INTERSECTION THEORY FOR CYCLES OF AN ALGEBRAIC VARIETY

I. Barsotti

Introduction. For a number of years intersection theory represented one of the most debated subjects in the field of algebraic geometry; also one of the main reasons for seeing in the whole structure of algebraic geometry an inherent flimsiness which even discouraged the study of this branch of mathematics. This situation came to an end when the methods of algebra began to be successfully applied to geometry, mainly by van der Waerden and Zariski; in the specific case of intersection theory, a completely general and rigorous treatment of the subject was given by Chevalley [3] in 1945. This rebuilding of algebraic geometry on firm foundations has often taken a form quite different from what the classical works would have led one to expect. Thus it is not surprising that Chevalley's solution of the problem has no evident link with the methods that, according to the suggestions of the classical geometers, should have been used in order to define the intersection multiplicity (for a sketch of these methods and suggestions see, for instance, [4]); rather, it is linked to the analytical approach, and it is therefore a strictly "local" theory, thus having the advantage of providing an intersection multiplicity also for algebroid varieties. The method by A. Weil [5] is another example of local theory.

The classical approach to the problem is illustrated in the introduction to [2] (see "first approach"), and carried out in the present paper. After an introduction dealing with algebraic correspondences (§1) we study in §2 a particular algebraic system related to any given cycle $\mathfrak{z}$ of a projective space, namely the system consisting of all the cycles obtained from $\mathfrak{z}$ by projective transformations of the ambient space, plus the "limit cycles" which must be added in order to complete the algebraic system (and which would correspond to the degenerate projective transformations). This system, called the homographic system of $\mathfrak{z}$, is used in §3 to obtain the principal results, namely Lemma 3.1 and Theorem 3.2. The wording of these results, as of the other results of §3, is complicated by the fact that we do not restrict ourselves to varieties over an algebraically

Received October 15, 1951. This paper is a result of research sponsored by the U.S. Navy (contract Nonr-10000 Office of Naval Research).

Pacific J. Math. 2 (1952), 473-521.

closed field, or to varieties in the sense of [5]; the gist of them, however, is the following:

Given the irreducible *cycles* $\mathfrak{y}$, $\mathfrak{z}$ of a projective space, let $\mathfrak{Z}$ be the "generic" element of the homographic system of $\mathfrak{z}$, and let P be an isolated component (of the right dimension) of the intersection of the *varieties* $\mathfrak{y}$ and $\mathfrak{z}$. Then the number of those intersections of the varieties $\mathfrak{y}$ and $\mathfrak{Z}$ which approach P when $\mathfrak{Z}$ approaches $\mathfrak{z}$ is, by definition, the intersection multiplicity of $\mathfrak{y}$ and $\mathfrak{z}$ at P; this number does not change if $\mathfrak{z}$ is allowed to vary in any "admissible" algebraic system rather than in its homographic system; and finally, the number is the same when $\mathfrak{z}$ varies in any algebraic system, provided that then we already count each intersection of $\mathfrak{y}$ and $\mathfrak{Z}$ with a certain multiplicity, to be computed by means of an "admissible" system. Also, the same number is obtained if $\mathfrak{y}$, or both $\mathfrak{y}$ and $\mathfrak{z}$, are allowed to vary.

The fact that we allow our varieties to be defined over an arbitrary field is not just a refinement of debatable usefulness, but a plain necessity: in fact, the general element of an algebraic system is *never* defined over an algebraically closed field (unless the system consists of just one element).

This definition takes care of the intersection of cycles of a projective space; the next step (carried out in §5) is the extension of the definition and of the related results to the cycles of an arbitrary (irreducible) variety V. Should it be possible to find, for any given cycle $\mathfrak{z}$ of V, an algebraic system of cycles of V, containing $\mathfrak{z}$, and playing the same role as the homographic system, then the theory on V would not differ from the theory on a projective space; more generally, it would be enough to find another cycle $\mathfrak{x}$ which does not contain the intersection U in which we are interested, and such that $\mathfrak{z} + \mathfrak{x}$ is contained in such a general algebraic system. Now, it is well known that this is not the case in general, but that one very wide class of cycles $\mathfrak{z}$ through U which fulfill the condition is the set of the cycles of V which are locally (at U) intersections of V and of a cycle of the ambient space; and this, in turn, is always the case if U is simple on V and the ground field is algebraically closed. As a consequence, we define the intersection multiplicity of $\mathfrak{y}$ and $\mathfrak{z}$ at U on V only for the case in which $\mathfrak{y}$ and $\mathfrak{z}$ are intersections, at U, of V with cycles Y, Z of the ambient space S; for this case the algebraic system containing $\mathfrak{z} + \mathfrak{x}$ (with $\mathfrak{x}$ not passing through U) which can be used in order to define the intersection multiplicity is the system of the intersections of V with the elements of the homographic system of $\mathfrak{Z}$; it is not even necessary, however, to consider this system: since the intersection of $\mathfrak{y}$ and $\mathfrak{Z}$ in S is already defined, the multiplicity of U in this intersection can be assumed to be, by definition, the multiplicity of U in the intersection of $\mathfrak{y}$ and $\mathfrak{z}$ on V. This is an outline of the content of §5, but one more detail needs to be mentioned here: it may happen, a priori at least, that although $\mathfrak{z}$ is not an intersection at U, it becomes such by a suitable birational transformation of V which is regular at U; this is taken into account after Theorem 5.9. Finally, since we are using rational cycles, it must be remarked that such cases as the vertex of a quadric cone are naturally taken care of by the theory: a line $\mathfrak{z}$ through the vertex U of a quadric cone V is the intersection at U of V with the cycle $\mathfrak{Z}/2$ of the 3-space containing V, $\mathfrak{Z}$ being the tangent plane to V along $\mathfrak{z}$.

Bezout's theorem is proved in §4 by means of one of the usual geometric methods, namely by letting the two cycles degenerate completely into cycles consisting of linear varieties only; other proofs of a more algebraic nature would display the relations of Bezout's theorem to that property of the divisors which is called the "product formula" by number theorists; the present proof, however, offers the advantage of being extremely simple.

The main advantage of the present geometrical theory of intersections is the fact that it can readily be applied to problems "in the large"; although throughout this paper the local intersection number is stressed, the theory finds easy and immediate application to the construction of the algebraic system determined by two cycles over any connected component of their intersection which happens to have a dimension larger than expected; in particular, the characteristic system of an irreducible subvariety of a variety and its virtual degree could easily be established. These topics, however, would find their natural place in a paper dealing with algebraic equivalence.

1. **Preliminary results.** We shall use the same definitions and notations as in [1] and [2], paying attention to the fact that some of the definitions or notations of [1] have been modified in [2]. A few additional modifications or generalizations will be explained now. In [1] "cycle" meant "integral effective cycle" (that is, with positive integers as coefficients); in [2] it meant "rational effective cycle"; it shall now mean "rational (effective or virtual) cycle". More precisely, a cycle is an expression of the form

$$\mathfrak{z} = \sum_{i=1}^{n} a_i V_i ,$$

where $n \geq 1$, the a_i's are nonzero rational numbers, and the V_i's are mutually distinct irreducible pseudosubvarieties of a pseudovariety over a field; $\mathfrak{z}$ is *unmixed* if all the V_i's have the same dimension (called the dimension of the cycle). The set of s-dimensional cycles becomes an additive group by addition of the zero cycle $0 = 0V$ for any s-dimensional irreducible pseudosubvariety V. The above expression $\sum_{i=1}^{n} a_i V_i$ is called the *minimal representation of* $\mathfrak{z}$; any expression $0V$ is a minimal representation of 0. If V is an s-dimensional irreducible pseudosubvariety, the *multiplicity of* V *in* $\mathfrak{z}$ is zero if $V \neq V_i$ for each i or if $\mathfrak{z} = 0$, and equals a_i if $V = V_i$. The cycle $\mathfrak{z}$ is *irreducible* if $n = a_1 = 1$. The identification, used in [1] and [2], of an irreducible cycle $\mathfrak{z} = 1V$ with the irreducible pseudovariety V is no longer valid. If $\sum_{i=1}^{n} a_i V_i$ is the minimal representation of the cycle $\mathfrak{z} \neq 0$, then each V_i is called a *component variety of* $\mathfrak{z}$, and each $1V_i$ is a *component of* $\mathfrak{z}$; the cycle $\mathfrak{y}$ whose minimal representation is $\sum_{j=1}^{m} b_j W_j$ is *part of* $\mathfrak{z}$ if $m \leq n$, and if it is possible to establish a $1-1$ correspondence $j \longrightarrow i(j)$ such that $a_{i(j)} = b_j$, $V_{i(j)} = W_j$ for $j = 1, \cdots, m$; the only part of 0 is 0.

If U is a subvariety of a projective space S over k, two cycles $\mathfrak{y}$, $\mathfrak{z}$ of S whose minimal representations are

$$\mathfrak{y} = \sum_{i=1}^{n} a_i V_i, \quad \mathfrak{z} = \sum_{j=1}^{m} b_j W_j$$

are said to *coincide locally at* U if either (1) no component of U is a subvariety of any V_i of of any W_j, or (2) if, say, $V_1, \cdots, V_r$ and $W_1, \cdots, W_s$ are the component varieties of $\mathfrak{y}$ and $\mathfrak{z}$ respectively which contain some component of U, then $r = s$, $V_i = W_i$ for $i = 1, \cdots, r$, and $a_i = b_i$ for $i = 1, \cdots, r$; the cycle

$$\sum_{i=1}^{r} a_i V_i = \sum_{j=1}^{s} b_j W_j$$

in case (2), or the cycle 0 in case (1), is called the *U-part of* $\mathfrak{z}$ (or *of* $\mathfrak{y}$); the *radical* rad $\mathfrak{z}$ *of* $\mathfrak{z}$ is the join of the component varieties of $\mathfrak{z}$ if $\mathfrak{z} \neq 0$, and is the empty variety if $\mathfrak{z} = 0$.

An algebraic correspondence is a cycle, not a pseudovariety. In the expressions $[D; V, G]$, $\{D; V, G]$, $(D; V, G)$, $D[G]$, $D(G)$, $\Delta[v]$, $\Delta(v)$, the symbols D and Δ are cycles, while the expressions themselves are pseudovarieties. In the expressions $\{D; V, G\}$, $\{D; V, G\}^*$, $D\{G\}$, $D\{G\}^*$, $\Delta\{v\}$, $\Delta\{v\}^*$, D and Δ are cycles, and so are the expressions themselves. In the expressions $e(D^*/D; V, G)$, $e(D^*/D; V, G)^*$, D is a cycle, D^* a pseudovariety. In the expressions ord $\mathfrak{z}$, deg $\mathfrak{z}$, red $\mathfrak{z}$, $\mathfrak{z}$ can be either a cycle or a variety; in the expressions ins $\mathfrak{z}$, exp $\mathfrak{z}$, $h(\mathfrak{z})$, $\mathfrak{z}$ can be either an irreducible cycle or an irreducible pseudovariety.

It is thus evident that if $\mathfrak{y}$, $\mathfrak{z}$ are cycles, then rad $\mathfrak{y} \cap$ rad $\mathfrak{z}$ is the variety which is the intersection of the varieties rad $\mathfrak{y}$ and rad $\mathfrak{z}$ (point-set theoretic), while $\mathfrak{y} \cap \mathfrak{z}$ has not been defined so far; and when it will be defined, it will be a cycle, not a variety.

Let V, F be varieties over k, F being irreducible, and let D be an unmixed algebraic correspondence between F and V, every component of which operates on the whole F; let G be an irreducible subvariety of F, D^* an irreducible component of $[D;\ V,\ G]$. The symbol $e(D^*/D;\ V,\ G)^*$ has been defined (when it exists) in [2] under the assumption that V and D be irreducible. We shall extend it now to a more general case. Let D be unmixed, and let $D = \sum_i a_i D_i$ be its minimal representation. Let v be a valuation of $k(F)$ over k, of the same dimension as G over k, and whose center on F is G; let $\{x^{(i)}\}$ be the h.g.p. (homogeneous general point) of D_i, and denote by $C_i(v)$ the complete set of extensions of v to $k(D_i)$ with respect to $\{x^{(i)}\}$ (see [2, § 3]). Assume $\dim D^* = \dim D - \dim F + \dim G$, and call $n_i(v)$ the number (≥ 0) of elements of $C_i(v)$ whose center on D_i is D^*. If

$$\sum_i a_i\, n_i(v) \operatorname{ins} D_i[F]\ (\operatorname{ord} D^*[G])^{-1}$$

does not depend on v, this number will be denoted by

$$e(D^*/D;\ V,\ G)^* = e(D^*/D;\ G,\ V)^*.$$

Clearly, if D' is another unmixed algebraic correspondence between F and V, having the same dimension as D, and if $e(D^*/D;\ V,\ G)^*$ and $e(D^*/D';\ V,\ G)^*$ both exist, then $e(D^*/aD + bD';\ V,\ G)^*$ exists and equals

$$ae(D^*/D;\ V,\ G)^* + be(D^*/D';\ V,\ G)^*$$

for any pair of rational numbers a, b. As a consequence of statement 5 of Theorem 3.1 of [2], we have the result: if $v_{ij}\ (j = 1, 2, \cdots)$ are the distinct elements of $C_i(v)$ whose center on D_i is D^*, then

$$(1)\qquad e(D^*/D;\ V,\ G)^* = \sum_{ij} a_i[\Gamma_{v_{ij}}:\ \Gamma_v]\,[K_{v_{ij}}:\ k(D^*)]\,[K_v:\ k(G)]^{-1}.$$

If D^* has the dimension $\dim D - \dim F + \dim G$, but it is not a component of $[D;\ V,\ G]$, then we set, by definition, $e(D^*/D;\ V,\ G)^* = 0$. This is in accordance with (1), since in this case no element of any $C_i(v)$ has the center D^* on D.

According to [2], instead of saying that $e(D^*/D;\ V,\ G)^* = \alpha$, we shall also say that α is the *multiplicity of* D^* *in* $\{D;\ V,\ G\}^*$, even if $\{D;\ V,\ G\}^*$ does not exist; this will be extended to the other expressions, like "$\mathfrak{z}$ is part of $\{D;\ V,\ G\}^*$" and similar ones.

Let $\sum_i a_i V_i$ be the minimal representation of an unmixed cycle $\mathfrak{y}$ over k. If K is an extension of k, and $V_{ij}\ (j = 1, 2, \cdots)$ are the distinct components of $(V_i)_K$, the extension of $\mathfrak{y}$ over K has been defined in [2] to be

$$\mathfrak{h}_K = \sum_{ij} a_i \exp V_i (\exp V_{ij})^{-1} V_{ij},$$

the exponent of V_{ij} being independent of j. This had the advantage that $\Psi_{t,y}\ \mathfrak{h} = \Psi_{t,y}\ \mathfrak{h}_K$, and that $\deg \mathfrak{h} = \deg \mathfrak{h}_K$. We shall often need, however, to consider the cycle

$$\mathfrak{h}' = \sum_{ij} a_i \text{ ins } V_i (\text{ins } V_{ij})^{-1} V_{ij};$$

this, as remarked in [2, §1] is an alternate definition of the extension of a cycle. The cycle $\mathfrak{h}'$ shall be called the *modified extension of* $\mathfrak{h}$ *over* K, and no special symbol will be used to denote it. We have ord $\mathfrak{h}'$ = ord $\mathfrak{h}$. Let finally $\mathfrak{Y}$ be a cycle over K. We say that $\mathfrak{Y}$ is a *partial extension of* $\mathfrak{h}$ *over* K if $\mathfrak{Y} = \sum_i \mathfrak{Y}_i$, where each component variety of $\mathfrak{Y}_i$ is a component of $(V_i)_K$, and a_i ord V_i = ord $\mathfrak{Y}_i$.

