for all ordinals $\alpha > 0$.

We characterize multifunctions F that are invariant in the sense of $C^r(F;\cdot) = F$. For every countable ordinal α, we describe the family of all iterated reduced cluster functions $C^{r,\alpha}(F;\cdot)$ of arbitrary multifunctions $F : X \to 2^Y$ and the family of all iterated reduced cluster functions $C^{r,\alpha}(f;\cdot)$ of arbitrary functions $f : X \to Y$, provided that X and Y are metrizable spaces and Y is separable.

1 Definitions and Basic Properties.

Let $F : X \to 2^Y$ be a multifunction mapping a topological space X into the subsets of a topological space Y. The empty set is allowed to be a value of F. The cluster set $C(F;x_0)$ and the reduced cluster set $C^r(F;x_0)$ of F at a point $x_0 \in X$ are defined by

$$C(F;x_0) = \bigcap_{U \in \mathcal{U}(x_0)} \mathrm{cl}(F(U))$$

and

$$C^r(F;x_0) = \bigcap_{U \in \mathcal{U}(x_0)} \mathrm{cl}(F(U \setminus \{x_0\})),$$

respectively, where $\mathrm{cl}(\cdot)$ is the closure operator, $\mathcal{U}(x_0)$ the family of all open neighborhoods of x_0, and $F(U)$ the union $F(U) = \bigcup_{x \in U} F(x)$. Obviously,

$$C(F;x_0) = C^r(F;x_0) \cup \mathrm{cl}(F(x_0)). \tag{1}$$

Key Words: Cluster set, reduced cluster set, reduced cluster function of order α.
Mathematical Reviews subject classification: Primary 54C50, 54C60; Secondary 54C08.
Received by the editors July 22, 2003
Communicated by: Clifford E. Weil
*This research was supported by DFG Grant RI 1087/2.

In the context of single-valued functions $f : X \to \mathbb{R}$ these concepts can be found in [4], p. 184, [5], pp. 188, 196, [1], pp. 140–141 (see also [11]). The application of the original definitions to set-valued maps follows the approach of [6], [7].

The sets $C(F; x_0)$ and $C^r(F; x_0)$ describe the local accumulation behavior of F in the entourage of x_0. Following the interpretation of [4], we consider $C(F; \cdot)$ and $C^r(F; \cdot)$ as multifunctions associated to F. We speak of the *cluster function* $C(F; \cdot)$ and the *reduced cluster function* $C^r(F; \cdot)$ of F.

The cluster function $C(F; \cdot)$ has the following simple meaning.

Proposition 1. *Let X and Y be topological spaces and let $F : X \to 2^Y$ be a multifunction. Then the graph of $C(F; \cdot)$ is the closure of the graph of F with respect to the product topology on $X \times Y$.*

PROOF. Let G be the multifunction whose graph is the closure of that of F. Then

$$\operatorname{graph}(G) = \big\{(x, y) \in X \times Y : \forall U \in \mathcal{U}(x)\, \big(y \in \operatorname{cl}(F(U))\big)\big\}.$$

Hence, for all $x \in X$,

$$G(x) = \big\{y \in Y : \forall U \in \mathcal{U}(x)\, \big(y \in \operatorname{cl}(F(U))\big)\big\} = \textstyle\bigcap_{U \in \mathcal{U}(x)} \operatorname{cl}(F(U)) = C(F; x).$$

$\square$

A basic consequence of Proposition 1 is

$$C(C(F; \cdot); \cdot) = C(F; \cdot)$$

for all $F : X \to 2^Y$. Accordingly, an iteration of the cluster function $C(F; \cdot)$ is not useful. However, we shall see that the behavior of the reduced cluster function $C^r(F; \cdot)$ is quite different, though its definition is very close to that of $C(F; \cdot)$.

Proposition 2. *Let X be a T_1-space, Y an arbitrary topological space, and $F : X \to 2^Y$ a multifunction. Then*

(a) *$C^r(F; \cdot) = C^r(C(F; \cdot); \cdot)$ and*

(b) *$C^r(F; \cdot) = C(C^r(F; \cdot); \cdot)$, that is, the graph of $C^r(F; \cdot)$ is closed.*

PROOF. (a) The inclusion $C^r(F; \cdot) \subseteq C^r(C(F; \cdot); \cdot)$ is obvious, because $F \subseteq C(F; \cdot)$. The converse can be obtained as follows.

$$\begin{aligned} C^r(C(F; \cdot); x_0) &= \textstyle\bigcap_{U \in \mathcal{U}(x_0)} \operatorname{cl}(C(F; U \setminus \{x_0\})) \\ &= \textstyle\bigcap_{U \in \mathcal{U}(x_0)} \operatorname{cl}\big(\bigcup_{x \in U \setminus \{x_0\}} \bigcap_{V \in \mathcal{U}(x)} \operatorname{cl}(F(V))\big) \\ &\subseteq \textstyle\bigcap_{U \in \mathcal{U}(x_0)} \operatorname{cl}\big(\bigcup_{x \in U \setminus \{x_0\}} \operatorname{cl}(F(U \setminus \{x_0\}))\big) \\ &= \textstyle\bigcap_{U \in \mathcal{U}(x_0)} \operatorname{cl}(F(U \setminus \{x_0\})) = C^r(F; x_0). \end{aligned}$$

The inclusion " $\subseteq$" is based on the T_1 property of X.

(b) Clearly, $C^r(F;\cdot) \subseteq C(C^r(F;\cdot);\cdot)$. On the other hand,

$$\begin{aligned} C(C^r(F;\cdot);x_0) &= \bigcap_{U\in\mathcal{U}(x_0)} \mathrm{cl}(C^r(F;U)) \\ &= \bigcap_{U\in\mathcal{U}(x_0)} \mathrm{cl}\left(\bigcup_{x\in U} \bigcap_{V\in\mathcal{U}(x)} \mathrm{cl}(F(V\setminus\{x\}))\right) \\ &\subseteq \bigcap_{U\in\mathcal{U}(x_0)} \mathrm{cl}\left(\bigcup_{x\in U} \mathrm{cl}(F(U\setminus\{x_0\}))\right) \\ &= \bigcap_{U\in\mathcal{U}(x_0)} \mathrm{cl}(F(U\setminus\{x_0\})) = C^r(F;x_0). \end{aligned}$$

This completes the proof. □

The following example justifies the restriction to T_1-spaces X in Proposition 2. Let $X=\mathbb{R}$ be equipped with the system of open sets $\{\emptyset,\mathbb{R}\}\cup\{(x,\infty) : x\in\mathbb{R}\}$. Then X is a T_0-space. We consider

$$f(x) = \begin{cases} 0, & x\neq 0, \\ 1, & x=0. \end{cases}$$

This yields

$$C^r(f;x) = \begin{cases} \{0,1\}, & x<0, \\ \{0\}, & x\geq 0, \end{cases} \qquad C(C^r(f;\cdot);x) = \begin{cases} \{0,1\}, & x\leq 0, \\ \{0\}, & x>0, \end{cases}$$

$$C(f;x) = \begin{cases} \{0,1\}, & x\leq 0, \\ \{0\}, & x>0, \end{cases} \qquad C^r(C(f;\cdot);x) = \begin{cases} \{0,1\}, & x\leq 0, \\ \{0\}, & x>0. \end{cases}$$

In particular, $C^r(f;0)\neq C^r(C(f;\cdot);0)$ and $C^r(f;0)\neq C(C^r(f;\cdot);0)$.

