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Tord Sjödin, Department of Mathematics, University of Umeå, 90187 Umeå, Sweden. email: `tord.sjodin@math.umu.se`

A NOTE ON THE HARMONIC DERIVATIVE

Abstract

We characterize ordinary differentiability in terms on the harmonic derivative and a local Lipschitz type condition and apply the result to $C^{k,1}-$functions.

1 Introduction.

We study the connection between ordinary differentiability and some weaker types of pointwise differentiability for functions f on $\mathbb{R}^n$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is ordinary differentiable of order m at a point $x = a$ if there is a polynomial $P(x) = \sum\limits_{|\alpha| \le m} c_\alpha \cdot x^\alpha$ such that

$$R_m(x, a) = f(x) - P(x - a) = o(|x - a|^m),$$

as $x \to a$. It was proved by the author in [4] that ordinary differentiability is equivalent to L^p-differentiability (or approximative differentiability) together with a certain local Lipschitz type condition. It is the purpose of this note to prove the analogous result for the harmonic derivative as defined in Stein [5], Ch VIII. Put $u(x, y) = P_y \star f(x)$, where $P_y(x)$ is the Poisson kernel. Then f is said to have a harmonic derivative $D_h^\alpha f$ at $x = a$ if $D^\alpha u$ has a non–tangential limit at $(a, 0)$. We prove that f is ordinary differentiable of order m at $x = a$ if and only if f has harmonic derivatives at $x = a$ of all orders up to m and satisfies the condition B_m, see definition below (Theorems 3.1 and 3.2).

We apply these results to $C^{k,1}-$functions in the following way. Assume that $f \in C^{m-1,1}(\Omega)$, $m \ge 1$ (see the definition in Section 2). Then f has

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an ordinary differential of order m a.e. in Ω by the Rademacher–Stepanov theorem. Theorem 3.3 gives a characterization of the points in Ω where such a differential exists in terms of harmonic derivatives. A somewhat different approach to these problems is found in [3]. The paper [4] also contains applications to Bessel potentials. Compare also Stocke [6].

Section 2 contains our notation and definitions and our results are stated in Section 3. Section 4 prepares for the proofs in Section 5. In Section 6 we describe the relations between the various derivatives considered in this paper. In particular, we show that the harmonic derivative and the approximative derivative are not related.

2 Notation and Definitions.

We let $\mathbb{R}^n$ denote the Euclidean space with points $x = (x_1, x_2, \ldots, x_n)$ and let $\mathbb{R}^{n+1}_+ = \{(x, y); x \in \mathbb{R}^n,\ y > 0\}$ denote the upper half space. Measure and integration is with respect to Lebesgue measure and is denoted by $|E|$ and $\int_E f(x)\,dx$ respectively. The Lebesgue spaces $L^p(\mathbb{R}^n)$ with norm $||f||_p$, $1 \le p < \infty$, are defined in the usual way. Let $f \in L^1(\mathbb{R}^n)$. Then we define $u(x, y) = P_y \star f(x)$, $(x, y) \in \mathbb{R}^{n+1}_+$, where

$$P_y(x) = c_n \cdot y \cdot (|x|^2 + y^2)^{-(n+1)/2}, (x, y) \in \mathbb{R}^{n+1}_+,$$

is the Poisson kernel for an appropriate constant c_n [5], Ch. III. Then $u(x, y)$ is harmonic in the upper half space $\mathbb{R}^{n+1}_+$ and has non–tangential trace f almost everywhere on the boundary. A function $g(x, y)$ defined on $\mathbb{R}^{n+1}_+$ has non–tangential limit A at $(a, 0)$ if $g(x, y) \to A$ as $(x, y) \to (a, 0)$ and $(x, y) \in V_s$, for every cone $V_s = \{(x, y); |x - a| < s \cdot y\}$, where $s > 0$. Differentiation of u with respect to the x-variable is written $D^\alpha u(x, y)$, where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ is a multi-index of length $|\alpha| = |\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|$. We also put $\alpha! = \alpha_1! \cdot \alpha_2! \cdot \cdots \cdot \alpha_n!$.