LEMMA 1.1. *Let D, D^*, F, V, G, k have the same meanings as in formula* (1). *Let F' be birationally equivalent to F, and such that if G' is any irreducible subvariety of F' which corresponds to G, and which has the same dimension as G, then $Q(G/F) \subseteq Q(G'/F')$. Let D' be the algebraic correspondence between F' and V such that $D'\{F'\}^* = D\{F\}^*$; for each G' let D_1^*, D_2^*, $\cdots$ be the pseudo-varieties which correspond to D^* and such that $1D_i^*$ operates on G', and assume F' to be such that $e(D_i^*/D'; V, G')^*$ exists for each G' and each i. Then $e(D^*/D; V, G)^*$ exists if and only if*

$$\alpha = \sum_i e(D_i^*/D'; V, G')^* \operatorname{ord}(1D_i^*)[G']$$

does not depend on G'; that is, if and only if $\sum_i e(D_i^*/D'; V, G')^*\ D_i^*$ is a partial extension of a fixed multiple of $1D^*$ over $k(G')$ for any G'. In such case, we have

$$e(D^*/D; V, G)^* = \alpha(\operatorname{ord}(1D^*)[G])^{-1}.$$

Proof. The proof of this lemma is an immediate application of (1), since the varieties G' are the centers on F' of the valuations v of formula (1).

COROLLARY. *Maintain the notations of Lemma 1.1, and let $\{\zeta\}$ be a set of parameters of $Q(G/F)$; then $\{\zeta\}$ is a set of parameters of each $Q(D^*/D_i)$. If $e(D^*/D; V, G)^*$ exists, it equals*

$$\sum_i a_i\ e(Q(D^*/D_i); \zeta)\ e(Q(G/F); \zeta)^{-1}.$$

Proof. In Lemma 1.1 choose for F' a normal associate to F, so that each $e(D_j^*/1D_i'; V, G')^*$ exists (by statement 1 of Theorem 5.3 of [2]) and equals $e(Q(D_j^*/D_i'); \zeta)\ e(Q(G'/F'); \zeta)^{-1}$. As a consequence of the lemma we then have

$$e(D^*/D\,;\,V,\,G)^* = \sum_{ij} a_i\; e(Q(D_j^*/D_i');\,\zeta)\; e(Q(G'/F');\,\zeta)^{-1} \times$$

$$\mathrm{ord}(1D_j^*)[G']\;(\mathrm{ord}(1D^*)[G])^{-1}$$

for any G'. There are finitely many varieties G' in this case, and we shall denote them by G_1', G_2', $\cdots$, while the D_j^*'s which operate on G_m' shall be denoted by D_{mj}^* $(j = 1, 2, \cdots)$. We have:

$$e(D^*/D;\,V,\,G)^* \sum_m e(Q(G_m'/F');\,\zeta)\,[k(G_m'):\,k(G)]\;\mathrm{ord}\,(1D^*)\,[G]$$

$$= \sum_i a_i \sum_{jm} e(Q(D_{mj}^*/D_i');\,\zeta)\;\mathrm{ord}\,(1D_{mj}^*)\,[G_m']\,[k(G_m'):\,k(G)],$$

or also

$$e(D^*/D\,;\,V,\,G)^* \sum_m e(Q(G_m'/F');\zeta)\,[k(G_m'):\,k(G)]$$

$$= \sum_i a_i \sum_{jm} e(Q(D_{mj}^*/D_i');\,\zeta)\,[k(D_{mj}^*):\,k(D^*)].$$

Now, by Lemma 2.2 of [2], we have

$$\sum_m e(Q(G_m'/F');\,\zeta)\,[k(G_m'):\,k(G)] = e(Q(G/F);\,\zeta)$$

and

$$\sum_{jm} e(Q(D_{mj}^*/D_i');\,\zeta)\,[k(D_{mj}^*):\,k(D^*)] = e(Q(D^*/D_i);\,\zeta),$$

Q.E.D.

We now maintain the same notations, and assume that V is irreducible and that each component of D, as well as $1D^*$, operates on the whole V. In this case $e(D^*/D;\,V,\,G)^*$ does not actually depend on D, but depends only on $D\{V\}^*$, by the above corollary, since $Q(D^*/D_i)$ contains $k(V)$. Accordingly, if Δ denotes $D\{V\}^*$ and Δ^* denotes $(1D^*)\,[V]$, we shall denote $e(D^*/D;\,V,\,G)^*$ also by $e(\Delta^*/\Delta)^*$. We remark that Δ^* can be described as a component of the intersection of rad Δ and $G_{k(V)}$ such that $1\Delta^*$ operates on the whole G. Let Δ_1^*, Δ_2^*, $\cdots$ be components of rad $\Delta \cap G_{k(V)}$ such that each $1\Delta_j^*$ operates on the whole G. If $\alpha_i = e(\Delta_i^*/\Delta)^*$ exists for each i, we shall say that $\sum_i \alpha_i \Delta_i^*$ is *part of the intersection* $G' \cap \Delta$ *of* G' *with* Δ, G' being the modified extension of $1G$ over $k(V)$. Notice that the symbol $\cap$ now links two cycles, so that no confusion may arise with rad $\Delta \cap$ rad G'. This notation, as will appear later, is in agreement with the general intersection theory.

LEMMA 1.2. *Let K be an algebraic function field over k, Δ an algebraic correspondence between K and an irreducible variety F over k, every component of which operates on the whole F. Let K' be an algebraic function field over k*

containing K, *and* Δ' *the modified extension of* Δ *over* K'. *Let* G *be an irreducible subvariety of* F, Z *and* Z' *the modified extensions of* $1G$ *over* K *and* K' *respectively. Let* Δ^* *be a component of* rad $\Delta \cap$ rad Z, *such that* $1\Delta^*$ *operates on the whole* G, *and let* Δ_i^* $(i = 1, 2, \cdots)$ *be the distinct components of* $\Delta^*_{K'}$; *then each* Δ_i^* *is a component of* rad $\Delta' \cap$ rad Z', *and each* $1\Delta_i^*$ *operates on the whole* G. *The multiplicity* $e(\Delta^*/\Delta)^*$ *exists if and only if* $e(\Delta_i^*/\Delta')^*$ *exists for some* i, *in which case this exists and is the same for each* i. *If this is the case, then the modified extension* $\Delta^{*\prime}$ *of* $e(\Delta^*/\Delta)^*\ \Delta^*$ *over* K' *is part of* $Z' \cap \Delta'$.

Proof. Obviously each Δ_i^* is a component of rad $\Delta' \cap$ rad Z', and dim Δ^* = dim Δ_i^* for each i. Therefore, if Δ^* has the dimension dim Δ + dim G − dim F, so does Δ_i^*, and conversely. The contention which needs to be proved is the last one. Now, if K' is purely transcendental over K, also this contention becomes obvious, since in such a case there is exactly one Δ_i^*. We shall therefore assume K' to be an algebraic extension of K. Again, a well-known artifice makes it possible to prove the last contention if it is known that it holds true for each K' which is normal over K. Hence we restrict our attention further to the case in which K' is normal over K (the word "normal" does not imply separability).

Under these assumptions, let v be a valuation of $k(F)$ over k of dimension equal to dim G, and whose center on F is G. Clearly we may further assume Δ to be irreducible. Let then w be an extension of v to $K(\text{rad } \Delta)$, having the center Δ^* on rad Δ; let Δ_i' $(i = 1, 2, \cdots)$ be the component varieties of Δ', and let w' be an extension of w to $K'(\Delta_1')$, whose center on Δ_1' will therefore be, say, Δ_1^*. Each automorphism σ of the Galois group $\mathfrak{G}$ of K' over K can be interpreted, in a natural way, as an operator which transforms, isomorphically and transitively, the fields $K'(\Delta_i')$ into each other. Then $\sigma w'$ has a meaning, and when σ ranges in $\mathfrak{G}$, $\sigma w'$ ranges among all the extensions of w to $K'(\Delta_i')$, for each i, while the centers of these range among all the Δ_j^*. As a consequence, $[\Gamma_{\sigma w'} : \Gamma_w]$ and $[K_{\sigma w'} : K_w]$ are the same for each σ. The ramification theory gives then

$$[\Gamma_{\sigma w'} : \Gamma_w]\,[K_{\sigma w'} : K_w] = [K'(\Delta_1') : K(\text{rad } \Delta)]\, m^{-1}\, n^{-1},$$

n being the number of distinct extensions of w to $K'(\Delta_1')$ whose center on Δ_1' is Δ_1^*, and m being the number of distinct $\sigma\Delta_1^*$ which are subvarieties of Δ_1'. Now, let $\alpha(w)$ be the sum of all the expressions $[\Gamma_{w''} : \Gamma_w]\,[K_{w''} : K'(\Delta_1^*)]$ when w'' ranges over the distinct extensions of w to $K'(\Delta_i')$ whose center on Δ_i' is Δ_1^*, and $i = 1, 2, \cdots$. If m' denotes the number of distinct Δ_i' which contain Δ_1^*, from what precedes we obtain

$$\alpha(w) = nm'[\Gamma_{w'} : \Gamma_w]\,[K_{w'} : K'(\Delta_1^*)]$$

$$= m'[K'(\Delta_1'):K(\mathrm{rad}\,\Delta)]\, m^{-1}\,[K_w:K(\Delta^*)]\,[K'(\Delta_1^*):K(\Delta^*)]^{-1}.$$

Now, there is the relation

$$m \times \text{number of distinct } \Delta_i' = m' \times \text{number of distinct } \Delta_j^*;$$

that is,

$$m' m^{-1} = \mathrm{red}\,\Delta\ \mathrm{red}\,\Delta_1^*(\mathrm{red}\,\Delta_1'\ \mathrm{red}\,\Delta^*)^{-1};$$

on the other hand,

$$[K'(\Delta_1^*):K(\Delta^*)] = \mathrm{ord}\,\Delta_1^*[K':K]\,(\mathrm{ord}\,\Delta^*)^{-1},$$

and likewise for $[K'(\Delta_1'):K(\mathrm{rad}\,\Delta)]$. Hence

$$\alpha(w) = \mathrm{ins}\,\Delta_1'\ \mathrm{ins}\,\Delta^*(\mathrm{ins}\,\Delta\ \mathrm{ins}\,\Delta_1^*)^{-1}\,[K_w:K(\Delta^*)].$$

If we denote by $\beta(v)$ the right side of formula (1), which would equal $e(\Delta^*/\Delta)^*$ if it were independent of v when $C(v/F) = G$, and by $\gamma(v)$ the similar expression for $e(\Delta_i^*/\Delta')^*$, then we have the relation:

$$\begin{aligned}\gamma(v) &= \textstyle\sum_w \mathrm{ins}\,\Delta(\mathrm{ins}\,\Delta_1')^{-1}\,\alpha(w)\,[\Gamma_w:\Gamma_v]\,[K_v:k(G)]^{-1}\\ &= \mathrm{ins}\,\Delta^*(\mathrm{ins}\,\Delta_1^*)^{-1}\textstyle\sum_w[\Gamma_w:\Gamma_v]\,[K_w:K(\Delta^*)]\,[K_v:k(G)]^{-1}\\ &= \mathrm{ins}\,\Delta^*(\mathrm{ins}\,\Delta_1^*)^{-1}\,\beta(v),\end{aligned}$$

where w ranges over all the extensions of v to $K(\mathrm{rad}\,\Delta)$ whose center on rad Δ is Δ^*. This proves that $\gamma(v)$ is independent of v if and only if $\beta(v)$ has the same property, and, because of (1), also proves all the statements of Lemma 1.2. Q.E.D.

THEOREM 1.1. *Let D be an unmixed algebraic correspondence between the irreducible variety F over k and the variety V over k, every component of which operates on the whole F. Let P and G be irreducible subvarieties of F, P also being a subvariety of G, and let D' be a component of $[D;\ V,\ P]$ such that $e(D'/D;\ V,\ P)^*$ exists. Let D_1^*, D_2^*, $\cdots$ be the components of $[D;\ V,\ G]$ which contain D'; then*

$$\dim D_i^* = \dim D - \dim F + \dim G.$$

Assume $e(D_i^*/D;\ V,\ G)^*$ to exist for each i, and set

$$D^* = \textstyle\sum_i e(D_i^*/D;\ V,\ G)^*\, D_i^*.$$

Then $e(D'/D^*;\ V,\ P)^*$ exists and equals $e(D'/D;\ V,\ P)^*$.

Proof. If $r = \dim D_i^*$, then we have $\dim D' \geq r - \dim G + \dim P$. Since

$\dim D' = \dim D - \dim F + \dim P$, it follows that $r \leq \dim D - \dim F + \dim G$, and therefore the equal sign must hold. This proves the statement concerning the dimension. We shall give a proof of the main result under the assumption that D is irreducible; the proof in the general case would proceed exactly in the same way.

Let v be a valuation of $k(F)$ over k, of dimension equal to dim G, whose center of F is G, and let w_1' be a valuation of K_v over k, of dimension equal to dim P, which compounded with v gives a valuation of $k(F)$, of dimension equal to dim P, and whose center on F is P. Let u be the valuation of $k(G) \subseteq K_v$ induced by w_1', and let $w_1', w_2', \cdots$ be the distinct extensions of u to K_v. Denote by w_i the valuation of $k(F)$ which is compounded of v and w_i', so that $\mathcal{C}(w_i/F) = P$. For each i, let $v_{i1}, v_{i2}, \cdots$ be the distinct extensions of v to $k(\text{rad } D)$ having the center D_i^* on rad D, and let $u_{i1}, u_{i2}, \cdots$ be the distinct extensions of u to $k(D_i^*)$ having the center D' on D_i^*. For given i, j, r, l, let w'_{lijrs} $(s = 1, 2, \cdots)$ be the distinct extensions of u_{ir} to $K_{v_{ij}}$ which induce w_l' in K_v, and call w_{lijrs} the valuation of $k(\text{rad } D)$ compounded of v_{ij} and w'_{lijrs}. For a given l, the w_{lijrs} are all the distinct extensions of w_l to $k(\text{rad } D)$ which have the center D' on D; therefore formula (1) gives

$$e(D'/D; P, V)^* [K_{w_l'} : k(P)] = \sum\nolimits_{ijrs} [\Gamma_{w_{lijrs}} : \Gamma_{w_l}] [K_{w'_{lijrs}} : k(D')];$$

now,

$$[\Gamma_{w_{lijrs}} : \Gamma_{w_l}] = [\Gamma_{v_{ij}} : \Gamma_v] [\Gamma_{w'_{lijrs}} : \Gamma_{w_l'}],$$

so that

$$\begin{aligned} &e(D'/D; P, V)^* [K_{w_l'} : K_u] [K_u : k(P)] [\Gamma_{w_l'} : \Gamma_u] \\ &= \sum\nolimits_{ijrs} [\Gamma_{v_{ij}} : \Gamma_v] [\Gamma_{w'_{lijrs}} : \Gamma_{u_{ir}}] [\Gamma_{u_{ir}} : \Gamma_u] [K_{w'_{lijrs}} : K_{u_{ir}}] \times \\ &\qquad [K_{u_{ir}} : k(D')]. \end{aligned}$$

We now sum with respect to l, and use the formulas

$$\sum\nolimits_l [K_{w_l'} : K_u] [\Gamma_{w_l'} : \Gamma_u] = [K_v : k(G)]$$

and

$$\sum\nolimits_{ls} [K_{w'_{lijrs}} : K_{u_{ir}}] [\Gamma_{w'_{lijrs}} : \Gamma_{u_{ir}}] = [K_{v_{ij}} : k(D_i^*)],$$

obtaining

$$e(D'/D;\, P,\, V)^*\,[K_u: k(P)]\,[K_v: k(G)]$$
$$= \sum_i \big(\sum_j [\Gamma_{v_{ij}}: \Gamma_v]\,[K_{v_{ij}}: k(D_i^*)]\big)\big(\sum_r [\Gamma_{u_{ir}}: \Gamma_u]\,[K_{u_{ir}}: k(D')]\big).$$

This proves Theorem 1.1, since

$$e(D_i^*/D;\, G,\, V)^*\,[K_v: k(G)] = \sum_j [\Gamma_{v_{ij}}: \Gamma_v]\,[K_{v_{ij}}: k(D_i^*)], \qquad \text{Q.E.D.}$$

It is hardly worth mentioning that if w is a valuation of $k(F)$ compounded of a valuation v of $k(F)$ and a valuation u of K_v, then

$$\Delta\{w\}^* = (\Delta\{v\}^*)\{u\}^*;$$

the proof of this fact is an immediate consequence of the obvious relation $\Delta\{w\} = (\Delta\{v\})\{u\}$. Another result which will be used later is the following: If Δ is an algebraic correspondence between the algebraic function field K over k and the variety V over k, let k' be an extension of k, K' a composite of K and k' over k (that is, the quotient field of the homomorphic image of $K \times k'$ over k modulo one of its prime ideals), Δ' the modified extension of Δ over K', so that Δ' is an algebraic correspondence between K' and $V' = V_{k'}$. If v is a valuation of K over k, v' any extension of v to K' over k', then $\Delta'\{v'\}^*$ is the modified extension of $\Delta\{v\}^*$ over $K_{v'}$. This fact also is derived from the analogous result concerning $\Delta\{v\}$, namely: if $\Delta' = \Delta_{K'}$, then $\Delta'\{v'\}$ is the extension of $\Delta\{v\}$ over $K_{v'}$.

