AN OPTIMUM PROBLEM IN THE WEINSTEIN METHOD FOR EIGENVALUES

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1. Introduction. The method of Weinstein [1] gives upper bounds for the eigenvalues $\lambda_1' \geq \lambda_2' \geq \cdots$ of the projection L' into a space $\mathfrak{G}$ of a completely continuous positive symmetric operator L in a Hilbert space $\mathfrak{H}$ with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots$. These upper bounds are the eigenvalues $\lambda_n^{(m)}$ of the projection of L into a space of finite index m,

$$\mathfrak{H} \ominus \{p_1, \cdots, p_m\}, \tag{1}$$

where $p_1, \cdots, p_m$ are any vectors in the space

$$\mathfrak{P} = \mathfrak{H} \ominus \mathfrak{G}. \tag{2}$$

The chief part of the Weinstein method is the explicit determination of the eigenvalues $\lambda_n^{(m)}$ in the space (1) in terms of the eigenvalues and eigenvectors of L in $\mathfrak{H}$. These satisfy

$$\lambda_n^{(m)} \geq \lambda_n'. \tag{3}$$

The values $\lambda_n^{(m)}$ will, of course, depend on the choice of the vectors $(p_1, \cdots, p_m)$. It is naturally desirable that the upper bound for a particular eigenvalue λ_n' should be as small as possible. This paper investigates how small it can be made for given n and m by a proper choice of the constraint vectors $(p_1, \cdots, p_m)$.

Because of the minimax principle, $\lambda_n^{(m)}$ must satisfy

$$\lambda_n^{(m)} \geq \lambda_{n+m}. \tag{4}$$

Our result is that the inequalities (3) and (4) are the only restrictions on the smallness of $\lambda_n^{(m)}$. In other words, for given n and m, there exist vectors $(p_1, \cdots, p_m)$ such that the weaker of the inequalities (3) and (4) becomes an equality.

Received December 12, 1951. This work was sponsored by the Office of Naval Research.

Pacific J. Math. 2 (1952), 413-418

2. The case of a single constraint. We first prove our result for the case of the first intermediate problem, that is, for $m = 1$.

THEOREM 1. *For any given n, there is a vector p in the space*

$$\mathfrak{P} = \mathfrak{H} \ominus \mathfrak{G} \tag{5}$$

such that, if the projection of L into $\mathfrak{H} \ominus \{p\}$ *has eigenvalues*

$$\lambda_1^{(1)} \geq \lambda_2^{(1)} \geq \cdots,$$

either

$$\lambda_n^{(1)} = \lambda_n' \tag{6}$$

or

$$\lambda_n^{(1)} = \lambda_{n+1}, \tag{7}$$

according as λ_n' *or* λ_{n+1} *is larger.*

Proof. If $\lambda_n' = \lambda_n$, then (6) is satisfied for any p and there is nothing to prove. Our theorem thus naturally splits into the two cases $\lambda_n > \lambda_n' \geq \lambda_{n+1}$ and $\lambda_n' < \lambda_{n+1}$, which we shall prove separately.

3. The case $\lambda_n > \lambda_n' \geq \lambda_{n+1}$. Let the eigenvector of L' corresponding to λ_n' be u_n'. Its eigenvalue equation can be written in terms of the operator L as

$$Lu_n' - \lambda_n' u_n' = p, \tag{8}$$

where p is some vector in $\mathfrak{P}$. Let us assume for the moment that p is not a null vector. Then (8) is an eigenvalue equation for the projection of L into $\mathfrak{G}$ but not for L. Any eigenvector of L corresponding to the eigenvalue λ_n' must, because of (8), be orthogonal to p and hence must belong to $\mathfrak{H} \ominus \{p\}$. Thus, the multiplicity of λ_n' as an eigenvalue of the projection of L into $\mathfrak{H} \ominus \{p\}$ is one greater than its multiplicity as an eigenvalue of L. Let the latter be $r \geq 0$. If $r = 0$, then $\lambda_n' > \lambda_{n+1}$, and λ_n' must be $\lambda_n^{(1)}$ by the minimax principle. If $r \geq 1$, then $\lambda_n > \lambda_{n+1} = \cdots = \lambda_{n+r} > \lambda_{n+r+1}$, and the minimax principle gives

$$\lambda_{n-1}^{(1)} \geq \lambda_n > \lambda_n' \geq \lambda_{n+1} > \lambda_{n+r+1} \geq \lambda_{n+r+1}^{(1)}. \tag{9}$$

Thus, since the multiplicity of λ_n' in $\mathfrak{H} \ominus \{p\}$ is $r + 1$, we must have

$$(10) \qquad \lambda_n^{(1)} = \lambda_n',$$

so that the vector p in (8) has the property stated in our theorem.