First order harmonic derivatives were defined in [5], Ch. VIII and the general case is found in [7], p. 93.

Definition 2.1. Let m be a positive integer and let f be locally integrable at $x = a$. Let f_1 be the restriction of f to a neighborhood U of a such that $f_1 \in L^1(U)$. We say that f has a harmonic derivative $D^\alpha_h f(a) = d_\alpha$ at $x = a$ of if $D^\alpha u(x, y)$ has non–tangential limit $\alpha! \cdot d_\alpha$ at $(a, 0)$, where $u(x, y) = P_y \star f_1(x)$. If f has harmonic derivatives $D^\alpha_h f(a)$ of all orders $|\alpha| \le m$ we say that f has a harmonic derivative at $x = a$ of order m and we call

$$P(x - a) = \sum_{|\alpha| \le m} d_\alpha \cdot (x - a)^\alpha$$

the harmonic differential of f at a of order m.

The definition of the harmonic differential is independent of the choice of U. Next we define a property called B_l which will be used in the statements of our theorems in Section 3.

Definition 2.2 (Sjödin [4]). Let $m-1 < l \leq m$, where m is a positive integer. We say that f has property B_l at $x = a$ if there is a polynomial $Q(x) = \sum_{1\leq|\alpha|\leq m-1} c_\alpha \cdot x^\alpha$ without constant term and of degree at most $m-1$, $Q(x) \equiv 0$ for $0 < l \leq 1$, such that if $f_m(x) = f(x) - Q(x)$, then for every $\epsilon > 0$ there are positive numbers t and δ, $0 < t < \min(\epsilon, 1)$, such that

$$0 < |x-a| < \delta \text{ and } |z-x| \leq t \cdot |x-a| \text{ imply } |f_m(z) - f_m(x)| \leq \epsilon \cdot |x-a|^l.$$

Remark. We note that the polynomial Q in Definition 2.2 is unique and if $0 < l \leq 1$, then $m = 1$, $Q \equiv 0$ and $f_1(z) = f(z)$.

A function $f : \Omega \to \mathbb{R}$, defined on an open subset Ω of $\mathbb{R}^n$, satisfies a Lipschitz condition on Ω if there is a number M such that $|f(x)-f(y)| \leq |x-y|$, for all $x, y \in \Omega$. We say f satisfies a local Lipschitz condition on Ω if every $x \in \Omega$ has a neighborhood U and a number M_U such that $|f(y) - f(z)| \leq M_U \cdot |y-z|$, for all $y, z \in U$.

Let k be a nonnegative integer and let $C^{k,1}(\Omega)$ denote the class of functions $f : \Omega \to \mathbb{R}$ which have continuous derivatives of order up to k such that $D^\alpha f$, $|\alpha| = k$, satisfy a local Lipschitz condition on Ω. Note that $C^{0,1}(\Omega)$ is the standard class of locally Lipschitz continuous functions on Ω.

3 Main Results.

Our main result is contained in the following theorems.

Theorem 3.1. *Let m be a positive integer and f a function defined in a neighborhood of $x = a$ in $\mathbb{R}^n$. Then f is differentiable at $x = a$ of order m if and only if*

$$f \text{ has a harmonic derivative of order } m \text{ at } x = a \tag{1}$$

and

$$f \text{ has property } B_m \text{ at } x = a. \tag{2}$$

Theorem 3.1 follows easily from the following slightly more general result.

Theorem 3.2. *Theorem 3.1 remains true if condition* (1) *is replaced by the condition that*

$$D_h^\alpha f(a) \text{ exist for all } |\alpha| = m. \tag{3}$$

Remark. The function $u(x, y)$ in Theorems 3.1 and 3.2 is well defined since property B_m implies that f is bounded in a neighborhood of $x = a$.

We give the following application of Theorem 3.1 to C^k-functions.