Finally, the extension of the meaning of $e(D^*/D;\, V,\, G)^*$ to the case in which D is reducible, and in particular the corollary to Lemma 1.1, affords a generalization of the reduction theorem (Theorem 5.4 of [2]) in the following sense:

THEOREM. 1.2. *In the statement of Theorem* 4.2 *of* [2], *let us replace the assumption of the existence of* $\{D;\, V_j,\, W_i\}^*$ *and* $\{D_h^{(i)};\, W_j,\, W_i\}^*$ *by the following assumption:*

$$e(D_h^{(i)}/D;\, V_j,\, W_i)^* \text{ exists for each } h,\, i,$$

and if

$$D^{(i)} = \sum_h e(D_h^{(i)}/D;\, V_j,\, W_i)^*\, D_h^{(i)},$$

then $e(U/D^{(i)};\, W_j,\, W_i)^*$ *exists for each* i. *Let us replace, moreover, the assumption that* D *is irreducible by the assumption that* D *is unmixed. Then* $e(U/D^{(i)};\, W_j,\, W_i)^*$ *does not depend on* i.

2. The homographic system. An irreducible algebraic system $\mathfrak{C}$ of integral effective cycles is in one-to-one correspondence with the irreducible variety $G = G(\mathfrak{C})$ (see [1]); therefore we shall apply to $\mathfrak{C}$ the language adapted to varieties. For instance, if G is a variety over k, we shall write $k(\mathfrak{C})$ in place of $k(G)$, $M(\mathfrak{C})$ in place of $M(G)$ (this denotes the set of the places of G; see [1]); the cycle $\Delta = \Delta(\mathfrak{C})$ shall be referred to as the *general element* or *general cycle of* $\mathfrak{C}$.

A *linear variety* is an irreducible variety L over a field k such that ord $L = 1$, or, equivalently, such that deg $L = 1$. From the definition of order or degree [1, § 2; 2, § 1], it appears that an r-dimensional irreducible subvariety V of the projective space $S = S_n(k)$ is linear if and only if $\wp(V/k[X])$ has a basis consisting of linear (i.e. of degree 1) forms in the X's $\{X\}$ being the h.g.p. of S; and a minimal basis will consist then of n-r linear forms. After an obvious identification, it also follows that a linear variety is a projective space. A *linear cycle* is an irreducible cycle whose radical is a linear variety.

Let S be an n-dimensional projective space over k, $\{x\}$ its h.g.p., and let X denote the one-column matrix $(x_0, \cdots, x_n)$, while $U = (u_{ij})$ is a square matrix of order $n+1$ with elements in k. Set $X' = UX$, and let $x_0', \cdots, x_n'$ be the elements of the one-column matrix X'; let $\mathfrak{U}$ be the homomorphic mapping of $k[x]$ such that $\mathfrak{U}a = a$ if $a \in k$, $\mathfrak{U}x_i = x_i'$ $(i = 0, \cdots, n)$; if det $U \neq 0$, $\mathfrak{U}$ is an automorphism and transforms in an obvious way an ideal of $k[x]$ into an ideal of $k[x]$, a subvariety of S into a subvariety of S, and a cycle of S into a cycle of S. U will be called the *matrix* of $\mathfrak{U}$; two $\mathfrak{U}$'s whose matrices have proportional elements have the same effect on homogeneous ideals, subvarieties, and cycles, and shall be identified; $\mathfrak{U}$ is called a *nondegenerate homography of* S. If $\mathfrak{z}$ is a cycle of S, then $\mathfrak{U}\mathfrak{z}$ is called a *homographic transform of* $\mathfrak{z}$.

Maintaining the same notations, assume the u_{ij}'s to be indeterminates; then $\mathfrak{U}$ is a nondegenerate homography of $S_n(k(u))$, and will be referred to as the *general homography of* S. Set $K' = k(u)$, so that K' is homogeneous for the set $\{u_{00}, \cdots, u_{nn}\}$; let K be the subfield of K' consisting of all the homogeneous elements of degree zero of K'. If $\mathfrak{z}$ is an unmixed cycle of S, set $\mathfrak{Z}' = \mathfrak{U}\mathfrak{z}_{K'}$; then $\mathfrak{Z}'$ is a cycle of $S_{K'}$, and it is the extension over K' of a cycle $\mathfrak{Z}$ of S_K. Clearly $\mathfrak{Z}$ is an unmixed algebraic correspondence between K and S, and is called the *general homographic transform of* $\mathfrak{z}$. Assume $\mathfrak{z}$ to be integral and effective; if $\bar{k}$ is the algebraic closure of k, and $\bar{\mathfrak{z}}$, $\bar{\mathfrak{Z}}$ are the extensions of $\mathfrak{z}$, $\mathfrak{Z}$ over $\bar{k}$, $K\bar{k}$ respectively, then $\bar{\mathfrak{Z}}$ is related to $\bar{\mathfrak{z}}$ as $\mathfrak{Z}$ is to $\mathfrak{z}$, and the set $\mathfrak{H}$ of the cycles $\bar{\mathfrak{Z}}\{v\}$, where v ranges over the places of $K\bar{k}$ over $\bar{k}$, is an algebraic system of cycles on[1] S, called the *homographic system of* $\mathfrak{z}$.

[1] Note that, according to [1] or [2], a cycle *on* S means a cycle *of* the extension of S over the algebraic closure of k.

LEMMA 2.1. *The homographic system of* $\mathfrak{z}$ *is the smallest algebraic system of cycles on* S *containing all the homographic transforms of* $\overline{\mathfrak{z}}$.

Proof. Set $\overline{K} = K\overline{k}$, and let $v \in M(\overline{K})$. Let u_{00} be such that $v(u_{ij}\, u_{00}^{-1}) \geq 0$ for every i, j. Let σ be the homomorphic mapping of R_v whose kernel is $\mathfrak{P}_v$, and set $u_{ij}(v) = \sigma(u_{ij}\, u_{00}^{-1})$; since u_{00} is not necessarily the only u_{rs} such that $v(u_{ij}\, u_{rs}^{-1}) \geq 0$ for each i, j, the set $\{u_{ij}(v)\}$ is determined but for a nonzero factor in $\overline{k}$. Let $U(v)$ be the matrix obtained after replacing, in U, each u_{ij} by the corresponding $u_{ij}(v)$: if $\det U(v) \neq 0$, then $U(v)$ is the matrix of a nondegenerate homography $\upsilon(v)$. These notations will be used throughout this section.

We contend that $\upsilon(v)\,\overline{\mathfrak{z}} = \overline{\mathfrak{Z}}\{v\}$, and this will completely prove the lemma. Let $\psi(t, y)$ be a determination of $\Psi t, y\overline{\mathfrak{z}} = \Psi t, y\mathfrak{z}$ (see [1, §2]); denote by Y the one-column matrix $(y_0, \cdots, y_{r+1})$, r being the dimension of $\mathfrak{z}$, and by T the matrix (t_{ij}), so that $Y = TX$; υ can be extended in a natural way to $\overline{k}(t, u, x)$, and we have

$$\upsilon Y = \upsilon(TX) = T(\upsilon X) = TUX = (\tau T)\,X = \tau(TX) = \tau Y,$$

where by τ we denote the automorphism of $\overline{k}(t, u, x)$ over $\overline{k}(u, x)$ such that $\tau T = TU$. If v has the previous meaning, $T(v)$ and $\tau(v)$ will be related to T, τ, v as $U(v)$, $\upsilon(v)$ are to U, υ, v. If $\overline{\mathfrak{z}}$ is irreducible, set

$$\mathfrak{p} = \wp(\operatorname{rad} \overline{\mathfrak{z}}/\overline{k}[x])\,\overline{K}'(t)\,[x],$$

where $\overline{K}' = K'\overline{k}$; we have, by definition,

$$\psi(t, y)\,\overline{K}'(t)\,[y] = \mathfrak{p} \cap \overline{K}'(t)\,[y],$$

hence

$$\psi(t, \upsilon y)\,\overline{K}'(t)\,[\upsilon y] = \upsilon\,\mathfrak{p} \cap \overline{K}'(t)\,[\upsilon y].$$

Applying τ^{-1}, and using the fact that $\upsilon y = \tau y$, we obtain

$$\psi(\tau^{-1}\, t, y)\,\overline{K}'(t)\,[y] = \upsilon\,\mathfrak{p} \cap \overline{K}'(t)\,[y],$$

which proves that $\psi(\tau^{-1}\, t, y)$ is a determination of $\Psi_{t,y}\,\overline{\mathfrak{Z}}'$; hence $\psi(\tau^{-1}(v)\, t, y)$ is a determination of $\Psi_{t,y}\,\overline{\mathfrak{Z}}\{v\}$. But, since

$$\tau^{-1}(v) = (\tau(v))^{-1},$$

we see in like manner that $\psi(\tau^{-1}(v), y)$ is a determination of $\Psi_{t,y}\,\upsilon(v)\,\overline{\mathfrak{z}}$. It is thus proved that $\upsilon(v)\,\overline{\mathfrak{z}} = \overline{\mathfrak{Z}}\{v\}$ if $\overline{\mathfrak{z}}$ is irreducible. If $\overline{\mathfrak{z}}$ is not irreducible, the same relation is easily established as a consequence of its validity for irreducible cycles, Q.E.D.

LEMMA 2.2. *The homographic system of $\mathfrak{z}$ contains the homographic system of each of its cycles.*

Proof. Let $\mathfrak{H}$ be the homographic system of $\mathfrak{z}$, and let $\mathfrak{z}_1 \in \mathfrak{H}$, so that

$$\Psi_{t,y}\ \mathfrak{z}_1 = \psi(\tau^{-1}(v)\ t, y)$$

(but for a proportionality coefficient) for some $v \in M(\overline{K})$; here $\tau^{-1}(v)t_{ij}$ has to be interpreted as the ij-th element of the matrix $TU^{-1}(v)$, which has a meaning even if $\det U(v) = 0$. Let $\mathfrak{Z}_1'$, $\mathfrak{Z}_1$ be obtained from $\mathfrak{z}_1$ as $\overline{\mathfrak{Z}}'$, $\overline{\mathfrak{Z}}$ are from $\overline{\mathfrak{z}}$; we have

$$\Psi_{t,y}\ \mathfrak{Z}_1' = \psi(\tau^{-1}(v)\ \tau^{-1} t, y).$$

For any $v' \in M(K)$ we have therefore

$$\Psi_{t,y}\ \mathfrak{Z}_1\{v'\} = \psi(\tau^{-1}(v)\ \tau^{-1}(v')\ t, y).$$

Now, there exists a place $v'' \in M(\overline{K})$ such that

$$\tau^{-1}(v)\ \tau^{-1}(v')T = \tau^{-1}(v'')\ T,$$

so that

$$\Psi_{t,y}\ \mathfrak{Z}_1\{v'\} = \psi(\tau^{-1}(v'')\ t, y) = \Psi_{t,y}\ \overline{\mathfrak{Z}}\{v''\},$$

or $\mathfrak{Z}_1\{v'\} = \overline{\mathfrak{Z}}\{v''\}$, Q.E.D.

LEMMA 2.3. *Let $\mathfrak{z}$ be an unmixed integral effective cycle of $S = S_n(k)$, and let $\mathfrak{Z}$ be the general homographic transform of $\mathfrak{z}$. Set $G = G_{\mathfrak{Z}}$ (see [1], Lemma 4.2); let Δ be the algebraic correspondence between G and S induced by $\mathfrak{Z}$ according to Lemma 4.2 of [1], and set $Z = D_{\Delta,G}$. Let $\overline{k}$ be the algebraic closure of k, and let P be a point of G such that $(Z\{P\})_{\overline{k}}$ is a homographic transform of $\mathfrak{z}_{\overline{k}}$. Then G is analytically irreducible at P.*

Proof. Let $\mathfrak{H}$ be the homographic system of $\mathfrak{z}$, and set $\overline{G} = G(\mathfrak{H})$; then $\overline{G}$ is a component of the extension of G over the algebraic closure $\overline{k}$ of k. Assume the lemma to be true when k is algebraically closed. In this case, $\overline{G}$ is analytically irreducible at each $\overline{P} \in \overline{G}$ such that $\overline{P}$ is the image point of a homographic transform of $\overline{\mathfrak{z}} = \mathfrak{z}_{\overline{k}}$. Let P be the point mentioned in the lemma, $R = Q(P/G)$, $\overline{P}$ the image (on $\overline{G}$) of $(Z\{P\})_{\overline{k}}$, $\overline{R} = Q(\overline{P}/\overline{G})$. If $\mathfrak{m} = \mathfrak{P}(P/G)$, $\overline{\mathfrak{m}} = \mathfrak{P}(\overline{P}/\overline{G})$, we have that $\mathfrak{m}\overline{k}$ is a primary ideal of $R\overline{k}$ belonging to $\overline{\mathfrak{m}} \cap R\overline{k}$, and that $\overline{\mathfrak{m}}^h \cap R \subseteq \mathfrak{m}^l$, where $l \longrightarrow \infty$ when $h \longrightarrow \infty$. Therefore the topology induced in R by the $\bar{R}$-topology is the R-topology, so that the completion R' of R is a subring of the completion $\bar{R}'$ of $\bar{R}$. Since, by assumption, $\bar{R}'$ is an integral domain, so is R'; that is, G is analytically irreducible at P. This shows that it is enough to prove the statement under the further assumption that k be algebraically closed.