Part (a) of Proposition 2 shows that all reduced cluster functions can be obtained as reduced cluster functions of multifunctions with closed graph. Moreover, since $C^r(F;\cdot)\subseteq C(F;\cdot)$, we obtain

$$C^r(C^r(F;\cdot);\cdot) \subseteq C^r(C(F;\cdot);\cdot) = C^r(F;\cdot).$$

The function $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x) = \begin{cases} 0, & x\notin\left\{\frac{1}{2},\frac{1}{3},\frac{1}{4},\dots\right\}, \\ 1-x, & x\in\left\{\frac{1}{2},\frac{1}{3},\frac{1}{4},\dots\right\} \end{cases}$$

illustrates that the inclusion $C^r(C^r(F;\cdot);\cdot)\subseteq C^r(F;\cdot)$ is a strict one in general. Indeed,

$$C^r(f;x) = \begin{cases} \{0\}, & x\neq 0, \\ \{0,1\}, & x=0 \end{cases} \qquad \text{and} \qquad C^r(C^r(f;\cdot);x)\equiv\{0\}.$$

This justifies the definition of iterated reduced cluster functions.

Let $F : X \to 2^Y$ be a multifunction between topological spaces X and Y and let $\alpha > 0$ be an ordinal number. We define the *reduced cluster function* $C^{r,\alpha}(F;\cdot)$ *of* F *of order* α by

$$C^{r,\alpha}(F;x) = \begin{cases} C^r(F;x) & \text{if } \alpha = 1, \\ C^r\big(C^{r,\beta}(F;\cdot);x\big) & \text{if } \alpha = \beta+1,\ \beta > 0, \\ \bigcap_{\beta<\alpha} C^{r,\beta}(F;x) & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

Proposition 2 shows that $(C^{r,\alpha}(F;\cdot))_{\alpha>0}$ is a decreasing transfinite sequence of multifunctions with closed graphs, provided that X is a T_1-space.

The reduced cluster function $C^r(F;\cdot)$ and its iterations $C^{r,\alpha}(F;\cdot)$ of a multifunction F with closed graph can be considered as a particular derivative within the system of all closed subsets of $X \times Y$ and as its iterations in the sense of [8], p. 270: One considers closed subsets of $X \times Y$ as graphs of multifunctions.

2 A Continuity Property of Invariant Multifunctions.

For every $F : X \to 2^Y$, there exists a smallest ordinal $\alpha_0 = \alpha_0(F)$ such that $C^{r,\alpha_0}(F;\cdot) = C^{r,\alpha_0+1}(F;\cdot)$ or, equivalently, $C^{r,\alpha_0}(F;\cdot) = C^{r,\alpha}(F;\cdot)$ for all $\alpha \geq \alpha_0$. The multifunction $C^{r,\alpha_0}(F;\cdot)$ shows a pleasant behavior in so far as it remains invariant under formation of its reduced cluster function. The ordinal α_0 indicates in some sense the distance between F and its "invariant derivative" $C^{r,\alpha_0}(F;\cdot)$. In the present section we study properties of invariant multifunctions.

The following local reformulation of the invariance $C^r(F;\cdot) = F$ is a simple consequence of (1), since $C^r(F;x_0)$ is a closed set.

Proposition 3. *Let X and Y be topological spaces, $F : X \to 2^Y$ a set-valued function, and $x_0 \in X$. Then $C^r(F;x_0) = F(x_0)$ if and only if $C(F;x_0) = F(x_0)$ and $F(x_0) \subseteq C^r(F;x_0)$.*

The inclusion $F(x_0) \subseteq C^r(F;x_0)$ can be considered as a local continuity property of F at the point x_0. In fact, in [10] the inclusion $f(x_0) \in C^r(f;x_0)$ in the case of a single-valued function $f : \mathbb{R} \to \mathbb{R}$ defines so-called S_∞-continuity of f at x_0, the reduced cluster set $C^r(f;x_0)$ coinciding with Thomson's S_∞-cluster set (S_∞)-$\Lambda(f,x_0)$ (see pp. 3, 4, 70).

Proposition 4. *Let X and Y be topological spaces, $F : X \to 2^Y$ a set-valued function, and $x_0 \in X$. Then the following are equivalent.*

(i) $F(x_0) \subseteq C^r(F;x_0)$.

(ii) $C^r(F;x_0) = C(F;x_0)$.

(iii) There is no open subset $V_0 \subseteq Y$ such that x_0 is an isolated point of the set $F^-(V_0) = \{x \in X : F(x) \cap V_0 \neq \emptyset\}$.

PROOF. The equivalence (i) $\Leftrightarrow$(ii) follows from (1).

$\neg$(i) $\Rightarrow$ $\neg$(iii). If $F(x_0) \not\subseteq C^r(F; x_0)$, we can fix $y_0 \in F(x_0) \setminus C^r(F; x_0)$. Since $y_0 \notin C^r(F; x_0)$, there exists a neighborhood $U_0 \in \mathcal{U}(x_0)$ such that $y_0 \notin \mathrm{cl}(F(U_0 \setminus \{x_0\}))$. We put $V_0 = Y \setminus \mathrm{cl}(F(U_0 \setminus \{x_0\}))$. Then $(U_0 \setminus \{x_0\}) \cap F^-(V_0) = \emptyset$. On the other hand, $x_0 \in F^-(V_0)$, for $y_0 \in F(x_0) \cap V_0$. Hence $U_0 \cap F^-(V_0) = \{x_0\}$, which shows that x_0 is an isolated point of $F^-(V_0)$.