Theorem 3.3. *Let m be a positive integer and let $f \in C^{m-1,1}(\Omega)$. Then f is ordinary differentiable of order m at a point $a \in \Omega$ if and only if f has harmonic derivatives $D_h^\alpha f(a)$ for all $|\alpha| = m$.*

4 Some Lemmas.

We start with some properties of the Poisson kernel which will be frequently used in the rest of this paper.

$$|D^\alpha P_y(x)| \le c(\alpha, n) \cdot y^{-n-|\alpha|}, (x, y) \in R_+^{n+1}, \tag{4}$$

$$\int\limits_{|x|>r} |D^\alpha P_y(x)| \cdot |x|^k \, dx \le c(\alpha, k, n) \cdot y \cdot r^{k-|\alpha|-1},$$

$$\text{for } r \ge y > 0, k < |\alpha| + 1, \tag{5}$$

$$\int |D^\alpha P_y(x)| \, dx \le c(\alpha, n) \cdot y^{-|\alpha|}, \; y > 0 \tag{6}$$

$$\int P_y(x) \, dx = 1, \; y > 0,$$

where as usual D^α denotes differentiation with respect to x. The proofs are straight forward consequences of the formula $P_y(x) = y^{-n} \cdot P_1(x/y)$ and are left to the reader. We now give three lemmas used in the proofs in Section 5 and begin with a lemma on functions having property B_l.

Lemma 4.1. *(a) If $0 < l_o < l_1$, then property B_{l_1} implies property B_{l_0},*

(b) If f has property B_l at $x = a$, then the limit $\lim\limits_{x \to a} f(x) = c_0$ exists,

(c) If f has property B_l at $x = a$ there are numbers $\delta_0 > 0$ and M such that

$$|f(x) - c_0 - Q(x - a)| \le M \cdot |x - a|^l,$$

for $0 < |x - a| < \delta_0$.

PROOF. The statement (a) is obvious from the definition, while (b) and (c) follow from [4], Lemma 5.3. $\square$

Lemma 4.2. *Let α and β be multi-indices, $|\alpha| \le |\beta|$. Then the integral*

$$\int D^\beta P_y(x-z) \cdot z^\alpha \, dz$$

converges and equals zero unless $\alpha = \beta$. In that case it equals $(-1)^\alpha \cdot \alpha!$.

The proof is through integration by parts, where we pass the derivatives D^β from $P_y(x-z)$ to z^α, see [5], p. 247.

Lemma 4.3. *Let m be an integer, $m \ge 2$, and let f be a function with compact support having property B_m at $x = a$, with polynomial $Q(x) = \sum_{1 \le |\alpha| \le m-1} c_\alpha \cdot x^\alpha$. Define $u(x,y) = P_y \star f(x)$. If $D^\beta u(x,y)$ has a non–tangential limit $\beta! \cdot d_\beta$ at (a,0) for some β with $1 \le |\beta| \le m-1$, then $d_\beta = c_\beta$.*

PROOF. It follows from Lemma 4.1 that there are numbers c_0, M and $\delta_0 > 0$ such that

$$|f(x) - \sum_{|\alpha| \le m-1} c_\alpha \cdot (x-a)^\alpha| \le M \cdot |x-a|^m, \tag{7}$$

for $|x-a| < \delta_0$. It is no loss of generality to assume that $f(x) = 0$, for $|x-a| > \delta_0$. Let $1 \le |\beta| = k \le m-1$ and define $f_k(x) = f(x) - \sum_{|\alpha| \le k} c_\alpha \cdot (x-a)^\alpha$. Let $0 < \epsilon < 1$ be arbitrary and define a cone V by $V = \{(x,t); t \ge \epsilon \cdot |x-a|\}$. We put $y = \epsilon \cdot |x-a|$. Then $(x,y) \in V$. Now consider the identity

$$I = \int D^\beta P_y(x-z) \cdot f_k(z) \, dz = D^\beta u(x,y) - \sum_{|\alpha| \le k} \int D^\beta P_y(x-z) \cdot c_\alpha \cdot (z-a)^\alpha \, dz.$$

