ON SECOND-ORDER NON-LINEAR OSCILLATIONS

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1. In this paper we establish criteria regarding the behaviour, oscillatory or otherwise, near $x=\infty$ of the solutions of

$$y''+fy^{2n-1}=0\ , \tag{1.1}$$

where $f=f(x)$ is positive and continuous for $x \geqslant 0$ and n is an integer greater than 1. A solution, not identically zero, will be said to be *oscillatory* if it has infinitely many zeros for $x \geqslant 0$.

The three possibilities to be distinguished are that the solutions of (1.1) might be (i) all oscillatory, (ii) some oscillatory and some not, and (iii) all nonoscillatory. We give here a necessary and sufficient condition for (i) to hold, and a sufficient condition for (iii).

In the linear case, $n=1$, a number of criteria have been found for cases (i) and (iii); in the linear case (ii) is impossible. A very sensitive procedure is afforded by the chain of logarithmic tests studied by J. C. P. Miller [3], P. Hartman [1], and W. Leighton [2]; some further developments in this field have been given recently by Ruth L. Potter [4], who has in particular a result [Theorem 5.1] bearing on the limitations of this procedure. There does not, however, seem to have been found any simple necessary and sufficient condition for (i) to hold in the linear case, so it is noteworthy that such a criterion exists in the nonlinear case.

2. The result in question is:

THEOREM 1. *Let $f=f(x)$ be positive and continuous for $x \geqslant 0$, and let n be an integer greater than unity. Then a necessary and sufficient condition for all solutions of* (1.1) *to be oscillatory is*

$$\int_0^\infty xf\,dx=\infty\ . \tag{2.1}$$

We remark that in the linear case the criterion is necessary but not sufficient.

It should be mentioned that no solution of (1.1) becomes infinite for any finite positive x-value; this is ensured by the positiveness of $f(x)$.

We prove first that if (1.1) has a nonoscillatory solution, then (2.1) cannot hold; this will prove the sufficiency of the criterion.

Received October 29, 1953.

Let then y denote a nonoscillatory solution of (1.1); y will then be ultimately of one sign, which we may without loss of generality take to be positive. It follows from (1.1) that y'' will be ultimately negative, so that y' will tend either to a positive limit, or to zero, or to a negative limit, or to $-\infty$. The last two cases can be excluded since they would imply that y is ultimately negative. Thus y must be ultimately monotonic increasing, and y' must tend to a finite nonnegative limit.

We next integrate (1.1) over $(0, x)$, getting

$$y'(x)-y'(0)+\int_0^x fy^{2n-1}dt=0 . \tag{2.2}$$

Since $y'(x)$ tends to a limit as $x\to\infty$, this implies that the integral on the left of (2.2) converges as $x\to\infty$; we may therefore integrate (1.1) over (x, ∞), getting now

$$y'(\infty)-y'(x)+\int_x^\infty fy^{2n-1}dt=0 ,$$

whence, since $y'(\infty)\geqslant 0$,

$$y'(x)\geqslant\int_x^\infty fy^{2n-1}dt \tag{2.3}$$

Still with the assumption that y is ultimately positive, let a be an x-value such that $y(x)>0$ for $x\geqslant a$. We integrate (2.3) over (a, x), where $x>a$, and get

$$y(x)-y(a)\geqslant\int_a^x du\int_u^\infty fy^{2n-1}dt=\int_a^x (t-a)fy^{2n-1}dt+(x-a)\int_x^\infty fy^{2n-1}dt ,$$

and hence, for $x>a$,

$$y(x)\geqslant\int_a^x (t-a)fy^{2n-1}dt ,$$

which we re-write in the form

$$(x-a)fy^{2n-1}\left\{\int_a^x (t-a)fy^{2n-1}dt\right\}^{1-2n}\geqslant(x-a)f . \tag{2.4}$$

We now take any x_1, x_2 such that $a<x_1<x_2$, and integrate (2.4) over (x_1, x_2). This gives

$$(2-2n)^{-1}\left[\left(\int_a^x (t-a)fy^{2n-1}dt\right)^{2-2n}\right]_{x_1}^{x_2}\geqslant\int_{x_1}^{x_2}(x-a)fdx .$$

If now we make $x_2\to\infty$, the left side remains finite; this proves that

$$\int_{x_1}^\infty (x-a)fdx<\infty ,$$

which is equivalent to

$$\int_0^\infty x f dx < \infty \ , \tag{2.5}$$

in contradiction of (2.1). Thus the sufficiency of the criterion is proved.