$\neg$(iii) $\Rightarrow$ $\neg$(i). Now we assume that x_0 is an isolated point of $F^-(V_0)$ for some open $V_0 \subseteq Y$. Then there exists a neighborhood $U_0 \in \mathcal{U}(x_0)$ such that $U_0 \cap F^-(V_0) = \{x_0\}$, that is, $F(U_0 \setminus \{x_0\}) \cap V_0 = \emptyset$ and $F(x_0) \cap V_0 \neq \emptyset$. Since V_0 is open, we obtain $\mathrm{cl}(F(U_0 \setminus \{x_0\}) \cap V_0 = \emptyset$ and therefore $C^r(F; x_0) \cap V_0 = \emptyset$. Thus

$$F(x_0) \setminus C^r(F; x_0) \supseteq (F(x_0) \cap V_0) \setminus (C^r(F; x_0) \cap V_0) \neq \emptyset,$$

which proves that $F(x_0) \not\subseteq C^r(F; x_0)$. □

Propositions 1, 3, and 4 yield the following characterization of invariant multifunctions.

Theorem 1. *A multifunction $F : X \to 2^Y$ between topological spaces X and Y is invariant in the sense of $C^r(F; \cdot) = F$ if and only if F has a closed graph and, for every open subset $V \subseteq Y$, the set $F^-(V) = \{x \in X : F(x) \cap V \neq \emptyset\}$ has no isolated points.*

Application to the particular open set $V = Y$ shows that, for an invariant multifunction $F = C^r(F; \cdot)$, the set $\{x \in X : F(x) \neq \emptyset\}$ does not contain isolated points, that is, $\{x \in X : F(x) \neq \emptyset\}$ is a perfect subset of X.

We close this section with the remarkable observation that every multifunction $F : X \to 2^Y$ between a metrizable space X and a separable metrizable space Y fails the continuity property $F(x) \subseteq C^r(F; x)$ only on a "small" set of points $x \in X$. A subset $A \subseteq X$ is called locally finite if every point $x_0 \in X$ possesses a neighborhood $U_0 \subseteq \mathcal{U}(x_0)$ whose intersection with A is finite. The following proposition generalizes a theorem from [2].

Proposition 5. *Let X be a metrizable space, Y a separable metrizable space, and $F : X \to 2^Y$ a multifunction. Then the set*

$$M = \{x \in X : F(x) \not\subseteq C^r(F; x)\}$$

is a countable union of locally finite subsets of X.

PROOF. We suppose that X and Y are equipped with corresponding metrics d_X and d_Y. For all $x \in M$, we choose a value $f(x) \in F(x) \setminus C^r(F;x)$. We define

$$M(k) = \big\{x \in M \colon \text{for every } \tilde{x} \in M \setminus \{x\},\ d_X(x,\tilde{x}) > \tfrac{1}{k} \text{ or } d_Y(f(x), f(\tilde{x})) > \tfrac{1}{k}\big\}$$

for integers $k \geq 1$. Then $M = \bigcup_{k \geq 1} M(k)$, for otherwise there would exist a point $x_0 \in M$ and a sequence $(x_k)_{k \geq 1} \subseteq M \setminus \{x_0\}$ such that $\lim_{k\to\infty} x_k = x_0$ and $\lim_{k\to\infty} f(x_k) = f(x_0)$. However, since $f(x_k) \in F(x_k)$, this would yield $f(x_0) \in C^r(F;x_0)$ contrary to the choice of $f(x_0)$.

Let $(y_l)_{l \geq 1}$ be a dense sequence in Y and

$$M(k,l) = \big\{x \in M(k) : f(x) \in B\big(y_l; \tfrac{1}{2k}\big)\big\},$$

$B\big(y_l; \frac{1}{2k}\big)$ denoting the closed ball of radius $\frac{1}{2k}$ centered at y_l. Accordingly, $M = \bigcup_{k \geq 1} M(k) = \bigcup_{k,l \geq 1} M(k,l)$. The sets $M(k,l)$ are locally finite. Indeed, any two distinct points $x_0, x_1 \in M(k,l)$ satisfy

$$d_Y(f(x_0), f(x_1)) \leq d_Y(f(x_0), y_l) + d_Y(y_l, f(x_1)) \leq \tfrac{1}{2k} + \tfrac{1}{2k} = \tfrac{1}{k}$$

and in turn $d_X(x_0, x_1) > \frac{1}{k}$, since $x_0, x_1 \in M(k)$. □

Proposition 5 requires several comments.

1. A countable union of locally finite subsets of a metric space (X, d_X) is a countable union of topologically discrete subsets of X and vice versa. Obviously, every locally finite set is discrete. Conversely, every discrete set $A \subseteq X$ is the countable union $A = \bigcup_{k \geq 1} A_k$ of the locally finite sets

$$A_k = \big\{x \in A : d_X(x, \tilde{x}) > \tfrac{1}{k} \text{ for all } \tilde{x} \in A \setminus \{x\}\big\}.$$

If X is separable, then every locally finite set is countable. Thus in this case a countable union of locally finite subsets is a countable subset of X.

2. Application of Proposition 5 to the multifunction $C^r(F;\cdot)$ shows that $\{x \in X : C^r(F;x) \neq C^r(C^r(F;\cdot);x)\} = \{x \in X : C^r(F;x) \not\subseteq C^r(C^r(F;\cdot);x)\}$ is a countable union of locally finite sets. By transfinite induction, all sets $\{x \in X : C^r(F;x) \neq C^{r,\alpha}(F;x)\}$, $\alpha > 0$ being a countable ordinal, have a representation of the same kind.

Consequently, for every countable $\alpha > 0$, $\{x \in X : F(x) \not\subseteq C^{r,\alpha}(F;x)\}$ is a countable union of locally finite sets, too.

Since $\{x \in X : C(F;x) \neq C^r(F;x)\} = \{x \in X : F(x) \not\subseteq C^r(F;x)\}$ by Proposition 4, the sets $\{x \in X : C(F;x) \neq C^{r,\alpha}(F;x)\}$ are countable unions of locally finite sets as well for all countable ordinals $\alpha > 0$.

3. In [2] Collingwood proves the claim of Proposition 5 for single-valued functions f mapping the plane unit disc D into the two-dimensional Euclidean

sphere S, then the exceptional set $\{x \in D : f(x) \notin C^r(f;x)\}$ describing as a countable set. He claims that this statement would be true for maps f into any complete metric space Y. This is not the case as the following example illustrates.