The first term on the right hand side tends to $\beta! \cdot d_\beta$ as $x \to a$ by our assumption, while the second term equals $\beta! \cdot c_\beta$ by Lemma 4.1. We finish the proof of the lemma by showing that $|I|$ tends to zero as $x \to a$. We have

$$|I| \le \int |D^\beta P_y(x-z)| \cdot |f_k(z)| \, dz = \int_{|z-x| \ge |x-a|} + \int_{|z-x| \le |x-a|} = A_1 + A_2. \tag{8}$$

It follows from (7) and the definition of f_k that there is M_1, independent of x, such that $|f_k(x)| \le M_1 \cdot \min(|x-a|^k, |x-a|^{k+1})$, for all $x \in \mathbb{R}^n$. Then for $|x-a| \le 1$

$$A_1 \le M_1 \cdot \int_{|z-x| \ge |x-a|} |D^\beta P_y(x-z)| \cdot |z-a|^k \, dz \le$$

$$\leq 2^k \cdot M_1 \cdot \int\limits_{|w| \geq |x-a|} |D^\beta P_y(w)| \cdot |w|^k \, dw \leq c(k,n) \cdot M_1 \cdot \epsilon,$$

by (5) and the properties of f_k. Further,

$$A_2 \leq M_1 \cdot \int\limits_{|z-x| \leq |x-a|} |D^\beta P_y(z-x)| \cdot |z-a|^{k+1} \, dz \leq$$

$$\leq 2^{k+1} \cdot M_1 \cdot |x-a|^{k+1} \int |D^\beta P_y(w)| \, dw \leq c(k,n) \cdot M_1 \cdot \epsilon^{-k} \cdot |x-a|,$$

by (6). It follows that I tends to zero as $x \to a$, which completes the proof of Lemma 4.3. □

5 Proofs of Theorems 3.2 and 3.3.

We start with the proof of Theorem 3.2. The necessity part is straight forward. Assume that f is differentiable at $x = a$ of order m. Then it is easy to see that f has property B_m at $x = a$ and f has a harmonic differential of order m. See [5], p. 247 for the case $m = 1$. The general case is proved analogously.

Now we turn to the proof of the sufficiency part of Theorem 3.2. Assume that (2) and (3) hold. Let $0 < \epsilon < 1$ be arbitrary and choose δ, t and the polynomial $Q(x)$ as in Definition 2.2, $Q(x) \equiv 0$ if $m = 1$. Without loss of generality we assume that f is zero outside a suitable disc centered at $x = a$. Then, as in the proof of Lemma 4.2, there is a unique number c_0 and $M_1 > 0$, independent of x, such that

$$|f(x) - \sum_{|\alpha| \leq m-1} c_\alpha \cdot x^\alpha| \leq M_1 \cdot \min(|x-a|^{m-1}, |x-a|^m), \tag{9}$$

for all $x \in \mathbb{R}^n$. Let $\alpha! \cdot a_\alpha$, $|\alpha| = m$, be the non-tangential limit of $D^\alpha u(x,y)$ at $(a,0)$ according to (3), where $u(x,y) = P_y \star f(x)$, and define $P(x) = \sum_{|\alpha| \leq m} c_\alpha \cdot x^\alpha$. We are going to prove that $f(x) - P(x-a) = o(|x-a|^m)$, as $x \to a$, and start from the identity

$$f(x) - P(x-a) = (f_m(x) - f_m(z)) + (f_m(z) - c_0) - \sum_{|\alpha| = m} c_\alpha \cdot (x-a)^\alpha, \tag{10}$$

where as usual $f_m(w) = f(w) - Q(w-a)$. Multiplying (10) with $P_y(x-z)$ and integrating with respect to z over $|z-x| \leq t \cdot |x-a|$ gives

$$(f(x) - P(x-a)) \cdot \int\limits_{|w| \leq t \cdot |x-a|} P_y(w) \, dw = I_1 + I_2 + I_3, \tag{11}$$