Under this assumption, let $\mathfrak{z}'$ be a homographic transform of $\mathfrak{z}$, and set $P = P(\mathfrak{z})$, $P' = P(\mathfrak{z}')$, so that $\mathfrak{z} = Z\{P\}$, $\mathfrak{z}' = Z\{P'\}$. Let K have the previous usual meaning. For each $v_0 \in M(K)$ whose center on G is P' we have $\det U(v_0) \neq 0$; let π be the automorphism of $k(u)$ over k such that $\pi U = U^{-1}(v_0)\, U$. We have, for $v \in M(K)$:

$$\Psi_{t,y}\, \mathfrak{Z}\{\pi v\} = \psi(\tau^{-1}(\pi v)\, t, y).$$

Now,

$$\tau T = TU, \quad \tau^{-1} T = TU^{-1},$$

$$\tau^{-1}(\pi v)\, T = TU^{-1}(\pi v) = T(\pi^{-1} U^{-1})(v) = TU^{-1}(v)\, U^{-1}(v_0)$$

$$= \tau^{-1}(v)\, \tau^{-1}(v_0)\, T,$$

so that

$$\Psi_{t,y}\, \mathfrak{Z}\{\pi v\} = \psi(\tau^{-1}(v)\, \tau^{-1}(v_0)\, t, y).$$

On the other hand, as we have already seen, $\Psi_{t,y}\, \upsilon(v_0)\, \mathfrak{Z}\{v\}$ is obtained from

$$\Psi_{t,y}\, \mathfrak{Z}\{v\} = \psi(\tau^{-1}(v)\, t, y)$$

by replacing $\{t\}$ with $\{\tau^{-1}(v_0)\, t\}$, so that

$$\Psi_{t,y}\, \upsilon(v_0)\, \mathfrak{Z}\{v\} = \psi(\tau^{-1}(v)\, \tau^{-1}(v_0)\, t, y) = \Psi_{t,y}\, \mathfrak{Z}\{\pi v\}.$$

It follows that

$$\mathfrak{Z}\{\pi v\} = \upsilon(v_0)\, \mathfrak{Z}\{v\},$$

and this proves that $\mathfrak{C}(\pi v/G)$ depends only on $\mathfrak{C}(v/G)$. Then the same is true for $\mathfrak{C}(v/\pi^{-1} G)$ and $\mathfrak{C}(v/G)$. Let H be the smallest subfield of K containing $k(G)$ and $\pi^{-1}(k(G)) = k(\pi^{-1} G)$; the embedding of $k(G)$ and $k(\pi^{-1} G)$ in H gives an irreducible algebraic correspondence C between G and $\pi^{-1}G$, and the above-proved property shows that C has the same dimension as G, and that $k(\text{rad } C)$ is purely inseparable over $k(G)$. Besides, if $P = P\{\mathfrak{z}\} \in G$, then $C[P]$ is the point $\pi^{-1}P'$ of $\pi^{-1}G$, and $P = C[\pi^{-1}P']$. Now, by Lemma 2.1, P' can be chosen in such a way that G is analytically irreducible at P', and therefore $\pi^{-1}G$ is analytically irreducible at $\pi^{-1}P'$. Let G^* be a normal associate to G, C^* the irreducible algebraic correspondence between $\pi^{-1}G$ and G^* generated by the embedding of $k(\pi^{-1}G)$ and $k(G^*)$ in H. Should G be not analytically irreducible at P, $C^*[\pi^{-1}P']$ would contain two distinct points, which is impossible by Theorem 4.1 of [1]. Hence G is analytically irreducible at P. By Lemma 2.1, however, we can choose for P the image of any cycle $\mathfrak{z}''$ of $\mathfrak{H}$ whose homographic system is $\mathfrak{H}$, Q.E.D.

THEOREM 2.1. *Maintain the same notation as in Lemma 2.3. If V is an irreducible subvariety of S, then $Z\{V\}$ exists, and each component of the total transform $\{Z; V, G]$ operates on the whole V.*

Proof. Let D be a component of $\{Z; V, G]$, P a point of V on which $1D$ operates, and assume

$$\dim D > \dim Z - \dim S + \dim V.$$

If then D' is a component of $[D; P, G]$, we have

$$\dim D' \geq \dim D - \dim V > \dim Z - \dim S,$$

and D' belongs to $(Z; P, G)$. If, therefore, we show that each component of $[Z; P, G]$ has dimension equal to $\dim Z - \dim S$, it is also proved that each component of $\{Z; V, G]$ has dimension equal to $\dim Z - \dim S + \dim V$, and that as a consequence $Z\{V\}$ exists, because V is simple on S (see statement f of Lemma 4.2 of [1]). In order to show that $\{Z; P, G]$ has the pure dimension $\dim Z - \dim S$, we proceed as follows: let $\bar{k}$ be the algebraic closure of k, and let $\mathfrak{Z}$ be the general homographic transform of $\mathfrak{z}$; let K have the usual meaning, and set $\bar{K} = K\bar{k}$, $\bar{\mathfrak{Z}} = \mathfrak{Z}_{\bar{K}}$, $\bar{\mathfrak{z}} = \mathfrak{z}_{\bar{k}}$; let $\bar{G}$, $\bar{Z}$ be related to $\bar{\mathfrak{Z}}$ as G, Z are to $\mathfrak{Z}$, so that $\bar{G}$ is a component of the extension of G over $\bar{k}$. Let $P_1, P_2, \cdots$ be the components of $P_{\bar{k}}$; we have $\bar{Q} \in [\bar{Z}; P_i, \bar{G}]$ for some i if and only if there exists a $Q \in [Z; P, G]$ such that $\bar{Q}$ is a component of $Q_{\bar{k}}$. Therefore $[Z; P, G]$ has the pure dimension $\dim Z - \dim S$ if and only if each $[\bar{Z}; P_i, \bar{G}]$ has the same property. As a consequence, it is sufficient to prove the statement under the further assumption that k is algebraically closed. Under this assumption, let $P' \in S$, and let π be a non-degenerate homography of S such that $\pi P = P'$. Let M be the matrix of π, so that $\pi X = MX$ (X being the one-column matrix $(x_0, \cdots, x_n)$). Let σ be the automorphism of $k(u)$ over k such that $\sigma U = MU$. Then it is possible to prove (by the same method used in the proof of Lemma 2.3) the following: if $v \in M(K)$ and $P \in \operatorname{rad}(\mathfrak{Z}\{v\})$, then $P' \in \operatorname{rad}(\mathfrak{Z}\{\sigma^{-1}v\})$; in other words, $Z[P']$ is the total transform of $\sigma^{-1}Z[P]$ in the algebraic correspondence C (between $\sigma^{-1}G$ and G) generated by the embedding of $k(G)$ and $k(\sigma^{-1}G)$ in K. Now, C is the same as the algebraic correspondence C used in the proof of Lemma 2.3, concerning which it was proved that it does not have fundamental points either on G or on $\sigma^{-1}G$. Therefore C has no fundamental variety either on G or on $\sigma^{-1}G$. Since P' can be chosen in such a way that $Z[P']$ has the pure dimension $\dim Z - \dim S$, it follows that $Z[P]$ also has the pure dimension $\dim Z - \dim S$, as asserted.

Suppose that a component D of $[Z; V, G]$ operates on $W \subset V$, so that it is also a component of $[Z; W, G]$. From the above proof it follows that

$$\dim D = \dim Z - \dim S + \dim W < \dim Z - \dim S + \dim V,$$

a contradiction, Q.E.D.

We say that a cycle or a subvariety $\mathfrak{z}$ of S is *degenerate* if each component of $\mathfrak{z}$ is a linear cycle or subvariety.

LEMMA 2.4. *The homographic system of an unmixed cycle of $S = S_n(k)$ contains some degenerate cycle.*

Proof. We may assume k to be algebraically closed, since we are dealing with an algebraic system. In view of Lemma 2.2, the statement is true if it is true when $\mathfrak{z}$ is irreducible. Therefore we assume $\mathfrak{z}$ to be irreducible. Set $r = \dim \mathfrak{z}$, and let F be a linear subvariety of S such that rad $\mathfrak{z} \cap F$ consists of finitely many points; we also require F to have dimension $n - r$. Such an F certainly exists, because by repeated application of the theorem according to which each minimal prime of a principal ideal is maximal dimensional, one can easily establish that the intersection of rad $\mathfrak{z}$ with a linear subvariety of S of dimension s has dimension $\geq r + s - n$, and that there exists some s-dimensional linear subvariety of S whose intersection with rad $\mathfrak{z}$ has the pure dimension $r + s - n$ if this number is not negative.

Let $\{x\}$ be the h.g.p. of S, and let $l_1, \cdots, l_r$ be the linear forms in the x's forming a basis of $\wp(F/k[x])$. The system of equations $l_i = 0 (i = 1, \cdots, r)$ can be solved for r among the x's, say $x_{n-r+1}, \cdots, x_n$, and the solution is written in the form

$$x_i = \sum_{j=0}^{n-r} a_{i-n+r,j}\, x_j \qquad (a_{p,q} \in k,\ i = n - r + 1, \cdots, n).$$

Let U' be the square matrix

$$\begin{pmatrix} 1 & 0 & \cdot & \cdot & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & \cdot & \cdot & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ a_{1,0} & \cdot & \cdot & \cdot & a_{1,n-r} & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{r,0} & \cdot & \cdot & \cdot & a_{r,n-r} & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix}$$

of order $n+1$. Let $v \in M(K)$ be such that $U(v) = U'$, and set

$$\mathfrak{p} = \wp(\operatorname{rad} \mathfrak{z}/k[x]).$$

Let T be the projective space over k whose h.g.p. is $\{u\}$, and set

$$u'_{ij} = u_{ij}\, u_{00}^{-1},$$

so that v is at finite distance for $\{u'\}$. If $\{p_1(x), p_2(x), \cdots\}$ is a basis of $\mathfrak{p}$, set

$$x'_i = \sum_j u'_{ij}\, x_j,$$

and let $\mathfrak{D}$ be the radical of the ideal of $k[x, u']$ whose basis is

$$\{p_1(x'), p_2(x'), \cdots\}.$$

If $D = \wp(\mathfrak{D})$, then $1D$ is an algebraic correspondence between T and S, and it differs from $Z = D_{\mathfrak{Z},T}$ at most for components which do not operate on the whole T. Set

$$P = \mathfrak{C}(v/T), \quad \mathfrak{q} = \mathfrak{C}(v/k[u']),$$

and let σ be the homomorphic mapping of $k[x, u']$ whose kernel is $\mathfrak{q}k[x, u']$. Then $\{\sigma x'\}$ is the h.g.p. of F, and $\{p(\sigma x')\}$ is the basis of an ideal of $k[x]$ whose radical is $\wp((1D)[P]/k[x])$. However, since $\{\sigma x'\}$ is the h.g.p. of F, $\{p(\sigma x')\}$ is also the basis of an ideal of $k[x_0, \cdots, x_{n-r}]$ whose radical $\mathfrak{R}$ is $\wp(\operatorname{rad} \mathfrak{z} \cap F/k[x_0, \cdots, x_{n-r}])$; therefore $\mathfrak{R}$ is purely 0-dimensional. Also, $\mathfrak{R}$ can be extended to an ideal $\mathfrak{R}k[x]$ of $k[x]$, and

$$\mathfrak{R}k[x] = \wp((1D)[P]/k[x]).$$

Now, $\mathfrak{R}k[x]$ is purely r-dimensional; besides, each minimal prime of $\mathfrak{R}$, being a 0-dimensional ideal of $k[x_0, \cdots, x_{n-r}]$, has a basis consisting of linear forms in the x's with coefficients in k, and the same must be true of each minimal prime of $\mathfrak{R}k[x]$. This proves that $(1D)[P]$ is a degenerate r-dimensional variety. Since $Z[P] \subseteq (1D)[P]$, and since each component of $Z[P]$ has dimension $\geq r$, this also proves that (1) $Z[P]$ is purely r-dimensional, and (2) $Z[P]$ is degenerate. From (1), and from the fact that T is locally normal at P, follows that $Z[P] = \mathfrak{Z}[v]$, so that (2) implies that $\mathfrak{Z}[v]$, which is the radical of a cycle of the homographic system of $\mathfrak{z}$, is degenerate, Q.E.D.

LEMMA 2.5. *Maintain the notations of Theorem* 2.1, *and assume k to be algebraically closed, and $\mathfrak{z}$ to be irreducible. Then $Z[V]$ is irreducible.*

Proof. Since, by Theorem 2.1, $Z\{V\}$ exists, and so does $Z\{P\}$ if $P \in V$, by Theorem 1.1 we have that $(Z\{V\})\{P\}$ exists, and that it is enough to prove the lemma under the additional assumption that V is a point. Besides, the same argument used in the proof of Theorem 2.1 shows that $Z[P]$ is either irreducible for each $P \in G$, or reducible for each $P \in G$. Set $D = D_{\mathfrak{Z},T}$, T having the same meaning as in the proof of Lemma 2.4. In order to prove that $Z[P]$ is irreducible, it is enough to prove that $D[P]$ is irreducible for some (hence for each) $P \in S$. Let W be the subvariety of T consisting of the centers on T of those $v \in M(T)$ for which det $U(v) = 0$. We shall show first that if it is true that $D[P]$ has only one component outside W for $P \in S$, then it is also true that $D[P]$ is irreducible. In fact, let $\mathfrak{A}$ be the prime algebraic system of cycles of T whose general element is $D\{S\}$ (after extending it over $k(S)$). If $D[P]$ is reducible for each $P \in S$, then $\mathfrak{A}$ is not simple; according to Theorem 5.4 of [1], $\mathfrak{A}$ is then composed with a simple algebraic system $\mathfrak{A}'$ and an involution $\mathfrak{I}$ on $G(\mathfrak{A}')$; $\mathfrak{A}'$ contains cycles which have no component variety on W (because not every element of $\mathfrak{A}$ has the radical in W), and $\mathfrak{I}$ contains cycles which have no component variety in any one given proper subvariety of $G(\mathfrak{A}')$. Therefore $\mathfrak{A}$ contains cycles which have no component variety in W, and this proves that for some (hence for each) $P \in S$, $D[P]$ is irreducible, as claimed.

For any point $Q \in T - W$ we shall write $\upsilon(Q)$ instead of $\upsilon(v)$, $v \in M(T)$, $\mathfrak{C}(v/T) = Q$. Then $D[P] - (D[P] \cap W)$ consists of the $Q \in T - W$ such that $\upsilon^{-1}(Q) \in$ rad $\mathfrak{z}$. Let $\mathfrak{P}$ be the general homographic transform of P constructed with the general homography υ^{-1} (rather than υ), and set $E = D_{\mathfrak{P},T}$; then $\upsilon^{-1}(Q)P = E[Q]$ if $Q \in T - W$, so that $D[P] - (D[P] \cap W) = L - (L \cap W)$, where L is the subvariety of T on which $E[\text{rad } \mathfrak{z}]$ operates. If we prove that $E[\text{rad } \mathfrak{z}]$ is irreducible, it will follow that L is irreducible, as desired. Now, the same argument used at the beginning of this proof shows that $E[\text{rad } \mathfrak{z}]$ is irreducible if $E[P']$ is irreducible for some (hence for each) $P' \in S$, or also if $E[P']$ has only one component outside W for a $P' \in S$, say $P' = P$. But this is obviously true, since the set of the $Q \in T - W$ for which $\upsilon^{-1}(Q)P = P$, that is, for which $\upsilon(Q)P = P$, is a linear variety less its intersection with W, Q.E.D.