If $\lambda_n' = \lambda_{n+1}$, it is possible that the vector p in (8) is a null vector. This means that the eigenvector of L' corresponding to λ_n' is also an eigenvector of L. Suppose that the same is also true of the eigenvalues $\lambda_{n+1}', \cdots, \lambda_{n+s-1}'$ but not of λ_{n+s}'. We then consider the projections $\overline{L}$ and $\overline{L}'$ into

$$\mathfrak{H} \ominus \left\{u_n', \cdots, u_{n+s-1}'\right\} \text{ and } \mathfrak{G} \ominus \left\{u_n', \cdots, u_{n+s-1}'\right\}$$

respectively, and call their eigenvalues $\overline{\lambda}_i$ and $\overline{\lambda}_i'$. Then $\overline{L}'$ has the same eigenvalues as L', except that the eigenvalues $\lambda_n', \cdots, \lambda_{n+s-1}'$ are removed. The same is true of $\overline{L}$ and L. Then

$$(11) \qquad \overline{\lambda}_n' = \lambda_{n+s}' \leq \lambda_n'.$$

If there is a vector p in $\mathfrak{P}$ so that the n-th eigenvalue of L in

$$\mathfrak{H} \ominus \left\{u_n', \cdots, u_{n+s-1}', p\right\}$$

is at most $\overline{\lambda}_n'$, then, because of (11), the n-th eigenvalue of L in $\mathfrak{H} \ominus \{p\}$ is λ_n' and equation (6) in our theorem will be proved. Now if

$$(12) \qquad \overline{\lambda}_n' = \lambda_{n+s}' \geq \overline{\lambda}_n,$$

then, since by definition of s the eigenvector of L' corresponding to λ_{n+s}' is not an eigenvector of L, the existence of such a vector p follows from the first part of this paragraph. If, on the other hand, we have

$$(13) \qquad \lambda_n' < \overline{\lambda}_{n+1},$$

the existence of this vector p will be assured by the results of the next paragraph.

A final possibility* is that there is no integer s such that the eigenvector u_{n+s} of L' is not also an eigenvector of L. In other words, all but the first $n-1$ eigenvectors of L' are also eigenvectors of L. Then, since $\lambda_n' = \lambda_{n+1}$, the vectors $u_1', \cdots, u_{n-1}'$ are the only eigenvectors of L' which are not orthogonal to $u_1, u_2, \cdots, u_n$. Therefore there is one linear combination p of $u_2, \cdots, u_n$ which is orthogonal to all eigenvectors of L' and hence belongs to $\mathfrak{P}$. There can

*This possibility was pointed out by C. Arf in the course of an alternative proof of the results here presented.

be at most $n-1$ eigenvectors of the projection of L into $\mathfrak{H} \ominus \{p\}$ which are not orthogonal to $u_1, \cdots, u_n$. Therefore, the n-th eigenvalue of this projection is $\lambda_{n+1} = \lambda'_n$, and both equalities (6) and (7) hold.

4. **The case** $\lambda'_n < \lambda_{n+1}$. We now show that if $\lambda'_n < \lambda_{n+1}$ then the equation (7) can be made to hold. This will be done by induction. We first replace the space $\mathfrak{P}$ by a finite space. Since L is completely continuous it follows that $\lambda_m \longrightarrow 0$ as $m \longrightarrow \infty$; therefore there is an integer m such that

$$\lambda_{n+1} \geq \lambda'_n + \lambda_{m+1}. \tag{14}$$

It has been shown by the author [1, 2] that if we let p_i be the projection in $\mathfrak{P}$ of u_i, then the eigenvalues $\lambda_n^{(m)}$ of L in $\mathfrak{H} \ominus \{p_1, \cdots, p_m\}$ satisfy

$$\lambda_n^{(m)} \leq \lambda'_n + \lambda_{m+1}\,. \tag{15}$$

Combining this with (14), we obtain

$$\lambda_n^{(m)} \leq \lambda_{n+1}. \tag{16}$$

Thus, it will suffice to show that if the inequality (16) holds where $\lambda_n^{(m)}$ is the n-th eigenvalue of L in a space $\mathfrak{H} \ominus \{p_1, \cdots, p_m\}$, then there is a linear combination p of the vectors $p_1, \cdots, p_m$ such that the n-th eigenvalue of L in $\mathfrak{H} \ominus \{p\}$ is λ_{n+1}. Our induction proof consists of showing that if (16) holds for $m > 1$ then there is a linear combination p' of p_{m-1} and p_m such that the n-th eigenvalue of L in $\mathfrak{H} \ominus \{p_1, \cdots, p_{m-2}, p'\}$ is at most λ_{n+1}. If $\lambda_n^{(m-1)} \leq \lambda_{n+1}$, this is obviously true, for we must only take $p' = p_{m-1}$. Thus, we need to examine only the case

$$\lambda_n^{(m-2)} \geq \lambda_n^{(m-1)} > \lambda_{n+1} \geq \lambda_n^{(m)}. \tag{17}$$

Since, by the minimax theorem,

$$\lambda_{n+1} \geq \lambda_{n+1}^{(m-2)}, \tag{18}$$

our induction step will be proved if we find p so that the n-th eigenvalue of L in $\mathfrak{H} \ominus \{p_1, \cdots, p_{m-2}, p'\}$ is equal to either $\lambda_n^{(m)}$ or $\lambda_{n+1}^{(m-2)}$. In other words, the induction step is just Theorem 1 in the special case in which $\mathfrak{P}$ is a 2-space.