where I_1, I_2 and I_3 correspond to the three terms in the right hand side of (10). We define a cone $V = \{(x, y); y \geq \epsilon \cdot t^m \cdot |x-a|\}$ and put $y = \epsilon \cdot t^m \cdot |x-a|$. Then $(x, y) \in V$ and the integral in the left hand side of (11) is at least $1 - c \cdot \epsilon$. Hence it suffices to estimate I_k, $k = 1, 2, 3$. First, Definition 2.2 implies that $|I_1| \leq \epsilon \cdot |x-a|^m$, for $|x-a| < \delta$. Next we split I_2 as

$$I_2 = \int\limits_{|z-a| \leq 2|x-a|} - \int\limits_{|z-a| \leq 2|x-a|, |z-x| > t|x-a|} = I_2' + I_2'',$$

where

$$|I_2''| \leq M_1 \cdot 2^m \cdot |x-a|^m \cdot \int\limits_{|w| \geq t|x-a|} P_y(w)\, dw \leq c(n, m) \cdot M_1 \cdot \epsilon \cdot |x-a|^m,$$

by (5) and (9). We now use Taylor's formula on the Poisson kernel to get

$$I_2' = \sum_{|\alpha| \leq m-1} \frac{1}{\alpha!} \cdot \int\limits_{|z-a| \leq 2 \cdot |x-a|} D^\alpha P_y(a-z) \cdot (x-a)^\alpha \cdot (f_m(z) - c_0)\, dz +$$

$$+ \int\limits_{|z-a| \leq 2 \cdot |x-a|} R_m(x, z) \cdot (f_m(z) - c_0)\, dz = A + B,$$

where

$$R_m(x, z) = \sum_{|\alpha| = m} \frac{1}{\alpha!} \cdot \int_0^1 D^\alpha P_y(a - z + s(x-a)) \cdot (x-a)^\alpha\, ds$$

is the remainder. We claim that A and $B + I_3$ are bounded by some constant times $\epsilon \cdot |x-a|^m$, as $x \to a$.

We split A as

$$A = \sum \int\limits_{|z-a| \leq r|x-a|} + \sum \int\limits_{r|x-a| < |z-a| \leq 2|x-a|}, \tag{12}$$

where the summation is over $|\alpha| \leq m-1$ and $r = t^m$. The first sum in (12) has terms bounded by

$$c(m) \cdot M_1 \cdot (r|x-a|)^m \cdot |x-a|^{|\alpha|} \cdot \int |D^\alpha P_y(w)|\, dw \leq$$

$$\leq c(m, n) \cdot M_1 \cdot r^m \cdot |x-a|^{m+|\alpha|} \cdot y^{-|\alpha|} \leq c(m, n) \cdot M_1 \cdot \epsilon \cdot |x-a|^m.$$

The terms in the second sum in (12) are bounded by

$$c(m) \cdot M_1 \cdot |x-a|^{|\alpha|} \cdot \int\limits_{r|x-a|<|z-a|\le 2r|x-a|} |D^\alpha P_y(a-z)| \cdot |z-a|^m \, dz \le$$

$$\le c(m) \cdot |x-a|^m \cdot \int\limits_{|z-a|>r|x-a|} |D^\alpha P_y(a-z)| \cdot |z-a|^{|\alpha|} \le c(m,n) \cdot \epsilon \cdot |x-a|^m.$$

In both cases we used (5), (9), $y = \epsilon \cdot t^m \cdot |x-a|$ and $r = t^m$. This proves our claim for A.