THEOREM 2.2. *Notations as in Theorem* 2.1. *Set* $n = \dim S$, $r = \dim \mathfrak{z}$, $s = \dim V$. *If* $r + s - n \geq 0$, *then each component of* $[Z;\ V,\ G]$ *operates on the whole* G, $V \cap \operatorname{rad} \mathfrak{z}$ *is not empty, and each of its components has dimension* $\geq r + s - n$.

Proof. The proof of this result, like that of Theorem 2.1, is readily reduced to the case in which k is algebraically closed and $\mathfrak{z}$ is irreducible. In this case, according to Lemma 2.5, $D = [Z;\ V,\ G]$ is irreducible, and, by Theorem 2.1, $D = \{Z;\ V,\ G]$. If P is a point of G such that $Z\{P\} = \mathfrak{z}'$, then $V \cap \operatorname{rad} \mathfrak{z}' = (1D)[P]$ by Theorem 2.1. Set $d = \dim D$, and let F be the irreducible subvariety of G on which $1D$ operates. Then $d = r + m - n + s$, where $m = \dim G$. Therefore, $(1D)[P]$ is empty if $P \notin F$, while if $P \in F$ each component of $(1D)[P]$, hence of $V \cap \operatorname{rad} \mathfrak{z}$, has dimension $\geq r + s - n + m - \dim F$. By Lemma 2.4, the homographic system $\mathfrak{H}$ of $\mathfrak{z}$ contains some degenerate cycle $\mathfrak{z}''$, and therefore, by Lemma 2.2, it contains the homographic system $\mathfrak{H}'$ of $\mathfrak{z}''$. According to the first part of the proof of Lemma 2.4, $\mathfrak{H}'$ contains some cycle $\mathfrak{z}_0$ such that $V \cap \operatorname{rad} \mathfrak{z}_0$ is nonempty and has pure dimension $r + s - n$. If $P_0 \in G$ is such that $\mathfrak{z}_0 = Z\{P_0\}$, it follows that $P_0 \in F$ and that

$$r + s - n + m - \dim F \leq r + s - n,$$

that is, that $\dim F = m$, $F = G$. Hence $1D$ operates on the whole G, as claimed, and each component of $(1D)[P]$, for any P, has dimension $\geq r + s - n$, Q.E.D.

3. **Intersection of cycles of a projective space.** *In this section* S *denotes an* n*-dimensional projective space over* k.

If $\mathfrak{y}$, $\mathfrak{z}$ are unmixed cycles of S, of dimensions r, s respectively, such that $r + s - n \geq 0$, then a component V of $\operatorname{rad} \mathfrak{z} \cap \operatorname{rad} \mathfrak{y}$ is said to be a *component variety of* $\mathfrak{z} \cap \mathfrak{y}$ or *of* $\mathfrak{y} \cap \mathfrak{z}$ if $\dim V = r + s - n$.

Let $\mathfrak{y}$, $\mathfrak{z}$ be unmixed integral effective cycles of S, of dimensions r, s respectively; assume V to be a component variety of $\mathfrak{y} \cap \mathfrak{z}$, and let $\mathfrak{y} = \sum_i a_i \mathfrak{y}_i$ be the minimal representation of $\mathfrak{y}$. Let $\mathfrak{Z}$ be the general homographic transform of $\mathfrak{z}$, $G = G_{\mathfrak{Z}}$, Z the algebraic correspondence between $k(G)$ and S induced by $\mathfrak{Z}$ according to Lemma 4.2 of [1]. Let P be the (unique) point of G such that $Z\{P\} = \mathfrak{z}$. Then P is a rational point, so that $V \times P$ is irreducible, and $V \times P$ is a pseudosubvariety of $\{Z;\ \mathfrak{y}_i,\ G]$ for some i; Theorems 2.1 and 2.2 imply then that $V \times P$ is a component of $[\{Z;\ \mathfrak{y}_i,\ G\};\ \mathfrak{y}_i,\ P]$ for some i. Now assume $Z = \sum_j c_j Z_j$ to be a minimal representation of Z, and let $\mathfrak{z}_1, \mathfrak{z}_2, \cdots$ be the distinct component varieties of $\mathfrak{z}$. Then each $\mathfrak{z}_i \times P$ is component of exactly one $[1Z_j; S, P]$, say $[1Z_{j(i)}; S, P]$; and if some Z_j is such that $[1Z_j; S, P]$ has more than one component, say $\mathfrak{z}_1$ and $\mathfrak{z}_2$, then $h(\mathfrak{z}_1) = h(\mathfrak{z}_2)$. This being established, set $c_j^* = c_j h(\mathfrak{z}_i)(h((1Z_j)[G]))^{-1}$, i being such that $j(i) = j$; set also $Z^* = \sum_j c_j^* Z_j$. Then we have $Z^*\{P\}^* = Z\{P\} = \mathfrak{z}$. By Theorem 2.1, $\{Z^*; \mathfrak{y}_i, G\}^*$ exists for each i. Since G is analytically irreducible at P by Lemma 2.3,

$$\alpha_i = e(V \times P/\{Z^*; \mathfrak{y}_i, G\}^*; \mathfrak{y}_i, P\}^*$$

exists for each i by Theorem 5.3 of [2]. The number $\sum_j a_j \alpha_j$ is denoted by $i(V, \mathfrak{y} \cap \mathfrak{z}, S)$ and called the *intersection multiplicity of* $\mathfrak{y}$ *with* $\mathfrak{z}$ *at* V *on* S. We set $i(V, \mathfrak{y} \cap \mathfrak{z}, S) = 0$ if $\dim V = r + s - n$ but V is not a subvariety of rad $\mathfrak{y} \cap$ rad $\mathfrak{z}$. If each component V_j of rad $\mathfrak{y} \cap$ rad $\mathfrak{z}$ has the dimension $r + s - n$, we set

$$\mathfrak{y} \cap \mathfrak{z} = \sum_j i(V_j, \mathfrak{y} \cap \mathfrak{z}, S) V_j ;$$

$\mathfrak{y} \cap \mathfrak{z}$ is called the *intersection of* $\mathfrak{y}$ *with* $\mathfrak{z}$ *on* S (although S does not appear, at this stage, in the symbol $\mathfrak{y} \cap \mathfrak{z}$). Evidently, if $i(V, \mathfrak{y}_1 \cap \mathfrak{z}, S)$ and $i(V, \mathfrak{y}_2 \cap \mathfrak{z}, S)$ both exist and have the same dimension, then $i(V, (\mathfrak{y}_1 + \mathfrak{y}_2) \cap \mathfrak{z}, S)$ also exists, and equals $\sum_j i(V, \mathfrak{y}_j \cap \mathfrak{z}, S)$.

A cycle $\mathfrak{W}$ of S whose minimal representation is $\mathfrak{W} = \sum_j e_j W_j$ is said to be a *part of* $\mathfrak{y} \cap \mathfrak{z}$ (whether $\mathfrak{y} \cap \mathfrak{z}$ exists or not) if (1) each W_j is a component variety of $\mathfrak{y} \cap \mathfrak{z}$, and (2) $e_j = i(W_j, \mathfrak{y} \cap \mathfrak{z}, S)$. The same cycle $\mathfrak{W}$ is said *to coincide locally at* U *with* $\mathfrak{y} \cap \mathfrak{z}$ (U being a subvariety of S) if (1) each W_j which contains some component of U is a component variety of $\mathfrak{y} \cap \mathfrak{z}$, (2) each component of rad $\mathfrak{y} \cap$ rad $\mathfrak{z}$ which contains some component of U coincides with some W_j, and (3) $e_j = i(W_j, \mathfrak{y} \cap \mathfrak{z}, S)$ for each W_j which contains some component of U. Also, $\mathfrak{y} \cap \mathfrak{z}$ is said *to exist locally* at U if $i(V, \mathfrak{y} \cap \mathfrak{z}, S)$ exists for each component V of rad $\mathfrak{y} \cap$ rad $\mathfrak{z}$ which contains some component of U; the *local part of* $\mathfrak{y} \cap \mathfrak{z}$ *at* U is $\sum_j i(X_j, \mathfrak{y} \cap \mathfrak{z}, S) X_j$, where X_j ranges among all the components of rad $\mathfrak{y} \cap$ rad $\mathfrak{z}$ each of which contains some component of U.

LEMMA 3.1. *Let* $\mathfrak{y}$, $\mathfrak{z}$ *be unmixed integral effective cycles of* $S = S_n(k)$, *of dimensions* r, s *respectively. If* $r + s - n \geq 0$, *let* V *be a component variety of* $\mathfrak{y} \cap \mathfrak{z}$. *Let* θ *be an unmixed algebraic correspondence between an algebraic function field* K *over* k *and* S, *such that the set* $N(\theta)$ *of the* $v \in M(K)$ *for which* $\theta\{v\}^*$ *is the modified extension of* $\mathfrak{z}$ *over* K_v *is nonempty. If* $\mathfrak{y}'$ *is the modified extension of* $\mathfrak{y}$ *over* K, *let* Λ_j $(j = 1, 2, \cdots)$ *be the component varieties of* $\mathfrak{y}' \cap \theta$, *and set*

$$\Lambda_\theta = \sum_j i(\Lambda_j, \mathfrak{y}' \cap \theta, S_K) \Lambda_j .$$

If $v \in N(\theta)$, *then a partial extension of* $i(V, \mathfrak{y} \cap \mathfrak{z}, S)\, V$ *over* K_v *is part of* $\Lambda_\theta\{v\}^*$.

Proof. The statement is clearly true if it is true when $\mathfrak{y}$ is irreducible; accordingly, we shall assume $\mathfrak{y}$ to be irreducible, and put $Y = \text{rad}\ \mathfrak{y}$.

In order to avoid lengthy repetitions, we shall say that the set $\{K, \theta\}$ is "admissible" if (1) every component of θ operates on the whole S, (2) $N(\theta)$ is not empty, and (3) each component of rad $\theta \cap Y_K$ has dimension $r + s - n$ and operates on the whole Y. And we shall say that an admissible set $\{K, \theta\}$ is "satisfactory" if the following statement is true: Set

$$\Gamma_\theta = \sum_j e(\Lambda_j/\theta)^* \Lambda_j \ ;$$

then, for each $v \in N(\theta)$, a partial extension of $i(V, \mathfrak{y} \cap \mathfrak{z}, S)\, V$ over K_v is part of $\Gamma_\theta\{v\}^*$.

Step 1. Let $\mathfrak{Z}$ be the general homographic transform of $\mathfrak{z}$, $G = G_{\mathfrak{Z}}$, $K = k(G)$, θ' the algebraic correspondence between K and S induced by $\mathfrak{Z}$ according to Lemma 4.2 of [1]. If $\theta' = \sum_j a'_j \theta_j$ is the minimal representation of θ', set

$$D_j = D_{1\theta_j, G}\,,$$

and let $P \in G$ be such that $\sum_j a'_j D_j\{P\} = \mathfrak{z}$. Set

$$a_j = a'_j h(\mathfrak{z}_i)\,(h(\theta_j))^{-1}\,,$$

$\mathfrak{z}_i$ being any component variety of $D_j\{P\}$; finally, put

$$\theta = \sum_j c_j\, \theta_j\,.$$

Then clearly $\{K, \theta\}$ is admissible by Theorems 2.1, 2.2, and $N(\theta)$ is the set of the $v \in M(K)$ whose center on G is P. If $\Gamma = \Gamma_\theta$ and $C = D_{\Gamma, G}$, then by definition we have

$$i(V, \mathfrak{y} \cap \mathfrak{z}, S) = e(V \times P/C\,;\, Y, P)^*;$$

if $v \in N(\theta)$, by formula (1) and by the corollary to Theorem 5.1 of [2] it follows that $\Gamma\{v\}^*$ is a partial extension of $C\{P\}^*$, and therefore $\{K, \theta\}$ is satisfactory. This is the contention of Step 1.

Step 2. Let K^* be an algebraic function field over K, θ^* the modified extension of θ over K^*. By means of Lemma 1.2 it is a simple matter to prove that $\{K^*, \theta^*\}$ is admissible if and only if $\{K, \theta\}$ is admissible; in this case, $N(\theta^*)$ consists of the extensions to K^* of the elements of $N(\theta)$; and clearly, if $\{K^*,$

$\theta^*\}$ and $\{K, \theta\}$ are admissible, then $\{K^*, \theta^*\}$ is satisfactory if and only if $\{K, \theta\}$ is such.

Step 3. We work again with two sets $\{K, \theta\}$ and $\{K^*, \theta^*\}$, on which we make the following assumptions: (1) if $G = G_\theta$, $G^* = G_{\theta^*}$, then $K = k(G)$, $K^* = k(G^*)$; (2) $\{K, \theta\}$ is (admissible and) satisfactory; (3) if $Z = D_{\theta, G}$, $Z^* = D_{\theta^*, G^*}$, then $G \subseteq G^*$ and $Z = \{Z^*; S, G\}^*$. We wish to prove that $\{K^*, \theta^*\}$ is admissible and satisfactory.

Clearly each component of θ^* operates on the whole S. If $N = N(\theta)$, $N^* = N(\theta^*)$, let $v \in N$, and let w be a valuation of K^* whose dimension equals dim G, and such that $\mathcal{C}(w/G^*) = G$. Then any valuation of K^* compounded of w and of an extension of v to K_w belongs to N^*, so that N^* is nonempty. Let C_i^* be a component of $\{Z^*; Y, G^*]$ such that $1\,C_i^*$ operates on the whole G^*, and let Y' be the subvariety of Y on which $1\,C_i^*$ operates. Since $(1\,C_i^*)\,[G^*]$ is a component of $Z^*[G^*] \cap Y_{K^*}$, by Theorem 2.2 it has dimension $\geq r+s-n$, so that dim $C_i^* \geq r+s-n+\dim G^*$. Let C_j be a component of $[1\,C_i^*; Y, G]$, so that

$$\dim C_j \geq \dim C_i^* - \dim G^* + \dim G \geq r + s - n + \dim G.$$

Since C_j is also a pseudosubvariety of $\{Z; Y, G]$, and since $1\,C_j$ operates on the whole G, by assumption C_j is also a pseudosubvariety of $[Z; Y, G]$, and therefore

$$\dim C_j \leq r + s - n + \dim G.$$

This proves that

$$\dim C_j = r + s - n + \dim G;$$

hence C_j is a component of $[Z; Y, G]$ and $1\,C_j$ operates on the whole G; therefore $1\,C_j$ operates on the whole Y, and the same must be true of $1\,C_i^*$. As a consequence, $\{K^*, \theta^*\}$ is admissible. We remark that we have also proved that G is not fundamental for $1\,C_i^*$.