Let $X = Y = D$ be the plane unit disc equipped with the Euclidean metric d_X and with the discrete metric d_Y defined by $d_Y(x_1, x_2) = 1$ for $x_1 \neq x_2$, respectively. Then the identity $f : X \to Y$, $f(x) = x$, yields $C^r(f;x) = \emptyset$ for all $x \in X$ and thus $\{x \in X : f(x) \notin C^r(f;x)\} = X$ is not countable.

3 Characteristic Properties of Iterated Reduced Cluster Functions.

In this section we ask for multifunctions $G : X \to 2^Y$ that can appear as reduced cluster functions $G = C^{r,\alpha}(F;\cdot)$ of fixed order $\alpha > 0$. We shall give a corresponding characterization if X is a metrizable space, Y a separable metrizable space, and α a countable ordinal.

The sequence $(C^{r,\alpha}(F;\cdot))_{\alpha>0}$ associated to a set-valued function F between second countable spaces X and Y stabilizes already for some countable ordinal α_0. Indeed, $(C^{r,\alpha}(F;\cdot))_{\alpha>0}$ can be seen as a decreasing transfinite sequence of closed graphs in the second countable space $X \times Y$ which, by Theorem 6.9 of [8], satisfies $C^{r,\alpha_0}(F;\cdot) = C^{r,\alpha_0+1}(F;\cdot)$ for some countable α_0. Consequently, in the case of second countable spaces X and Y the study of countable ordinals α is not a restriction, but in fact covers arbitrary ordinals α.

The theorems to be presented make use of the concept of the iterated Cantor-Bendixson derivatives X^α of a topological space X (see Definition 6.10 of [8]). The first Cantor-Bendixson derivative X' is defined by

$$X' = \{x \in X : x \text{ is not isolated in } X\}.$$

The iterated Cantor-Bendixson derivatives for arbitrary ordinals α are

$$X^\alpha = \begin{cases} X & \text{if } \alpha = 0, \\ \left(X^\beta\right)' & \text{if } \alpha = \beta + 1,\ \beta \geq 0, \\ \bigcap_{\beta<\alpha} X^\beta & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

The derivatives X^α can easily be described by the aid of iterated reduced cluster functions. Given two topological spaces X and Y, we consider the multifunction

$$\mathbf{I}_A(x) = \begin{cases} Y, & x \in A, \\ \emptyset, & x \notin A \end{cases}$$

as an indicator function of a subset $A \subseteq X$. Obviously, $C^r(\mathbf{I}_X;\cdot) = \mathbf{I}_{X'}$. Transfinite induction then yields

$$C^{r,\alpha}(\mathbf{I}_X;\cdot) = \mathbf{I}_{X^\alpha}$$

for all ordinals $\alpha > 0$.

Now we characterize the variety of iterated reduced cluster functions of a fixed countable order α attainable from arbitrary multifunctions (Theorem 2) and from single-valued functions (Theorem 3). These are the central results of the present paper. The proofs will be given in Sections 4 and 5.

Theorem 2. *Let X be a metrizable space, Y a separable metrizable space, $\alpha > 0$ a countable ordinal, and $G : X \to 2^Y$ a multifunction. Then the following are equivalent.*
(i) There exists $F : X \to \{\{y\} : y \in Y\} \cup \{\emptyset\}$ such that $C^{r,\alpha}(F;\cdot) = G$.
(ii) There exists $F : X \to 2^Y$ such that $C^{r,\alpha}(F;\cdot) = G$.
(iii) $C(G;\cdot) = G$ and $G(x) = \emptyset$ for all $x \in X \setminus X^\alpha$.

It is remarkable that multifunctions F whose values are restricted to singletons and the empty set give rise to the same iterated reduced cluster functions as arbitrary multifunctions do.

Theorem 3. *Let X be a metrizable space, Y a separable metrizable space, $\alpha > 0$ a countable ordinal, and $G : X \to 2^Y$ a multifunction. Then the following are equivalent.*
(i) There exists $f : X \to Y$ such that $C^{r,\alpha}(f;\cdot) = G$.
(ii) There exists $F : X \to 2^Y \setminus \{\emptyset\}$ such that $C^{r,\alpha}(F;\cdot) = G$.
If Y is compact then (i) and (ii) are equivalent to
(iii) $C(G;\cdot) = G$ and $\{x \in X : G(x) = \emptyset\} = X \setminus X^\alpha$.
If Y is not compact then (i) and (ii) are equivalent to
(iii)′ $C(G;\cdot) = G$, $G(x) = \emptyset$ for all $x \in X \setminus X^\alpha$, and $\{x \in X : G(x) = \emptyset\}$ is a countable union of locally finite subsets of X.

Let us point out once more that multifunctions with arbitrary non-empty values do not give rise to a larger class of iterated reduced cluster functions than single-valued functions do.

In contrast with that, the class of multifunctions G that can be obtained as cluster functions $G = C(F;\cdot)$ of multifunctions $F : X \to 2^Y \setminus \{\emptyset\}$ in general is strictly larger than the family of cluster functions of single-valued functions $f : X \to Y$. Proposition 1 shows that every multifunction $G : X \to 2^Y$ with closed graph appears as the cluster function of some $F : X \to 2^Y$. Indeed, we can put $F = G$. Set-valued functions G that can be obtained as the cluster function $G = C(f;\cdot)$ of a single-valued function $f : X \to Y$, however, are more restrictive. Clearly, f must be a selection of $G = C(f;\cdot)$, so that necessarily $G(x) \neq \emptyset$ for all $x \in X$. We cite the following characterization from the forthcoming paper [9]. Therein $\operatorname{card}(\cdot)$ denotes the cardinality of a set.

Theorem 4. *Let X be a completely metrizable space, Y a Polish space, and G a multifunction mapping X into the non-empty subsets of Y. Then G is the cluster function of a function $f : X \to Y$ if and only if*

$$C^r(G;x) \subseteq G(x) \text{ and } \operatorname{card}(G(x) \setminus C^r(G;x)) \leq 1$$

for all $x \in X$.

For example, the multifunction $G : \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}$,

$$G(x) = \begin{cases} \{0\}, & x \neq 0, \\ \{-1, 0, 1\}, & x = 0, \end{cases}$$

clearly has a closed graph, but G is not the cluster function $G = C(f;\cdot)$ of a function $f : \mathbb{R} \to \mathbb{R}$, because $\operatorname{card}(G(0) \setminus C^r(G;0)) = 2$.

4 Proof of Theorem 2.

The implication (i) $\Rightarrow$(ii) is trivial.