Thus if $\lambda_n^{(m)} \geq \lambda_{n+1}^{(m-2)}$ the induction is proved by the results of §3. Note that in the case of a common eigenvector where one had to reduce the proof in §3 to the proof of this section, the reduction is to the case $\lambda_n^{(m)} < \lambda_{n+1}^{(m-2)}$, which will now be treated.

If $\lambda_n^{(m)} < \lambda_{n+1}^{(m-2)}$, we must construct a linear combination p' of p_{m-1} and p_m so that $\lambda_{n+1}^{(m-2)}$ is the n-th eigenvalue of L in $\mathfrak{H} \ominus \{p_1, \cdots, p_{m-2}, p'\}$. To do this, we take for p' the linear combination of p_{m-1} and p_m which is orthogonal to the eigenvector corresponding to $\lambda_{n+1}^{(m-2)}$. Then $\lambda_{n+1}^{(m-2)}$ is an eigenvalue of L in $\mathfrak{H} \ominus \{p_1, \cdots, p_{m-2}, p'\}$. By the minimax principle, the $(n+1)$-st eigenvalue in this space is at most $\lambda_n^{(m)} < \lambda_{n+1}^{(m-2)}$. Therefore $\lambda_{n+1}^{(m-2)}$ must be the n-th eigenvalue in this space, and p' has the desired property.

Thus, our induction step is proved and Theorem 1 has been shown to hold in all possible cases.

5. The general intermediate problem.

We are now in a position to prove the more general result announced in the introduction.

THEOREM 2. *For any fixed integers m and n, there are vectors $p_1, \cdots, p_m$ in $\mathfrak{P}$ which, if used as constraints in the n-th intermediate problem, yield either*

$$\lambda_n^{(m)} = \lambda_n' \tag{19}$$

or

$$\lambda_n^{(m)} = \lambda_{n+m}. \tag{20}$$

Proof. We first prove the possibility of the equality (20) when

$$\lambda_{n+m} > \lambda_n' \tag{21}$$

According to Theorem 1 with $n + m - 1$ substituted for n, there is a vector p_1 such that

$$\lambda_{n+m-1}^{(1)} = \lambda_{n+m}. \tag{22}$$

We then apply Theorem 1 to the projection of L into $\mathfrak{H} \ominus \{p_1\}$ to assert the existence of a vector p_2 such that

$$\lambda_{n+m-2}^{(2)} = \lambda_{n+m}. \tag{23}$$

This process is repeated until the equality (20) is obtained. Inequality (21) assures us that the equality (7) of Theorem 1 will always be attainable.

If $\lambda_{n+m} \leq \lambda_n'$, then there is an integer $l \leq m$ such that

$$\lambda_{n+l} \leq \lambda'_n < \lambda_{n+l-1}. \tag{24}$$

We shall show that there are l vectors $p_1, \cdots, p_l$ for which

$$\lambda_n^{(l)} = \lambda'_n \tag{25}$$

The equality (19) will then hold for any $m-1$ vectors $p_{l+1}, \cdots, p_m$ appended to the first l.

Since $\lambda'_n \geq \cdots \geq \lambda'_{n+l-1}$, we can proceed as in the proof of (20) to show that there are $l-1$ vectors $p_1, \cdots, p_{l-1}$ for which

$$\lambda_{n+1}^{(l+1)} = \lambda_{n+l}. \tag{26}$$

We now apply Theorem 1 to L in $\mathfrak{H} \ominus \{p_1, \cdots, p_{l-1}\}$. According to (24) and (26) it is the equality (6) which can be made to hold by a constraint p_l. We thus obtain (25), and Theorem 2 is proved.

References

1. H. F. Weinberger, *An error estimate for the method of Weinstein.* Abstract of a paper presented at the Minneapolis meeting of the American Mathematical Society, 1951.

2. ———, *An error estimate for the Weinstein method for eigenvalues.* Submitted to Proc. Amer. Math. Soc.

3. A. Weinstein, *Etude des spectres des équations aux dérivées partielles de la théorie des plaques elastiques,* Memorial des Sciences Mathematiques, 88, Gauthier-Villars, Paris, 1937.

4. N. Aronszajn, *Rayleigh-Ritz and A. Weinstein methods for approximation of eigenvalues.* Proc. Nat. Acad. Sci. 34 (1948) 474-480 and 594-601.

5. N. Aronszajn and A. Weinstein, *On the unified theory of eigenvalues of plates and membranes.* Amer. J. Math. 64 (1942) 623-645.

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