Let now G'^* be a normal associate to G^*, and call G' an irreducible subvariety of G'^* which corresponds to G in the birational correspondence between G^* and G'^*. Set

$$Z'^* = D_{\theta^*, G'^*}.$$

If $C_i'^*$ is any component of $\{Z'^*; Y, G'^*]$, and if $1\,C_i'^*$ operates on the whole G'^*, since $\{K^*, \theta^*\}$ is admissible and S is normal we have that $e(C_j'^*/Z'^*; Y, G'^*)^*$ exists. Set

$$C'^* = \textstyle\sum_i e(C_i'^*/Z'^*; Y, G'^*)^* \, C_i'^*.$$

Then $\Gamma^* = \Gamma_{\theta*}$ equals $C'^*\{G'^*\}^*$. Set also $Z' = \{Z'^*; S, G'\}^*$, so that $Z'\{G'\}^*$ is the modified extension of $Z\{G\}^*$ over $k(G')$. Since G is not fundamental for $1C_i^*$ (as previously remarked), G' is not fundamental for C'^*, and G'^* is locally normal at G'. Hence, by Theorem 4.1 of [1], $C' = \{C'^*; Y, G'\}^*$ exists. The component varieties C_j' of C' are those components A of $\{Z'; Y, G'\}$ such that $1A$ operates on the whole G'; but then C_j', Z'^*, Y, G' can replace respectively U, D, W_1, W_2 in Theorem 1.2, and the result is that C' equals the cycle obtained from $[Z'; Y, G']$ in the same way as C'^* is obtained from $[Z'^*; Y, G'^*]$.

Let C be obtained in the same way from $[Z; Y, G]$; then

$$\Gamma = \Gamma_\theta = C\{G\}^*,$$

and, by Lemma 1.2, $C'\{G'\}^*$ is the modified extension of Γ over $k(G')$.

If $v^* \in N^*$ and $P = \mathbb{C}(v^*/G^*)$, then $Z^*\{P\}^*$ is the modified extension of $\mathfrak{z}$ over $k(P)$; therefore $P \in G$; hence $P' = \mathbb{C}(v^*/G'^*)$ belongs to one of the irreducible subvarieties of G'^* (say G') which correspond to G. Since $C'\{G'\}^*$ is the modified extension of Γ over $k(G')$, and because of formula (1), there are components V_j $(j = 1, 2, \cdots)$ of $V \times P$ such that $e(V_j/C'; Y, P')^*$ exists, and such that

$$i(V, \mathfrak{h} \cap \mathfrak{z}, S) \operatorname{ord} V = \textstyle\sum_j e(V_j/C'; Y, P')^* \operatorname{ord}(1V_j)\{P'\}^*.$$

Hence each V_j is a component of $[C'^*; Y, P']$, and $e(V_j/C'^*; T, P')^*$ exists since G'^* is normal (see Theorem 5.3 of [2]). But then Theorem 1.1 yields

$$e(V_j/C'^*; Y, P')^* = e(V_j/C'; Y, P')^*.$$

As a consequence, a partial extension of

$$\textstyle\sum_j e(V_j/C'^*; Y, P')^* (1V_j)\{P'\}^*$$

(which is also a partial extension of $i(V, \mathfrak{h} \cap \mathfrak{z}, S)V$) over K_{v*} is part of $\Gamma^*\{v^*\}^*$. This means that $\{K^*, \theta^*\}$ is satisfactory, as announced.

Step 4. If $\{K, \theta\}$ is the set given in the statement of Lemma 3.1, let θ' be the general homographic transform of $\mathfrak{z}$, and set

$$K' = k(\cdots, u_{ij}u_{00}^{-1}, \cdots),$$

the u_{ij}'s playing the usual role in the definition of θ'. Let K^*, θ^* be obtained from K, θ as K', θ' are from k, $\mathfrak{z}$, and by means of the same u_{ij}'s. Set also

$$G = G_{\theta'}, \quad G^* = G_{\theta*}, \quad K_1 = k(G), \quad K_1^* = k(G^*),$$

and let θ_1, θ_1^* be the algebraic correspondences between respectively, K_1, K_1^*, and S of which θ', θ^* are modified extensions over respectively, K' and K^*.

If $v \in N = N(\theta)$ and w is the unique extension of v over K^*, since $\theta\{v\}^*$ is the modified extension of $\mathfrak{z}$ over K_v it follows that $\theta^*\{w\}^*$ is the modified extension of θ' over K_w. This also means that $G \subseteq G^*$, and that $Z_1 \doteq \{Z_1^*; S, G\}^*$, where we have set

$$Z_1 = D_{\theta_1, G}, \quad Z_1^* = D_{\theta_1^*, G^*}.$$

By Step 1, $\{K_1, \theta_1\}$ is satisfactory; since $\{K_1^*, \theta_1^*\}$ has been shown to be related to $\{K_1, \theta_1\}$ as $\{K^*, \theta^*\}$ is related to $\{K, \theta\}$ in Step 3, it follows that $\{K_1^*, \theta_1^*\}$ is satisfactory. Step 2 implies then that $\{K^*, \theta^*\}$ is satisfactory.

Step 5. Let K, θ, K^*, θ^*, N have the same meanings as in Step 4, and set

$$N^* = N(\theta^*), \quad \Gamma^* = \Gamma_{\theta^*}.$$

Let w be a valuation of K^* over K such that $u_{ij}(w) = 0$ if $i \neq j$, $u_{ii}(w) = 1$, and $K_w = K$. If $v \in N$, let v^* be the place of K^* over k which is compounded of w and v, so that $v^* \in N^*$. Since $\{K^*, \theta^*\}$ is satisfactory by Step 4, a modified extension of $i(V, \mathfrak{y} \cap \mathfrak{z}, S)\, V$ over $K_{v^*} = K_v$ is part of

$$\Gamma^*\{v^*\}^* = (\Gamma^*\{w\}^*)\{v\}^*,$$

Since w is a place of K^* over K, and $K_w = K$, and since θ^* is the general homographic transform of θ, the following statement is true by definition: If Λ_j is a component variety of $\mathfrak{y}' \cap \theta$, then $i(\Lambda_j, \mathfrak{y}' \cap \theta, S_K)\, \Lambda_j$ is part of $\theta^*\{w\}^*$. As a consequence, $\Lambda = \Lambda_\theta$ is part of $\Gamma^*\{w\}^*$, and its component varieties are all the components of rad $\mathfrak{y}' \cap$ rad θ' of dimension $r + s - n$. If V_l is a component of the extension of V over K_v, and if Λ' is a component variety of $\Gamma^*\{w\}^*$ such that $(1\,\Lambda')\{v\}^*$ has V_l as a component variety, certainly

$$\dim \Lambda' = \dim V_l = r + s - n;$$

that is, Λ' is a component variety of Λ. This proves that $(\Gamma^*\{w\}^*)\{v\}^*$ and $\Lambda\{v\}^*$ coincide locally at V_l; hence a partial extension of $i(V, \mathfrak{y} \cap \mathfrak{z}, S)\, V$ over K_v is part of $\Lambda\{v\}^*$, Q.E.D.

REMARK 1. Maintaining the same notations, by comparing Steps 3 and 5 we see that if $\{K, \theta\}$ is admissible, then $\Gamma_\theta = \Lambda_\theta$, and $\{K, \theta\}$ is satisfactory.

REMARK 2. Remark 1 shows that the use of the word "intersection" and of the symbol $\cap$ in Lemma 1.2 agrees with the present definition of intersection of cycles.

REMARK 3. Remark 1 also shows that in defining the intersection $\mathfrak{h} \cap \mathfrak{z}$, any admissible set $\{K, \theta\}$ can be used in place of the set $\{K, \theta\}$ of Step 1 of the proof of Lemma 3.1. Step 1 itself shows that admissible sets do exist.

THEOREM 3.1. *If $\mathfrak{h}_1$, $\mathfrak{h}_2$ are r-dimensional cycles of S, and $\mathfrak{z}_1$, $\mathfrak{z}_2$ are s-dimensional cycles of S, and V is a component variety of*

$$(\mathfrak{h}_1 + \mathfrak{h}_2) \cap (\mathfrak{z}_1 + \mathfrak{z}_2),$$

then

$$i(V, (\mathfrak{h}_1 + \mathfrak{h}_2) \cap (\mathfrak{z}_1 + \mathfrak{z}_2), S) = \sum\nolimits_{lj} i(V, \mathfrak{h}_l \cap \mathfrak{z}_j, S).$$

Proof. (See Remark 4 at the end of this proof). Assume $\mathfrak{h}_j$, $\mathfrak{z}_j$ $(j = 1, 2)$ to be integral effective cycles. By definition,

$$i(V, (\mathfrak{h}_1 + \mathfrak{h}_2) \cap (\mathfrak{z}_1 + \mathfrak{z}_2), S) = \sum\nolimits_l i(V, \mathfrak{h}_l \cap (\mathfrak{z}_1 + \mathfrak{z}_2), S).$$

Hence it is enough to prove that if $\mathfrak{h}$ denotes either $\mathfrak{h}_l$, then

$$i(V, \mathfrak{h} \cap (\mathfrak{z}_1 \cap \mathfrak{z}_2), S) = \sum\nolimits_j i(V, \mathfrak{h} \cap \mathfrak{z}_j, S).$$

Now, let θ_j be the general homographic transform of $\mathfrak{z}_j$, and set

$$K = k(\cdots, u_{ij}\, u_{00}^{-1}, \cdots).$$

Then $\theta_1 + \theta_2$ is the general homographic transform of $\mathfrak{z}_1 + \mathfrak{z}_2$. In the notations of Lemma 3.1 and its proof,

$$N = N(\theta_1 + \theta_2) = N(\theta_1) \cap N(\theta_2),$$

and

$$\{K, \theta_j\} \ (j = 1, 2), \ \{K, \theta_1 + \theta_2\}$$

are satisfactory. We also have

$$\Gamma_{\theta_1 + \theta_2} = \Gamma_{\theta_1} + \Gamma_{\theta_2},$$

so that Lemma 3.1 itself and Remark 3 imply

$$i(V, \mathfrak{h} \cap (\mathfrak{z}_1 + \mathfrak{z}_2), S) = \sum\nolimits_j i(V, \mathfrak{h} \cap \mathfrak{z}_j, S), \text{ Q.E.D.}$$

REMARK 4. So far, Theorem 3.1 has a meaning only if $\mathfrak{h}_j$, $\mathfrak{z}_j$ $(j = 1, 2)$ are integral effective cycles. This particular case is sufficient, however, to give a meaning to $i(V, \mathfrak{h} \cap \mathfrak{z}, S)$ when $\mathfrak{h}$, $\mathfrak{z}$ are rational virtual cycles: in fact, for some integer m it is true that

$$m\mathfrak{y} = \mathfrak{y}_1 - \mathfrak{y}_2, \quad m\mathfrak{z} = \mathfrak{z}_1 - \mathfrak{z}_2,$$

where $\mathfrak{y}_j$, $\mathfrak{z}_j$ $(j = 1, 2)$ are integral effective cycles; Theorem 3.1 shows that the number

$$m^{-1}[i(V, \mathfrak{y}_1 \cap \mathfrak{z}_1, S) - i(V, \mathfrak{y}_1 \cap \mathfrak{z}_2, S) - i(V, \mathfrak{y}_2 \cap \mathfrak{z}_1, S) + i(V, \mathfrak{y}_2 \cap \mathfrak{z}_2, S)]$$

depends only on V, $\mathfrak{y}$, $\mathfrak{z}$. This number will be denoted by $i(V, \mathfrak{y} \cap \mathfrak{z}, S)$ and called the *intersection multiplicity of* $\mathfrak{y}$ *with* $\mathfrak{z}$ *at* V *on* S; all the other notations and definitions concerning $\mathfrak{y} \cap \mathfrak{z}$ are extended likewise. With this definition it is easily proved that Theorem 3.1 remains true in general. As a matter of fact, Lemma 3.1 itself remains true after removing the assumption that $\mathfrak{y}$ and $\mathfrak{z}$ are integral effective cycles.

Let $\mathfrak{z}$ be an unmixed r-dimensional cycle of an irreducible n- dimensional variety U over k, and let V be an irreducible subvariety of U of dimension $\leq r$, $R = Q(V/U)$; let $\mathfrak{z} = \sum_i b_i \mathfrak{z}_i$ be the minimal representation of $\mathfrak{z}$, and set $\mathfrak{p}_i = \mathfrak{P}(\mathfrak{z}_i/U) \cap R$ for each i such that $R \subseteq Q(\mathfrak{z}_i/U)$ (that is, such that $V \subseteq \mathfrak{z}_i$). We say that $\mathfrak{z}$ is a *complete intersection at* V *on* U if there exists a subset $\{\zeta\}$ of a set of parameters of R such that (1) the $\mathfrak{p}_i$'s are all the distinct minimal primes of the ideal of R whose basis is $\{\zeta_1, \zeta_2, \cdots\}$, and (2) we have

$$b_i = e(R_{\mathfrak{p}_i}; \zeta) = e(Q(\mathfrak{z}_i/U); \zeta)$$

for each i for which $\mathfrak{p}_i$ exists. Any such set $\{\zeta\}$ is called a *set of representatives of* $\mathfrak{z}$ *at* V *on* U. Also, $\{\zeta\}$ is assumed to consist of units of R if $V \not\subseteq \operatorname{rad} \mathfrak{z}$. A complete intersection at V obviously coincides, locally at V, with an integral effective cycle.

LEMMA 3.2. *Maintain the notations of Lemma* 3.1; *assume* $\mathfrak{z}$ *to be irreducible, and* $\mathfrak{y}$ *to be a complete intersection at* V *on* S. *Let* $\{\zeta\}$ *be a set of representatives of* $\mathfrak{y}$ *at* V *on* S, *and set* $\mathfrak{p} = \mathfrak{P}(\operatorname{rad} \mathfrak{z}/S) \cap Q(V/S)$. *If* σ *is the homomorphic mapping of* $Q(V/S)$ *whose kernel is* $\mathfrak{p}$, *then* $\{\sigma\zeta\}$ *is a set of parameters of* $Q(V/\operatorname{rad} \mathfrak{z})$, *and* $i(V, \mathfrak{y} \cap \mathfrak{z}, S) = e(Q(V/\operatorname{rad} \mathfrak{z}); \sigma\zeta)$.

Proof. Let θ be the general homographic transform of $\mathfrak{z}$,

$$K = k(\cdots, u_{ij} u_{00}^{-1}, \cdots),$$

T the projective space whose h.g.p. is $\{u\}$, P the point of T at which $u_{ij} = 0$ for $i \neq j$, $u_{ii} = 1$. Let $\{\eta\}$ be a regular set of parameters of $Q(P/T)$. Set $Z = D_{\theta, T}$, and let $\mathfrak{y} = \sum_j a_j \mathfrak{y}_j$ be the minimal representation of $\mathfrak{y}$. Set $C_j = \{Z; \mathfrak{y}_j, T\}^*$. If C_{jl} ($l = 1, 2, \cdots$) are the component varieties of C_j, then from the corollary to Lemma 1.1 follows that

$$C_j = \sum_l e(Q(C_{jl}/\text{rad } Z); \zeta)\, e(Q(\mathfrak{y}_j/S); \zeta)^{-1}\, C_{jl}.$$

According to Lemma 3.1 and its proof, we also have, by Theorem 3.1:

$$\begin{aligned} i(V, \mathfrak{y} \cap \mathfrak{z}, S) &= \sum_j a_j\, e(V \times P/C_j; \mathfrak{y}_j, P)^* \\ &= \sum_{jl} a_j\, e(Q(V \times P/C_{jl}); \eta)\, e(Q(P/T); \eta)^{-1}\, e(Q(C_{jl}/\text{rad } Z); \zeta) \\ &\qquad \times e(Q(\mathfrak{y}_j/S); \zeta)^{-1}. \end{aligned}$$

Since $a_j = e(Q(\mathfrak{y}_j/S); \zeta)$, this also gives

$$i(V, \mathfrak{y} \cap \mathfrak{z}, S) = \sum_{jl} e(Q(V \times P/C_{jl}); \eta)\, e(Q(C_{jl}/\text{rad } Z); \zeta).$$

The ideals $\mathfrak{P}(C_{jl}/\text{rad } Z) \cap Q(V \times P/\text{rad } Z)$ are all the minimal primes of the ideal of $Q(V \times P/\text{rad } Z)$ whose basis is $\{\zeta\}$; therefore the associativity formula (Theorem 2.1 of [2]) gives

$$i(V, \mathfrak{y} \cap \mathfrak{z}, S) = e(Q(V \times P/\text{rad } Z), \zeta, \eta).$$

The only minimal prime of the ideal of $Q(V \times P/\text{rad } Z)$ whose basis is $\{\eta\}$ is the ideal $\mathfrak{P}(\text{rad } \mathfrak{z} \times P/\text{rad } Z) \cap Q(V \times P/\text{rad } Z)$, and

$$e(Q(\text{rad } \mathfrak{z} \times P/\text{rad } Z); \eta) = e(\text{rad } \mathfrak{z} \times P/\text{rad } Z; S, P)^* = 1;$$

therefore, if τ denotes the homomorphic mapping of $Q(V \times P/\text{rad } Z)$ whose kernel is said prime, the associativity formula gives

$$i(V, \mathfrak{y} \cap \mathfrak{z}, S) = e(Q(V \times P/\text{rad } \mathfrak{z} \times P; \tau\zeta) = e(Q(V/\text{rad } \mathfrak{z}); \sigma\zeta), \text{ Q.E.D.}$$

Notice that the fact expressed in Lemma 3.2 is the basic reason for which $\Gamma_\theta = \Lambda_\theta$ when θ is admissible (see Remark 1 and the proof of Lemma 3.1).