If (ii) is satisfied, then, by Proposition 2 (b) and by the inductive definition of $C^{r,\alpha}(\cdot;\cdot)$, $G = C^{r,\alpha}(F;\cdot)$ has a closed graph. Proposition 1 yields $C(G;\cdot) = G$. The claim $G(x) = C^{r,\alpha}(F;x) = \emptyset$ for all $x \in X \setminus X^\alpha$ follows inductively from the simple fact that $C^r(F;x) = \emptyset$ for $x \in X \setminus X'$. This way we obtain (ii) $\Rightarrow$(iii).

The implication (iii) $\Rightarrow$(i), however, requires more efforts.

Lemma 1. *Let (X, d_X) be a metric space and let $\varepsilon > 0$. Then there exists a disjoint family $\{H(i,j) : i,j \in \{1,2,3,\ldots\}\}$ of locally finite sets $H(i,j) \subseteq X$ such that, for every $x_0 \in X'$ and all $i,j \geq 1$, there is $x_1 \in H(i,j)$ such that $d_X(x_0,x_1) \leq \frac{\varepsilon}{i}$.*

PROOF. For $(i_1,j_1),(i_2,j_2) \in \{(i,j) : i,j \geq 1\}$, we say that $(i_1,j_1) \prec (i_2,j_2)$ if either $i_1 + j_1 < i_2 + j_2$ or $i_1 + j_1 = i_2 + j_2$ and $i_1 < i_2$. The sets $H(i,j)$ are going to be defined by induction with respect to the order $\prec$. Let $i_0, j_0 \geq 1$ be fixed and assume that $H(i,j)$ is already defined for all $(i,j) \prec (i_0,j_0)$. Since X is paracompact (see [3], p. 300), there exists a locally finite cover $\mathcal{C}$ of X by open subsets C of diameter $\operatorname{diam}(C) \leq \frac{\varepsilon}{i_0}$. Let

$$\mathcal{C}' = \{C \in \mathcal{C} : C \cap X' \neq \emptyset\}.$$

We can fix a point $x_C \in C \setminus \bigcup_{(i,j)\prec(i_0,j_0)} H(i,j)$ for every $C \in \mathcal{C}'$, because C contains a non-isolated point and the set $\bigcup_{(i,j)\prec(i_0,j_0)} H(i,j)$ is locally finite. Then we put $H(i_0,j_0) = \{x_C : C \in \mathcal{C}'\}$.

By definition, $H(i_0,j_0)$ is locally finite and disjoint with all sets $H(i,j)$, $(i,j) \prec (i_0,j_0)$. If $x_0 \in X'$, then there is $C \in \mathcal{C}'$ such that $x_0 \in C$. The corresponding point $x_1 = x_C \in H(i_0,j_0)$ satisfies the required estimate $d_X(x_0,x_1) \leq \operatorname{diam}(C) \leq \frac{\varepsilon}{i_0}$. □

Lemma 2. *Let (X,d_X) be a metric space, (Y,d_Y) a separable metric space, $G : X \to 2^Y$ a multifunction such that $C(G;\cdot) = G$ and $G(x) = \emptyset$ for all $x \in X \setminus X'$, and let $\varepsilon > 0$. Then there exists $F : X \to \{\{y\} : y \in Y\} \cup \{\emptyset\}$ such that $F(x) \subseteq G(B(x;\varepsilon))$ for all $x \in X$ and $C^r(F;\cdot) = G$.*

PROOF. Let $i \geq 1$ be an integer. Since Y is separable and paracompact (see [3], p. 300), there exists a locally finite countable open cover $\mathcal{C}_i = \{C_{i,j} : j \geq 1\}$ of Y such that $\operatorname{diam}(C_{i,j}) \leq \frac{\varepsilon}{i}$ for all $C_{i,j} \in \mathcal{C}_i$. We reduce the sets $H(i,j) \subseteq X$ from Lemma 1 by putting

$$H'(i,j) = \left\{x \in H(i,j) : G\left(B\left(x;\tfrac{\varepsilon}{i}\right)\right) \cap C_{i,j} \neq \emptyset\right\}.$$

For every $x \in H'(i,j)$, we choose a value

$$f(x) \in G\left(B\left(x;\tfrac{\varepsilon}{i}\right)\right) \cap C_{i,j}. \tag{2}$$

Then we define

$$F(x) = \begin{cases} \{f(x)\} & \text{if } x \in \bigcup_{i,j\geq 1} H'(i,j), \\ \emptyset & \text{if } x \in X \setminus \bigcup_{i,j\geq 1} H'(i,j). \end{cases}$$

The definition immediately implies $F(x) \subseteq G(B(x;\varepsilon))$ for all $x \in X$.

For the proof of $C^r(F;\cdot) = G$ let $x_0 \in X$ be fixed. First we suppose $y_0 \in G(x_0)$ for showing that $G(x_0) \subseteq C^r(F;x_0)$. For every $i \geq 1$, there exists $j_i \geq 1$ such that $y_0 \in C_{i,j_i}$, since $\mathcal{C}_i$ covers Y. We obtain $x_0 \in X'$, because $G(x_0) \supseteq \{y_0\} \neq \emptyset$. Therefore, by Lemma 1, we find $x_i \in H(i,j_i)$ such that $d_X(x_0,x_i) \leq \frac{\varepsilon}{i}$. In fact, $x_i \in H'(i,j_i)$, because

$$G\left(B\left(x_i;\tfrac{\varepsilon}{i}\right)\right) \cap C_{i,j_i} \supseteq G(x_0) \cap C_{i,j_i} \supseteq \{y_0\} \neq \emptyset.$$

Moreover, $d_Y(y_0,f(x_i)) \leq \frac{\varepsilon}{i}$, since $y_0 \in C_{i,j_i}$, $f(x_i) \in G\left(B\left(x_i;\frac{\varepsilon}{i}\right)\right) \cap C_{i,j_i} \subseteq C_{i,j_i}$, and $\operatorname{diam}(C_{i,j_i}) \leq \frac{\varepsilon}{i}$.

The sequence $(x_i)_{i\geq 1}$ satisfies $\lim_{i\to\infty} x_i = x_0$ and $\lim_{i\to\infty} f(x_i) = y_0$. The points x_i, $i \geq 1$, are mutually distinct, for the sets $H(i,j_i)$ are pairwise disjoint by Lemma 1. This yields $y_0 \in C^r(F;x_0)$ and in turn proves the inclusion $G(x_0) \subseteq C^r(F;x_0)$.