We are now only left with $(B + I_3)$ and first note that

$$I_3 = -\sum_{|\alpha|=m} c_\alpha \cdot (x-a)^\alpha + I_4,$$

where$|I_4| \le c \cdot \sum_{|\alpha|=m} |c_\alpha| \cdot \epsilon \cdot |x-a|^m$ by (5). We split B into two terms

$$B = \int\limits_{R^n} - \int\limits_{|z-a|>2|x-a|} = B_1 + B_2.$$

To prove our claim for $(B + I_3)$, and thereby complete the proof of Theorem 3.2, it suffices to show that $|B_2| \le c(m,n) \cdot M_1 \cdot \epsilon \cdot |x-a|^m$ and

$$B_1 - \sum_{|\alpha|=m} c_\alpha \cdot (x-a)^\alpha = o(|x-a|^m), \tag{13}$$

as $x \to a$. Estimating the remainder $R_m(x,z)$ in Taylors formula gives that $|B_2|$ is at most

$$\sum_{|\alpha|=m} \frac{1}{\alpha!} \cdot \int\limits_{|u|>|x-a|} du \int\limits_0^1 ds |D^\alpha P_y(u)| \cdot |x-a|^m \cdot |f_m(a+s(x-a)-u) - a_0| \le$$

$$\le c(m,n) \cdot M_1 \cdot \sum_{|\alpha|=m} \int\limits_{|u|>|x-a|} du \, |D^\alpha P_y(u)| \cdot |x-a|^m \cdot (|u|^m + |x-a|^m) \le$$

$$\le c(m,n) \cdot M_1 \cdot \epsilon \cdot |x-a|^m,$$

by a change of variables and using (5) and (9). This settles the estimate for B_2. For B_1 we have the formula

$$B_1 = \sum_{|\alpha|=m} \frac{1}{\alpha!} \cdot \int\limits_0^1 ds \, D^\alpha u(a+s(x-a),y) \cdot (x-a)^\alpha \tag{14}$$

by Lemma 4.2. The point $(a + s(x - a), y)$ belongs to the cone V. Hence the integral in (14) tends to $\alpha! \cdot c_\alpha$ as $x \to a$, for all $|\alpha| = m$, since the integrand converges uniformly to $\alpha! \cdot c_\alpha$, $0 \le s \le 1$. This proves (13) and completes the proof of Theorem 3.2. □

Proof of Theorem 3.3. We first show that f has property B_m everywhere in Ω. This is obvious if $m = 1$ and the general case follows from Taylors formula. Theorem 3.3 now follows from Theorem 3.2. □

6 Examples and Remarks.

We start with two definitions. A function f is L^p-differentiable of order l, $m \le l < m + 1$, at $x = a$ if there is a polynomial $P(x - a)$ of order at most m such that

$$(r^{-n} \cdot \int\limits_{|x-a|\le r} |f(x) - P(x-a)|^p \, dx)^{1/p} = o(r^l), \quad as\ r \to 0,$$

and f is approximately differentiable of order l at $x = a$ if

$$r^{-n} \cdot |\{x; |f(x) - P(x-a)| > \lambda \cdot |x - a|^l \text{ and } |x - a| < r\}| \to 0, \text{ as } r \to 0,$$

for every $\lambda > 0$. It is easy to see that L^p-differentiability implies approximative differentiability. That also L^p-differentiability implies the existence of a harmonic differential is proved in [5], p. 247 for $m = 1$. The same proof applies in the general case. In this section we show that the existence of a harmonic differential is not related to approximative differentiability. Consequently, our Theorems 3.1 and 3.2 do not follow from [4].

There are functions f that have approximate differentials of any order at $x = a$ but are not locally integrable there, c.f. [1], p. 150. Such functions cannot have harmonic derivatives at $x = a$. In the rest of this section we construct an example of a function f on $\mathbb{R}$ that has a harmonic differential but is not approximately differentiable at the origin.

Example. Let $I = [a, b]$ be an interval and k a positive integer. Define

$$g_{I,k}(x) = (-1)^j, \text{ if } x \in [a + (j-1)v, a + jv), \, 1 \le j \le 2k,$$

where $v = (b - a)/2k$. The oscillating function $g_{I,k}$ has a cancelling effect as expressed in the following lemma.