LEMMA 3.3. *Let $\mathfrak{y}$, $\mathfrak{z}$ be unmixed cycles of S, and let V be a component variety of $\mathfrak{y} \cap \mathfrak{z}$. Let Δ be the general homographic transform of $\mathfrak{y}$,*

$$K = k(\cdots, u_{ij}\, u_{00}^{-1}, \cdots),$$

v any place of K over k such that $K_v = k$, $u_{ij}(v) = 0$ if $i \neq j$, $u_{ii}(v) = 1$. If $\mathfrak{z}'$ is the modified extension of $\mathfrak{z}$ over K, and Λ_j ($j = 1, 2, \cdots$) are all the component varieties of $\Delta \cap \mathfrak{z}'$, set

$$\Lambda = \sum_j i(\Lambda_j, \Delta \cap \mathfrak{z}', S_K)\, \Lambda_j.$$

Then $i(V, \mathfrak{y} \cap \mathfrak{z}, S)\, V$ *is part of* $\Lambda\{v\}^*$.

Proof. Assume first $\mathfrak{y}$ and $\mathfrak{z}$ to be integral effective cycles. Let u'_{ij} be the reciprocal element of u_{ij} in the matrix $U = (u_{ij})$; if $\{X\}$ is the h.g.p. of S, let σ be the non-degenerate homography of S_K such that

$$\sigma X_i = \sum_j u'_{ij}\, u_{00}\, X_j.$$

Then $\sigma\Delta$ is the modified extension $\mathfrak{y}'$ of $\mathfrak{y}$ over K, and $\theta = \sigma\, \mathfrak{z}$ is the general homographic transform of $\mathfrak{z}$. If Λ_j is a component variety of $\Delta \cap \mathfrak{z}'$, then $\sigma\Lambda_j$ is a component variety of $\mathfrak{y}' \cap \theta$, and

$$i(\Lambda_j, \Delta \cap \mathfrak{z}', S_K) = i(\sigma\Lambda_j, \mathfrak{y}' \cap \theta, S_K).$$

If

$$\psi(\cdots, u_{ij}\, u_{00}^{-1}, \cdots, t, y) = \Psi_{t,y}\, \Lambda,$$

and τ is constructed from $\{u'_{ij}\, u_{00}\}$ as τ is constructed from $\{u_{ij}\}$ in § 2, then by § 2 we have

$$\Psi_{t,y}(\sigma\Lambda) = \psi(\cdots, u_{ij}\, u_{00}^{-1}, \cdots, \tau^{-1}t, y).$$

If we replace here each u_{ij} by $u_{ij}(v)$, we obtain

$$\Psi_{t,y}((\sigma\Lambda)\{v\}^*) = \psi(u_{ij}(v), t, y) = \Psi_{t,y}(\Lambda\{v\}^*),$$

which goes to show that

$$(\sigma\Lambda)\{v\}^* = \Lambda\{v\}^*.$$

But $i(V, \mathfrak{y} \cap \mathfrak{z}, S)\, V$ is part of $(\sigma\Lambda)\{v\}^*$ by Lemma 3.1, so that it is also part of $\Lambda\{v\}^*$, as asserted.

If $\mathfrak{y}$ or $\mathfrak{z}$ are not integral effective cycles, the proof of Lemma 3.3 is easily derived from the above special case, Q.E.D.

THEOREM 3.2. *Let* K *be an algebraic function field over* k, Δ *and* θ *two unmixed algebraic correspondences between* K *and* S, *of dimensions* r, s *respectively. Let* $\mathfrak{y}$, $\mathfrak{z}$ *be two cycles of* S *such that the set* N *of the* $v \in M(K)$ *for which* $\Delta\{v\}^*$, $\theta\{v\}^*$ *are the modified extensions over* K_v *of* $\mathfrak{y}$, $\mathfrak{z}$ *respectively is nonempty. Let* V *be a component variety of* $\mathfrak{y} \cap \mathfrak{z}$, *and let* Λ_j $(j = 1, 2, \cdots)$ *be all the component varieties of* $\theta \cap \Delta$; *set*

$$\alpha_j = i(\Lambda_j, \theta \cap \Delta, S_K), \ \Lambda = \sum_j \alpha_j \Lambda_j .$$

Then the set $\{\Lambda_j\}$ *is nonempty, and, for each* $v \in N$, *a partial extension of* $i(V, \mathfrak{z} \cap \mathfrak{y}, S)$ *over* K_v *is part of* $\Lambda\{v\}^*$.

THEOREM 3.3. *If* $\mathfrak{y}$, $\mathfrak{z}$ *are unmixed cycles of* S, *and* V *is a component variety of* $\mathfrak{y} \cap \mathfrak{z}$, *then*

$$i(V, \mathfrak{y} \cap \mathfrak{z}, S) = i(V, \mathfrak{z} \cap \mathfrak{y}, S).$$

Proof. Theorems 3.2 and 3.3 will be proved together in a number of steps. We shall prove them under the additional assumption that $\mathfrak{y}$, $\mathfrak{z}$ are integral effective cycles. The transition to the general case is obvious.

Step 1. In the notation of Theorem 3.2, let k' be the algebraic closure of k in K, and let K' be a field isomorphic to K over k'; then the direct product $K \times K'$ over k' is an integral domain. Let E be the quotient field of $K \times K'$. Let Δ' be a "copy" of Δ over K'. Given a $v \in N$, select elements $x_1, \cdots, x_m \in K$ such that (1) $K = k'(x)$, (2) $k'[x] \subseteq R_v$, (3) if $\mathfrak{p} = \mathfrak{C}(v/k'[x])$, then $k'[x]_{\mathfrak{p}}$ contains all the coefficients of

$$\Psi_{t,y}\, \theta \in K[t, y],$$

after one of these has been made equal to 1, and (4) $k'[x]_{\mathfrak{p}}$ contains all the coefficients of

$$\Psi_{t,y}\, \Delta \in K[t, y],$$

after one of these has been made equal to 1.

Let $x'_1, \cdots, x'_m$ be the elements of K' which correspond to $x_1, \cdots, x_m$ in the isomorphism between K and K'. Then $E = k'(x, x')$. The ideal $\mathfrak{P}$ of $k'[x, x']$ whose basis is $\{x'_1 - x_1, \cdots, x'_m - x_m\}$ is prime; let u be any valuation of E over k whose center on $k'[x, x']$ is $\mathfrak{P}$, and whose dimension over k equals $\dim \mathfrak{P}/k = \operatorname{transc} K/k$. Then u is a place of E over K. Let Δ^* be the modified extension of Δ' over K: then we see that $\Delta^*\{u\}^*$ is the modified extension of Δ over K_u. Let Λ^* be obtained from Δ^* and θ as Λ_θ (in Lemma 3.1) is obtained from θ and $\mathfrak{y}$. Then, by Lemma 3.1, a partial extension of Λ over K_u is part of $\Lambda^*\{u\}^*$.

Step 2. Let $k'[z]$ be the integral closure of $k'[x, x']$, and let v' be the place of K' which corresponds to v in the isomorphism between K and K'. Set $\mathfrak{q} = \mathfrak{C}(v'/k'[x'])$, and let $\mathfrak{Q}$ be the minimal prime of $\mathfrak{q}k'[x, x']$. Denote by v^* the place of E over k which is compounded of u and of an extension of v to K_u.

Then $\mathfrak{Q} \subset \mathfrak{C}(v^*/k'[x, x'])$, and therefore some minimal prime $\mathfrak{Q}'$ of $\mathfrak{Q}k'[z]$ is contained in $\mathfrak{C}(v^*/k'[z])$. We select a place w of E over K whose center on $k'[z]$ is $\mathfrak{Q}'$; then there exists a place w^* of E over k whose center on $k'[z]$ is $\mathfrak{C}(v^*/k'[z])$, and which is compounded of w and of some place v_1 of K_w over k. If v_0 is the place of K over k induced by v_1, we have

$$\mathfrak{C}(w^*/k'[x, x']) = \mathfrak{C}(v^*/k'[x, x']),$$

hence

$$\mathfrak{C}(v_0/k'[x]) = \mathfrak{C}(w^*/k'[x]) = \mathfrak{C}(v^*/k'[x]) = \mathfrak{C}(v/k'[x]).$$

As a consequence, because of the choice of $\{x\}$, v_0 and v have, on G_θ and G_Δ, the same centers; since $v \in N$, we deduce that $v_0 \in N$. We also have

$$\mathfrak{C}(w/k'[x']) = \mathfrak{Q}' \cap k'[x'] = \mathfrak{q} = \mathfrak{C}(v'/k'[x']);$$

therefore, since $v \in N$, it follows that $\Delta^*\{w\}^*$ is the modified extension of $\mathfrak{h}$ over K_w. Let Λ' be obtained from θ, $\mathfrak{h}'$ (= modified extension of $\mathfrak{h}$ over K) as Λ_θ (in Lemma 3.1) is obtained from $\mathfrak{h}$, θ respectively. Now we can replace, in Lemma 3.1 $\mathfrak{h}$ by θ, $\mathfrak{z}$ by $\mathfrak{h}'$, θ by Δ^*, Λ_θ by Λ^*, and the result is that a partial extension of Λ' over K_w is part of $\Lambda^*\{w\}^*$.

Step 3. We now make the assumption that a partial extension of $i(V, \mathfrak{z} \cap \mathfrak{h}, S)V$ over K_{v_0} is part of $\Lambda'\{v_0\}^*$. Since we also have that

$$\Lambda^*\{w^*\}^* = (\Lambda^*\{w\}^*)\{v_1\}^*$$

is the modified extension of $\Lambda'\{v_0\}^*$ over K_{w^*}, we deduce that a partial extension of $i(V, \mathfrak{z} \cap \mathfrak{h}, S)\,V$ over K_{v_1} is part of $\Lambda^*\{w^*\}^*$.

Let F be the irreducible variety over k' whose n.h.g.p. is $\{z\}$; set

$$L = D_{\Lambda^*, F}, \ \mathfrak{P} = \mathfrak{C}(w^*/F) = \mathfrak{C}(v^*/F),$$

and let U be the subvariety of S on which L operates. Since F is normal, for any component C of $[L;\, U, P]$ of dimension $r + s - n$ the number $e(C/L;\, U, P)^*$ exists. The previous result shows that among the C's there are pseudosubvarieties V_j of $S_{k'} \times F$ such that $(1V_j)[P]$ is a component of $V_{k(P)}$; and it also shows that if

$$V' = \sum_j e(V_j/L;\, U, P)^*\,(1V_j)[P],$$

then a partial extension of $i(V, \mathfrak{z} \cap \mathfrak{h}, S)\,V$ over $k(P)$ is part of V'.

The concluding statement of Step 1 shows that a partial extension of $\Lambda\{v\}^*$ over K_{v^*} is part of $\Lambda^*\{v^*\}^*$. If V'' is the part of $\Lambda\{v\}^*$ whose component varieties are components of the extension of V over K_v, this also means that a partial extension of V'' over K_{v*} coincides with a partial extension of V' over K_{v*}, and therefore also with a partial extension of $i(V, \mathfrak{z} \cap \mathfrak{y}, S)\,V$ over K_{v*}. But then V'' itself is a partial extension of $i(V, \mathfrak{z} \cap \mathfrak{y}, S)\,V$ over K_v, and this proves that *Theorem* 3.2 *is true if the assumption made at the beginning of Step* 3 *is true.*

Step 4. We now apply the content of Steps 1, 2, 3, to the following case: assume $\mathfrak{y}$, $\mathfrak{z}$ to be irreducible; let θ' be the general homographic transform of $\mathfrak{z}$,

$$H = k(\cdots, u_{ij}\, u_{00}^{-1}, \cdots);$$

let Δ' be the general homographic transform of $\mathfrak{y}$, constructed with an independent set $\{u'_{ij}\}$ of indeterminates, and set

$$H' = k(\cdots, u'_{ij}\, u'^{-1}_{00}, \cdots);$$

set

$$K = k(\cdots, u_{ij}\, u_{00}^{-1}, \cdots, \cdots, u'_{ij}\, u'^{-1}_{00}, \cdots),$$

and let Δ, θ be the modified extensions of Δ', θ' over K. We select places $\mathfrak{p}$, $\mathfrak{p}'$ of H, H' over k such that

$$K_{\mathfrak{p}} = K_{\mathfrak{p}'} = k,\ u_{ij}(\mathfrak{p}) = u'_{ij}(\mathfrak{p}') = 0 \text{ if } i \neq j,\ u_{ii}(\mathfrak{p}) = u'_{ii}(\mathfrak{p}') = 1.$$

We further select for v the place of K over k which is compounded of the unique extension $\mathfrak{p}^*$ of $\mathfrak{p}$ over H', and of $\mathfrak{p}'$. In this case the set $\{x\}$ can be selected to coincide with the set

$$\{\cdots, u_{ij}\, u_{00}^{-1}, \cdots, \cdots, u'_{ij}\, u'^{-1}_{00}, \cdots\}$$

(see Step 1), and $k' = k$. Besides, $k[x, x']$ is integrally closed, so that $\{z\} = \{x, x'\}$ (see Step 2). The fact that $k[z] = k[x, x']$ implies that we can select $v_0 = v$ in Step 2. Hence we can replace, in Lemma 3.3, S by $S_{H'}$, $\mathfrak{y}$ by the modified extension $\mathfrak{z}''$ of $\mathfrak{z}$ over H', $\mathfrak{z}$ by the modified extension $\mathfrak{y}''$ of $\mathfrak{y}$ over H', K by K, v by $\mathfrak{p}^*$, Δ by θ, Λ by Λ', V by the extension V'' of V over H', and Lemma 3.3 yields that $i(V'', \mathfrak{z}'' \cap \mathfrak{y}'', S_{H'})\,V''$ is part of $\Lambda'\{\mathfrak{p}^*\}^*$. Now, from the definition of intersection multiplicity follows that

$$i(V'', \mathfrak{z}'' \cap \mathfrak{y}'', S_{H'}) = i(V, \mathfrak{z} \cap \mathfrak{y}, S);$$

therefore $i(V, \mathfrak{z} \cap \mathfrak{y}, S)\,V$ is part of

$$(\Lambda'\{\mathfrak{p}^*\}^*)\{\mathfrak{p}'\}^* = \Lambda'\{v\}^* = \Lambda'\{v_0\}^*.$$