For the proof of the converse we consider $y_0 \in C^r(F;x_0)$. There exists a sequence of distinct points $(x_k)_{k\geq 1} \subseteq \bigcup_{i,j\geq 1} H'(i,j)$ such that

$$\lim_{k\to\infty} x_k = x_0 \text{ and } \lim_{k\to\infty} f(x_k) = y_0. \tag{3}$$

We determine integers $i(k), j(k) \geq 1$ by $x_k \in H'(i(k), j(k))$. Next we infer $\lim_{k\to\infty} i(k) = \infty$.

Let us suppose the contrary, that is, there exists a subsequence $(x_{k_l})_{l\geq 1}$ of $(x_k)_{k\geq 1}$ such that

$$\sup\{i(k_l) : l \geq 1\} = i_0 < \infty.$$

The cover $\mathcal{C}_0 = \mathcal{C}_1 \cup \ldots \cup \mathcal{C}_{i_0} = \{C_{i,j} : 1 \leq i \leq i_0, j \geq 1\}$ is locally finite being a finite union of locally finite covers. Thus there exists a neighborhood $V \subseteq Y$ of y_0 that intersects only finitely many sets from $\mathcal{C}_0$, that is, there is $j_0 \geq 0$ such that

$$V \cap C_{i,j} = \emptyset \text{ for all } 1 \leq i \leq i_0, j > j_0. \tag{4}$$

Since $\lim_{l\to\infty} f(x_{k_l}) = y_0$, we obtain $(f(x_{k_l}))_{l\geq l_0} \subseteq V$ for suitable $l_0 \geq 1$. Equation (2) then shows that $f(x_{k_l}) \in V \cap C_{i(k_l),j(k_l)}$ for $l \geq l_0$. Now (4) yields $j(k_l) \leq j_0$ for $l \geq l_0$. This gives

$$(x_{k_l})_{l\geq l_0} \subseteq \bigcup_{1\leq i\leq i_0, 1\leq j\leq j_0} H(i,j).$$

However, $(x_{k_l})_{l\geq l_0}$ contains infinitely many points in every neighborhood of the limit x_0, whereas $\bigcup_{1\leq i\leq i_0, 1\leq j\leq j_0} H(i,j)$ is a locally finite set. This contradiction shows that $\lim_{k\to\infty} i(k) = \infty$.

Using (3), the property $f(x_k) \in G\big(B\big(x_k; \frac{\varepsilon}{i(k)}\big)\big)$ coming from (2), and $\lim_{k\to\infty} i(k) = \infty$, we conclude $y_0 \in C(G;x_0) = G(x_0)$. This completes the proof of the claim $C^r(F;x_0) \subseteq G(x_0)$. □

Lemma 3. *Let (X, d_X) be a metric space, (Y, d_Y) a separable metric space, $\alpha > 0$ a countable ordinal, $G : X \to 2^Y$ a multifunction such that $C(G;\cdot) = G$ and $G(x) = \emptyset$ for all $x \in X \setminus X^\alpha$, and let $\varepsilon > 0$. Then there exists $F : X \to \{\{y\} : y \in Y\} \cup \{\emptyset\}$ such that $F(x) \subseteq \mathrm{cl}(G(B(x;\varepsilon)))$ for all $x \in X$ and $C^{r,\alpha}(F;\cdot) = G$.*

PROOF. We proceed by induction on α. Lemma 2 gives the claim for $\alpha = 1$. We suppose that the claim is true for all ordinals β with $0 < \beta < \alpha$.

First we assume that $\alpha = \beta + 1$ for some $\beta > 0$. We consider the closed subspace X^β of X. The restriction $G|_{X^\beta} : X^\beta \to 2^Y$ obviously satisfies $C_{X^\beta}(G|_{X^\beta};\cdot) = G|_{X^\beta}$, $C_{X^\beta}(\cdot;\cdot)$ denoting the cluster function with respect to the space X^β, and $G|_{X^\beta}(x) = \emptyset$ for all $x \in X^\beta \setminus \big(X^\beta\big)'$. Thus Lemma 2 provides a multifunction $F_0|_{X^\beta} : X^\beta \to 2^Y$ such that

$$F_0|_{X^\beta}(x) \subseteq G\big(B\big(x; \tfrac{\varepsilon}{3}\big)\big) \text{ for all } x \in X^\beta \tag{5}$$

and

$$C^r_{X^\beta}(F_0|_{X^\beta};\cdot) = G|_{X^\beta}. \tag{6}$$

We extend $F_0|_{X^\beta}$ to $F_0 : X \to 2^Y$ by putting $F_0(x) = \emptyset$ for all $x \in X \setminus X^\beta$. Then we define $F_1 : X \to 2^Y$ by $F_1 = C(F_0;\cdot)$. We obtain

$$C(F_1;\cdot) = F_1 \tag{7}$$

by Proposition 1,

$$F_1(x) = \emptyset \text{ for all } x \in X \setminus X^\beta, \tag{8}$$

because $F_0(x) = \emptyset$ for all $x \in X \setminus X^\beta$ and X^β is closed,

$$F_1(x) \subseteq \mathrm{cl}\,\big(G\big(B\big(x; \tfrac{2\varepsilon}{3}\big)\big)\big) \text{ for all } x \in X \tag{9}$$

by (5), and

$$C^r(F_1;\cdot) = G \tag{10}$$

by (6) and (8) together with $G(x) = \emptyset$ for all $x \in X \setminus X^\beta$.

Application of the induction hypothesis to (7) and (8) finally yields a multifunction $F : X \to \{\{y\} : y \in Y\} \cup \{\emptyset\}$ such that $F(x) \subseteq \mathrm{cl}\,\big(F_1\big(B\big(x; \tfrac{\varepsilon}{3}\big)\big)\big)$ for all $x \in X$ and $C^{r,\beta}(F;\cdot) = F_1$. Properties (9) and (10) immediately imply the claims $F(x) \subseteq \mathrm{cl}(G(B(x;\varepsilon)))$ for all $x \in X$ and $C^{r,\alpha}(F;\cdot) = G$. This completes the case $\alpha = \beta + 1$.