As to the necessity, we shall show that if (2.5) is the case then for any prescribed value of $y(\infty)$, for example 1, there exists a solution of (1.1) such that

$$y(\infty)=1 \ , \quad y'(\infty)=0 \ , \tag{2.6}$$

which is obviously nonoscillatory.

It is easily verified that if the integral equation

$$y(x)=1-\int_x^\infty (t-x)f(t)\{y(t)\}^{2n-1}dt \tag{2.7}$$

has a solution y which is continuous and uniformly bounded as $x\to\infty$, then it is also a solution of (1.1) with the supplementary conditions (2.6). The existence of a bounded continuous solution of (2.7) may be established by the Picard method of successive approximation. We define a sequence of functions

$$y_m(x) \qquad (m=0,\ 1,\ \cdots)\ , \quad x\geqslant 0\ ,$$

by

$$y_0(x)\equiv 0\ ,$$

$$y_{m+1}(x)=1-\int_x^\infty (t-x)f(t)\{y_m(t)\}^{2n-1}dt \qquad (m=0,\ 1,\ \cdots)\ .$$

The remainder of the argument need only be sketched. We can prove by induction that if x is so large that

$$\int_x^\infty (t-x)f(t)dt<1\ ,$$

assuming now (2.5), then $0\leqslant y_m(x)\leqslant 1$. We have also

$$y_{m+2}(x)-y_{m+1}(x)=\int_x^\infty (t-x)f(t)\{(y_m(t))^{2n-1}-(y_{m+1}(t))^{2n-1}\}dt\ ,$$

whence, for sufficiently large x,

$$|y_{m+2}(x)-y_{m+1}(x)|\leqslant(2n-1)\max_{t\geqslant x}|y_m(t)-y_{m+1}(t)|\int_x^\infty (t-x)f(t)dt\ .$$

From this we deduce the convergence of the sequence $y_m(x)$ $(m=0,1,\cdots)$, for x so large that

$$(2n-1)\int_x^\infty (t-x)f(t)dt<1 ;$$

the continuity of the limiting function is easily established. This proves the existence of a nonoscillatory solution of (1.1) for sufficiently large x, which is enough for our purpose.

This completes the proof of Theorem 1.

3. **We conclude** with a simple sufficient criterion for nonoscillatory solutions which happens also to be true in the linear case [4, Lemma 1.2].

THEOREM 2. *Let $f(x)$ be positive and continuously differentiable for $x\geqslant 0$, and let $f'\leqslant 0$. Let also*

$$\int_0^\infty x^{2n-1}f\,dx<\infty . \tag{3.1}$$

Then (1.1) *has no oscillatory solutions.*

We observe first of all that the result

$$\frac{d}{dx}\left\{\frac{1}{2}y'^2+\frac{1}{2n}fy^{2n}\right\}=\frac{1}{2n}f'y^{2n}\leqslant 0$$

implies that, for any solution, y' remains bounded as $x\to\infty$.

Supposing if possible that (1.1) had an oscillatory solution, let x_0, $x_1, \cdots$ be its successive zeros. Let x_m be for convenience a zero for which $y'(x_m)>0$, and let x_m' be the unique zero of y' in (x_m, x_{m+1}). Integrating (1.1) over (x_m, x_m'), we have

$$y'(x_m')-y'(x_m)+\int_{x_m}^{x_m'} fy^{2n-1}dx=0 ,$$

or

$$y'(x_m)=\int_{x_m}^{x_m'} fy^{2n-1}dx . \tag{3.2}$$

Now y' is positive and decreasing in (x_m, x_m'), and $y(x_m)=0$; hence for $x_m\leqslant x\leqslant x_m'$ we have

$$0\leqslant y\leqslant y'(x_m)(x-x_m) .$$

Thus from (3.2) we derive

$$y'(x_m)\leqslant\{y'(x_m)\}^{2n-1}\int_{x_m}^{x_m'} f(x)(x-x_m)^{2n-1}dx ,$$

and so

$$1 \leqslant \{y'(x_m)\}^{2n-2} \int_{x_m}^{\infty} f x^{2n-1} dx .$$

This however becomes impossible as x_m becomes large, since $y'(x_m)$ has been proved to remain bounded as $x_m \to \infty$, while by (3.1) we have

$$\int_{x_m}^{\infty} f x^{2n-1} dx \to 0 .$$

Since we have obtained a contradiction it follows that (1.1) has under these assumptions no oscillatory solutions. This proves the theorem.

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