Lemma 6.1. *Let $h \in C^1(I)$. Then*

$$\Big|\int_I h(x)\cdot g_{I,k}(x)\,dx\Big| \le \frac{1}{4k}\cdot \sup_I |h'(x)|\cdot |I|^2.$$

The proof of Lemma 6.1 is elementary and therefore omitted. Now let f_i, $i = 1, 2, \ldots$, be the function $g_{I,k}$ with $I = [2^{-i}, 2^{1-i}]$, $f_i(x) = 0$ outside I, where $k = k_i$ will be defined later. Define

$$f(x) = \sum_{i=1}^{\infty} 2^{-i}\cdot f_i(x),\ x > 0,$$

$f(0) = 0$ and $f(-x) = f(x)$. Then f does not have a first order approximative differential at $x = 0$. If such a differential exists it must equal zero, because f is an even function. This is however impossible, since $|f(x)| \ge |x|/2$ for $|x| \le 1$. It remains to prove that f has a first order harmonic derivative at $x = 0$ if the sequence $\{k_i\}$ increases fast enough.

Now we define $u(x, y) = P_y \star f(x)$. Then u is harmonic in the upper half plane and $u(x, y) \to 0$, as $(x, y) \to (0, 0)$ non–tangentially, since $x = 0$ is a Lebesgue point of f and $f(0) = 0$ [5], Ch. VII, Theorem 1. Further,

$$\frac{\partial u}{\partial x} = \int P_y'(x - z)\cdot f(z)\,dz,$$

since the differentiated integral converges uniformly on the entire real axis. Let $s > 0$ and define a cone $V_s = \{(x, y); |x| < s\cdot y \text{ and } y > 0\}$. Let $\epsilon > 0$ be arbitrary, $0 < \epsilon < \min(1/s, 1)$, and let $0 < y < 1$. Now assume that $(x, y) \in V_s$ and write

$$\frac{\partial u}{\partial x} = \int P_y'(x - z)\cdot f(z)\,dz = \int\limits_{|z|\ge r} + \int_{-r}^{0} + \int_{0}^{r} = A + B + C,$$

where $r = 2/(\epsilon\cdot\sqrt{y})$. Then

$$|A| \le \int\limits_{|z-x|\ge r/2} |P_y'(x - z)|\cdot |f(z)|\,dz \le \int\limits_{|w|\ge r/2} |P_y'(w)|\,dw \le c\cdot\epsilon,$$

by (5). Lemma 6.1 and the estimate $|P_y''(w)| \le c\cdot y^{-3}$ gives

$$|B| \le \sum_{I=1}^{\infty} 2^{-I}\cdot \Big|\int_{-r}^{0} P_y'(x - z\,dz\Big| \le \sum_{N}^{\infty} 2^{-3i}\cdot y^{-3}\cdot k_i^{-1},$$

where $N = N(\epsilon, y) = [\log_2 1/r]$. Choosing $k_i = 2^{4i}$ we get $|B| \leq c \cdot \epsilon^{-7} \cdot \sqrt{y}$ and hence $\lim_{y \to 0} |B| = 0$. We get an analogous estimate for the term C. It follows that $\limsup_{y \to 0} |\frac{\partial u}{\partial x}| \leq c \cdot \epsilon$. Since ϵ and s are arbitrary we conclude that $\frac{\partial u}{\partial x}$ tends non–tangentially to zero at $(0, 0)$. This proves that f has a first order harmonic derivative at $x = 0$ and completes our example.

Remark. It is possible to choose the sequence $\{k_i\}_1^\infty$ such that $\frac{\partial^j u}{\partial x^j}$ tends to zero non–tangentially, for all $j = 1, 2, \ldots$, i.e. f has harmonic derivatives of all orders equal to zero at $x = 0$. We simply choose $k_i = 2^{2^i}$. Then for $s > 0$, $0 < \epsilon < \min(s^{-j-1}, 1)$, $0 < y < 1$ and $(x, y) \in V_s$ we can proceed as above with $r = 2 \cdot (y/\epsilon)^{1/(j+1)}$ to prove that $\partial^j u/\partial x^j$ tends to zero non–tangentially at (0,0), for $j = 1, 2, \ldots$.

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