Now let α be a limit ordinal, that is, $\alpha = \lim_{\beta<\alpha} \beta$. We fix an enumeration $\alpha = \{\beta_1, \beta_2, \beta_3, \ldots\}$ of α. Since $G(x) = \emptyset$ for all $x \in X \setminus X^{\beta_i}$, $i \geq 1$, the induction hypothesis gives us multifunctions $F_i : X \to 2^Y$, $i \geq 1$, such that $F_i(x) \subseteq \mathrm{cl}\,\big(G\big(B\big(x; \tfrac{\varepsilon}{3i}\big)\big)\big)$ for all $x \in X$ and $C^{r,\beta_i}(F_i;\cdot) = G$. We define $F_0 : X \to 2^Y$ by $F_0(x) = \bigcup_{i\geq 1} F_i(x)$. Obviously,

$$F_0(x) \subseteq \mathrm{cl}\,\big(G\big(B\big(x; \tfrac{\varepsilon}{3}\big)\big)\big) \text{ for all } x \in X. \tag{11}$$

Next we show that

$$C^{r,\alpha}(F_0;\cdot) = G. \tag{12}$$

Indeed,

$$C^{r,\alpha}(F_0;\cdot) = \bigcap_{i\geq 1} C^{r,\beta_i}(F_0;\cdot) \supseteq \bigcap_{i\geq 1} C^{r,\beta_i}(F_i;\cdot) = G.$$

For the verification of the converse inclusion let $j \geq 1$ be fixed and let i_j be determined by $\beta_{i_j} = \max\{\beta_1, \ldots, \beta_j\}$. We obtain

$$\begin{aligned} C^{r,\alpha}(F_0;x_0) &\subseteq C^{r,\beta_{i_j}}(F_0;x_0) \\ &= C^{r,\beta_{i_j}}\big(\textstyle\bigcup_{1\leq i\leq j} F_i(\cdot) \cup \bigcup_{i>j} F_i(\cdot); x_0\big) \\ &= \textstyle\bigcup_{1\leq i\leq j} C^{r,\beta_{i_j}}(F_i;x_0) \cup C^{r,\beta_{i_j}}\big(\bigcup_{i>j} F_i(\cdot); x_0\big). \end{aligned}$$

For $1 \le i \le j$, we have $C^{r,\beta_{i_j}}(F_i;x_0) \subseteq C^{r,\beta_i}(F_i;x_0) = G(x_0)$, since $\beta_i \le \beta_{i_j}$. Moreover, $C^{r,\beta_{i_j}}\big(\bigcup_{i>j} F_i(\cdot);x_0\big) \subseteq C\big(\bigcup_{i>j} F_i(\cdot);x_0\big) \subseteq \operatorname{cl}\big(G\big(B\big(x_0;\frac{\varepsilon}{3j}\big)\big)\big)$, for $F_i(x) \subseteq \operatorname{cl}\big(G\big(B\big(x;\frac{\varepsilon}{3i}\big)\big)\big)$. This yields

$$C^{r,\alpha}(F_0;x_0) \subseteq G(x_0) \cup \operatorname{cl}\big(G\big(B\big(x_0;\tfrac{\varepsilon}{3j}\big)\big)\big) = \operatorname{cl}\big(G\big(B\big(x_0;\tfrac{\varepsilon}{3j}\big)\big)\big)$$

for all $j \ge 1$. Consequently,

$$C^{r,\alpha}(F_0;x_0) \subseteq \textstyle\bigcap_{j\ge 1} \operatorname{cl}\big(G\big(B\big(x_0;\tfrac{\varepsilon}{3j}\big)\big)\big) = C(G;x_0) = G(x_0),$$

which completes the verification of (12).

The multifunction F_0 defined so far satisfies the required claims of Lemma 3 up to the restriction of its values to $\{\{y\} : y \in Y\} \cup \{\emptyset\}$. Therefore we put $G_0 = C^r(F_0;\cdot)$. Then, by (11),

$$G_0(x) \subseteq \operatorname{cl}\big(G\big(B\big(x;\tfrac{2\varepsilon}{3}\big)\big)\big) \text{ for all } x \in X. \tag{13}$$

Proposition 2 (b) yields

$$C(G_0;\cdot) = G_0 \text{ and } G_0(x) = \emptyset \text{ for all } x \in X \setminus X'.$$

Thus, by Lemma 2, there exists $F : X \to \{\{y\} : y \in Y\} \cup \{\emptyset\}$ such that

$$F(x) \subseteq G_0\big(B\big(x;\tfrac{\varepsilon}{3}\big)\big) \text{ for all } x \in X \text{ and } C^r(F;\cdot) = G_0.$$

The first property and (13) yield the claim $F(x) \subseteq \operatorname{cl}(G(B(x;\varepsilon)))$ for all $x \in X$. The second equation shows $C^r(F;\cdot) = C^r(F_0;\cdot)$ and, by (12), in turn $C^{r,\alpha}(F;\cdot) = G$. This completes the proof. $\square$

Lemma 3 clearly proves the implication (iii) $\Rightarrow$(i) of Theorem 2.

5 Proof of Theorem 3.

The implication (i) $\Rightarrow$(ii) is trivial.

Let us suppose that F is a set-valued function as in claim (ii). Theorem 2 shows that $C(G;\cdot) = G$ and $G(x) = \emptyset$ for all $x \in X \setminus X^\alpha$.

If Y is compact, then $C^r(F;x) \ne \emptyset$ for every $x \in X'$, since

$$C^r(F;x) = \textstyle\bigcap_{U \in \mathcal{U}(x)} \operatorname{cl}(F(U \setminus \{x\}))$$

is an intersection of compact sets such that every finite intersection of them is non-empty. By induction, we obtain $G(x) = C^{r,\alpha}(F;x) \ne \emptyset$ for all $x \in X^\alpha$. This yields (ii) $\Rightarrow$(iii).

Now assume Y not to be compact. For proving (ii) $\Rightarrow$(iii)$'$ it remains to show that $\{x \in X : C^{r,\alpha}(F;x) = \emptyset\}$ is a countable union of locally finite sets for every $F : X \to 2^Y \setminus \{\emptyset\}$. This follows from the second remark after Proposition 5 which says that even the larger set $\{x \in X : F(x) \not\subseteq C^{r,\alpha}(F;x)\}$ can be expressed as a countable union of locally finite sets.

The following lemma is the main tool for the proof of the remaining implications (iii) $\Rightarrow$(i) and (iii)$'$$\Rightarrow$(i), respectively.

Lemma 4. *Let (X, d_X) be a metric space, (Y, d_Y) a separable metric space, $\alpha > 0$ a countable ordinal, and $G : X \to 2^Y \setminus \{\emptyset\}$ a multifunction that satisfies condition (iii) of Theorem 3 if Y is compact or condition (iii)$'$ of Theorem 3 if Y is not compact, respectively. Then there exists a function $g : X \to Y$ such that $C^{r,\alpha}(g;x) \subseteq G(x)$ for all $x \in X$.*

PROOF. It suffices to find $g : X \to Y$ such that

$$C^r(g;x) \subseteq G(x) \text{ for all } x \in X^\alpha, \tag{14}$$

since then $C^{r,\alpha}(g;x) \subseteq C^r(g;x) \subseteq G(x)$ for $x \in X^\alpha$ and $C^{r,\alpha}(g;x) = \emptyset = G(x)$ for $x \in X \setminus X^\alpha$.

First we assume that Y is compact. Moreover, we assume that $X^\alpha \neq \emptyset$, for otherwise claim (14) is trivially satisfied by any function $g : X \to Y$. Since X^α is closed and, by (iii), $G(x) \neq \emptyset$ for all $x \in X^\alpha$, we can choose a function g such that

$$g(x) \in \begin{cases} G(x) & \text{if } x \in X^\alpha, \\ G(B(x; 2d_X(x; X^\alpha))) & \text{if } x \notin X^\alpha, \end{cases}$$

where $d_X(x; X^\alpha) = \inf\{d_X(x, \tilde{x}) : \tilde{x} \in X^\alpha\}$.

For the proof of (14) let $x_0 \in X^\alpha$ and $y_0 \in C^r(g;x_0)$. Then there exists a sequence $(x_i)_{i \geq 1} \subseteq X$ such that $\lim_{i\to\infty} x_i = x_0$ and $\lim_{i\to\infty} g(x_i) = y_0$. The choice of g yields

$$g(x_i) \in G(B(x_i; 2d_X(x_i, x_0))) \subseteq G(B(x_0; 3d_X(x_i, x_0)))$$

and hence

$$y_0 \in \bigcap_{j\geq 1} \mathrm{cl}((g(x_i))_{i\geq j}) \subseteq \bigcap_{\varepsilon>0} \mathrm{cl}(G(B(x_0;\varepsilon))) = C(G;x_0) = G(x_0),$$

which proves (14).

Now we assume that Y is not compact. Then there exists a sequence $(y_k)_{k\geq 1}$ of distinct points of Y such that every subset of $(y_k)_{k\geq 1}$ is closed. According to condition (iii)$'$ there exists a representation

$$\{x \in X : G(x) = \emptyset\} = \bigcup_{k\geq 1} M(k)$$

with locally finite disjoint sets $M(k) \subseteq X$. We choose g such that

$$g(x) \in \begin{cases} G(x) & \text{if } x \in X \setminus \bigcup_{k\geq 1} M(k), \\ \{y_k\} & \text{if } x \in M(k). \end{cases}$$

Again we consider $y_0 \in C^r(g; x_0)$ for proving (14). We obtain a sequence $(x_i)_{i\geq 1} \subseteq X \setminus \{x_0\}$ of distinct points such that $\lim_{i\to\infty} x_i = x_0$ and $\lim_{i\to\infty} g(x_i) = y_0$. Without loss of generality, either $(x_i)_{i\geq 1} \subseteq X \setminus \bigcup_{k\geq 1} M(k)$ or $(x_i)_{i\geq 1} \subseteq \bigcup_{k\geq 1} M(k)$. In the first case

$$y_0 \in C(G; x_0) = G(x_0),$$

because $g(x_i) \in G(x_i)$. The latter case, however, does not appear. Indeed, if $(x_i)_{i\geq 1} \subseteq \bigcup_{k\geq 1} M(k)$, then $(g(x_i))_{i\geq 1}$ would be a convergent subsequence of $(y_k)_{k\geq 1}$. The choice of $(y_k)_{k\geq 1}$ then would imply $y_0 = \lim_{i\to\infty} g(x_i) = y_{k_0}$ for some $k_0 \geq 1$ and $g(x_i) = y_{k_0}$ for all $i \geq i_0$. Thus $(x_i)_{i\geq i_0} \subseteq M(k_0)$. This is impossible, since $(x_i)_{i\geq i_0}$ contains infinitely many points of every neighborhood of x_0, whereas $M(k_0)$ is locally finite. This completes the proof of (14). $\square$

Now the implications (iii) $\Rightarrow$(i) and (iii)$'\Rightarrow$(i) are easy to infer. According to part (iii) $\Rightarrow$(i) of Theorem 2 there exists $F : X \to \{\{y\} : y \in Y\} \cup \{\emptyset\}$ such that $C^{r,\alpha}(F; \cdot) = G$. Lemma 4 provides a function $g : X \to Y$ with $C^{r,\alpha}(g; \cdot) \subseteq G$. We define $f : X \to Y$ by

$$\{f(x)\} = \begin{cases} F(x) & \text{if } F(x) \neq \emptyset, \\ \{g(x)\} & \text{if } F(x) = \emptyset. \end{cases}$$

Then $F(x) \subseteq \{f(x)\} \subseteq F(x) \cup \{g(x)\}$ and

$$G(x) = C^{r,\alpha}(F; x) \subseteq C^{r,\alpha}(f; x) \subseteq C^{r,\alpha}(F; x) \cup C^{r,\alpha}(g; x) = G(x)$$

for all $x \in X$. This yields $C^{r,\alpha}(f; \cdot) = G$ and completes the proof of (iii) $\Rightarrow$(i) and (iii)$'\Rightarrow$(i), respectively.

References

[1] G. Aumann, *Reelle Funktionen*, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen 68, Springer-Verlag, Berlin, Göttingen, Heidelberg, 1954.

[2] E. F. Collingwood, *Cluster set theorems for arbitrary functions with applications to function theory*, Ann. Acad. Sci. Fenn. Ser. A I, **336/8** (1963), 15 pp.

[3] R. Engelking, *General topology*, Sigma Series in Pure Mathematics 6, Heldermann Verlag, Berlin, 1989.

[4] H. Hahn, *Theorie der reellen Funktionen*, Band I, Verlag Julius Springer, Berlin, 1921.

[5] H. Hahn, *Reelle Funktionen, Erster Teil: Punktfunktionen*, Akademische Verlagsgesellschaft m.b.H., Leipzig, 1932, reprinted by Chelsea Publishing Company, New York, 1948.

[6] J. E. Joseph, *Regularity, normality and multifunctions*, Proc. Amer. Math. Soc., **70** (1978), 203–206.

[7] J. E. Joseph, *Multifunctions and cluster sets*, Proc. Amer. Math. Soc., **74** (1979), 329–337.

[8] A. S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics 156, Springer-Verlag, Berlin, Heidelberg, New York, 1994.

[9] C. Richter, *The cluster function of single-valued functions*, submitted.

[10] B. S. Thomson, *Real functions*, Lecture Notes in Mathematics 1170, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1985.

[11] J. D. Weston, *Some theorems on cluster sets*, J. London Math. Soc., **33** (1958), 435–441.