Hence, in this particular case, the assumption made at the beginning of Step 3 is true, and therefore, by Step 3, $i(V, \mathfrak{z} \cap \mathfrak{y}, S)\, V$ is part of $\Lambda\{v\}^*$. If Λ_j is any component variety of Λ, then $1\Lambda_j$, considered as an algebraic correspondence between K and S_H, operates on the whole rad θ' (see Remark 1), while, considered as an algebraic correspondence between K and $S_{H'}$, $1\Lambda_j$ operates on the whole rad Δ'. If $\{\zeta\}$, $\{\delta\}$ are sets of regular parameters of $Q(\text{rad } \theta'/S_H)$ and $Q(\text{rad } \Delta'/S_{H'})$ respectively, it follows that θ, Δ are complete intersections at Λ_j on S_K, and that $\{\zeta\}$, $\{\delta\}$ are sets of representatives of θ, Δ, respectively, on S_K. Set

$$R = Q(\Lambda_j/S_K).$$

If σ, τ are the homomorphic mappings of R whose kernels are $\mathfrak{P}(\text{rad } \Delta/S_K) \cap R$ and $\mathfrak{P}(\text{rad } \theta/S_K) \cap R$ respectively, Lemma 3.2 implies that

$$i(\Lambda_j, \theta \cap \Delta, S_K) = e(\sigma R; \sigma\zeta),$$

and this equals $e(R; \zeta, \delta)$ by the associativity formula (Theorem 2.1 of [2]); therefore, again by the associativity formula and Lemma 3.2, we have

$$i(\Lambda_j, \theta \cap \Delta, S_K) = e(R; \zeta, \delta) = e(\tau R; \tau\delta) = i(\Lambda_j, \Delta \cap \theta, S_K).$$

This shows that Λ is unaffected when Δ, θ are interchanged, that is, when $\mathfrak{y}$, $\mathfrak{z}$ are interchanged; hence $i(V, \mathfrak{y} \cap \mathfrak{z}, S)\, V$ is also part of $\Lambda\{v\}^*$, and this amounts to saying that

$$i(V, \mathfrak{y} \cap \mathfrak{z}, S) = i(V, \mathfrak{z} \cap \mathfrak{y}, S).$$

Theorem 3.3 is thus completely proved when $\mathfrak{y}$ and $\mathfrak{z}$ are irreducible, and therefore also when they are not irreducible, because of Theorem 3.1.

Step 5. We go back to the general case considered in Steps 1, 2, 3, and prove that the assumption made at the beginning of Step 3 is always true. According to Theorem 3.3, proved in Step 4, the equality

$$i(\Lambda_j', \theta \cap \mathfrak{y}', S_K) = i(\Lambda_j', \mathfrak{y}' \cap \theta, S_K)$$

is true for any component variety Λ_j' of Λ'. Therefore we can replace, in Lemma 3.1, $\mathfrak{y}$ by $\mathfrak{y}$, $\mathfrak{z}$ by $\mathfrak{z}$, θ by θ, Λ_θ by Λ', and the result is that a partial extension of

$$i(V, \mathfrak{y} \cap \mathfrak{z}, S)\, V = i(V, \mathfrak{z} \cap \mathfrak{y}, S)\, V$$

over K_{v_0} is part of $\Lambda'\{v_0\}^*$, since, as it was proved in Step 2, $v_0 \in N$. This completes the proof of Theorem 3.2, Q.E.D.

THEOREM 3.4. *Let $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$ be three unmixed cycles of $S = S_n(k)$, of dimensions r, s, t respectively such that*

$$r + s + t - 2n \geq 0;$$

let V be a component of rad $\mathfrak{x}$ ∩ rad $\mathfrak{y}$ ∩ rad $\mathfrak{z}$ *of dimension* $r+s+t-2n$. *Let* $U_1, U_2, \cdots$ *be the components of* rad $\mathfrak{x}$ ∩ rad $\mathfrak{y}$ *which contain V, and let* $W_1, W_2, \cdots$ *be the components of* rad $\mathfrak{y}$ ∩ rad $\mathfrak{z}$ *which contain V. Then*

$$\dim U_j = r + s - n, \ \dim W_j = s + t - n,$$

so that

$$U = \sum_j i(U_j, \mathfrak{x} \cap \mathfrak{y}, S)\, U_j \quad and \quad W = \sum_j i(W_j, \mathfrak{y} \cap \mathfrak{z}, S)\, W_j$$

exist. Moreover,

$$i(V, \mathfrak{x} \cap W, S) = i(V, U \cap \mathfrak{z}, S).$$

This number shall be denoted by $i(V, \mathfrak{x} \cap \mathfrak{y} \cap \mathfrak{z}, S)$, *and a similar notation will be used when more that three cycles are involved.*

Proof. We may assume, by Theorem 3.1, $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$ to be irreducible. Let $\mathfrak{X}'$, $\mathfrak{Y}'$, $\mathfrak{Z}'$ be the general homographic transforms of $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$, respectively, constructed with three independent sets of indeterminates $\{u_{ij}\}$, $\{v_{ij}\}$, $\{w_{ij}\}$, and set

$$H = k(\cdots, u_{ij}\, u_{00}^{-1}, \cdots), \ J = k(\cdots, v_{ij}\, v_{00}^{-1}, \cdots), \ L = k(\cdots, w_{ij}\, w_{00}^{-1}, \cdots),$$
$$K = k(\cdots, u_{ij}\, u_{00}^{-1}, \cdots, \cdots, v_{ij}\, v_{00}^{-1}, \cdots, \cdots, w_{ij}\, w_{00}^{-1}, \cdots).$$

Let $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ be the modified extensions of $\mathfrak{X}'$, $\mathfrak{Y}'$, $\mathfrak{Z}'$ respectively over K. Then $(\mathfrak{X} \cap \mathfrak{Y}) \cap \mathfrak{Z}$ and $\mathfrak{X} \cap (\mathfrak{Y} \cap \mathfrak{Z})$ exist. Let $\{\xi\}$, $\{\eta\}$, $\{\zeta\}$ be sets of regular parameters of $Q(\text{rad}\ \mathfrak{X}'/S_H)$, $Q(\text{rad}\ \mathfrak{Y}'/S_J)$, $Q(\text{rad}\ \mathfrak{Z}'/S_L)$ respectively. If Λ_j is any component variety of $\Lambda = \mathfrak{X} \cap \mathfrak{Y}$, $1\Lambda_j$ operates on the whole rad $\mathfrak{X}'$ and the whole rad $\mathfrak{Y}'$, so that $\mathfrak{X}$ and $\mathfrak{Y}$ are complete intersections at Λ_j on S_K, and $\{\xi\}$, $\{\eta\}$ are their sets of representatives at Λ_j on S_K. Therefore, by Lemma 3.2, Theorem 3.3, and the associativity formula (Theorem 2.1 of [2]), we have

$$i(\Lambda_j, \mathfrak{X} \cap \mathfrak{Y}, S_K) = e(Q(\Lambda_j/S_K); \xi, \eta).$$

If Γ_l is a component variety of $\Gamma = \Lambda \cap \mathfrak{Z}$, this also shows that Λ is a complete intersection at Γ_l on S_K, and that $\{\xi, \eta\}$ is a set of representatives of Λ at Γ_l on S_K; since $\mathfrak{Z}$ is also a complete intersection at Γ_l on S_K, and $\{\zeta\}$ is a set of representatives of $\mathfrak{Z}$ at Γ_l on S_K, the same argument gives

$$i(\Gamma_l, \Lambda \cap \mathfrak{Z}, S_K) = e(Q(\Gamma_l/S_K); \xi, \eta, \zeta).$$

If now $\Delta = \mathfrak{Y} \cap \mathfrak{Z}$, we can prove by the same method that

$$i(\Gamma_l, \mathfrak{X} \cap \Delta, S_K) = e(Q(\Gamma_l/S_K); \xi, \eta, \zeta),$$

so that $\Lambda \cap \mathfrak{Z} = \mathfrak{X} \cap \Delta$. Let now v be a place of K over k such that,

$$u_{ij}(v) = v_{ij}(v) = w_{ij}(v) = 0 \text{ if } i \neq j, u_{ii}(v) = v_{ii}(v) = w_{ii}(v) = 1, K_v = k.$$

Theorem 3.2 implies that U is part of $\Lambda\{v\}^*$, and therefore also that $i(V, U \cap \mathfrak{z}, S)V$ is part of $\Gamma\{v\}^*$; for the same reason, $i(V, \mathfrak{x} \cap W, S)V$ is part of $\Gamma\{v\}^*$, Q.E.D.

4. Further properties of the intersection multiplicity in a projective space. *Throughout this section, S will be an n-dimensional projective space over the field k.*

THEOREM 4.1. *If $\mathfrak{x}$, $\mathfrak{y}$ are unmixed integral effective cycles of S, and V is a component variety of $\mathfrak{x} \cap \mathfrak{y}$, then $i(V, \mathfrak{x} \cap \mathfrak{y}, S)$ is a positive integer.*

Proof. In the proof of Theorem 3.4 it has been shown that

$$i(\Lambda_j, \mathfrak{X} \cap \mathfrak{Y}, S_K) = e(Q(\Lambda_j/S_K); \xi, \eta),$$

so that Λ is an integral effective cycle. But then $i(V, \mathfrak{x} \cap \mathfrak{y}, S)$ is an integer because it is the multiplicity of V in $\Lambda\{v\}^*$ (v having the same meaning as in the proof of Theorem 3.4), Q.E.D.

From Lemma 3.2, Theorems 3.2 and 3.4, and Lemma 2.3 of [2], it is now possible to see that a cycle $\mathfrak{z}$ is a complete intersection at V on S, and has the set of representatives $\{\zeta\}$ at V on S, if and only if $\mathfrak{z}$ coincides locally at V with $\mathfrak{x}_1 \cap \mathfrak{x}_2 \cap \cdots$, where $\mathfrak{x}_i$ is the $(n-1)$-dimensional cycle

$$\mathfrak{x}_i = \sum_j v_{ij}(\zeta_i) [K_{v_{ij}}: k(\mathfrak{C}(v_{ij}/S))] \, \mathfrak{C}(v_{ij}/S);$$

here v_{ij} $(j = 1, 2, \cdots)$ are all the discrete normalized valuations of $k(S)$ over k of rank 1 and dimension $n-1$ such that $v_{ij}(\zeta_i) > 0$.

THEOREM 4.2. *Let $\mathfrak{y}$, $\mathfrak{z}$ be unmixed cycles of S of dimensions r, s such that $r + s - n \geq 0$; let V be a component variety of $\mathfrak{y} \cap \mathfrak{z}$. Let k' be an extension of k, and $\mathfrak{y}'$, $\mathfrak{z}'$ the modified extensions of $\mathfrak{y}$, $\mathfrak{z}$ over k'. Then each component V_j of $V_{k'}$ is a component variety of $\mathfrak{y}' \cap \mathfrak{z}'$, and*

$$\sum_j i(V_j, \mathfrak{h}' \cap \mathfrak{z}', S_{k'}) V_j$$

is the modified extension over k' *of* $i(V, \mathfrak{h} \cap \mathfrak{z}, S)\, V$.

Proof. The first assertion is evidently true. In order to prove the second statement, let $\mathfrak{Y}^*$, $\mathfrak{Z}^*$ be the general homographic transforms of $\mathfrak{h}$, $\mathfrak{z}$ respectively, constructed with two independent sets of indeterminates $\{u_{ij}\}$, $\{v_{ij}\}$. Set

$$K = k(\cdots, u_{ij}\, u_{00}^{-1}, \cdots, \cdots, v_{ij}\, v_{00}^{-1}, \cdots),$$

and let $\mathfrak{Y}$, $\mathfrak{Z}$ be the modified extensions of $\mathfrak{Y}^*$, $\mathfrak{Z}^*$ respectively over K. Then $\Lambda = \mathfrak{Y} \cap \mathfrak{Z}$ exists, and if v is a place of K over k such that

$$K_v = k,\ u_{ij}(v) = v_{ij}(v) = 0 \ \text{ if } \ i \neq j,\ u_{ii}(v) = v_{ii}(v) = 1,$$

then $i(V, \mathfrak{h} \cap \mathfrak{z}, S)\, V$ is part of $\Lambda\{v\}^*$ by Theorem 3.2. Now let $\mathfrak{Y}'$, $\mathfrak{Z}'$, K' be obtained from $\mathfrak{h}'$, $\mathfrak{z}'$ as $\mathfrak{Y}$, $\mathfrak{Z}$, K are from $\mathfrak{h}$, $\mathfrak{z}$; then $\mathfrak{Y}'$, $\mathfrak{Z}'$ are the modified extensions of $\mathfrak{Y}$, $\mathfrak{Z}$ respectively over K'. If $\Lambda' = \mathfrak{Y}' \cap \mathfrak{Z}'$, assume for a moment Λ' to be the modified extension of Λ over k'. If v' is any extension of v to K' over k' such that $K_{v'} = k'$, then $\Lambda'\{v'\}^*$ is the modified extension of $\Lambda\{v\}^*$ over k', and therefore

$$\sum_j i(V_j, \mathfrak{h}' \cap \mathfrak{z}', S_{k'}) V_j$$

is the modified extension over k' of $i(V, \mathfrak{h} \cap \mathfrak{z}, S)\, V$, as claimed. We conclude that the theorem is true if it is true when applied to $\mathfrak{Y}$, $\mathfrak{Z}$, or also, *a fortiori*, if it is true under the additional assumption that $\mathfrak{h}$, $\mathfrak{z}$ are complete intersections at V. This, in turn, is equivalent, by Lemma 3.2, to the following assertion: Let A be an irreducible subvariety of S, $\{\zeta\}$ a set of parameters of $R = Q(A/S)$; let A' be the modified extension of $1A$ over k', $A_1, \cdots, A_m$ its component varieties, and set

$$R_i = Q(A_i/S_{k'}).$$

Then

$$A' = \sum_i e(R_i; \zeta)\ e(R; \zeta)^{-1} A_i.$$

Now, if k' is an algebraic function field over k the proof of this statement is implicitly contained in the proof of Lemma 1.2; otherwise, it can be obtained by a well-known limiting process, Q.E.D.

THEOREM 4.3 (BEZOUT'S THEOREM). *Let* $\mathfrak{h}$, $\mathfrak{z}$ *be unmixed cycles of* S *such that* $\mathfrak{h} \cap \mathfrak{z}$ *exists. Then*

$$\operatorname{ord}(\mathfrak{h} \cap \mathfrak{z}) = (\operatorname{ord} \mathfrak{h})(\operatorname{ord} \mathfrak{z}).$$

Proof. By Theorem 3.1, we may assume without loss of generality that $\mathfrak{h}$ and $\mathfrak{z}$ are irreducible; and, by Theorem 4.2, we may assume k to be algebraically closed. Let $\mathfrak{Y}$, $\mathfrak{Z}$, K, Λ have the same meanings as in the proof of Theorem 4.2. Then

$$\operatorname{ord}(\mathfrak{h} \cap \mathfrak{z}) = \operatorname{ord} \Lambda.$$

Since k is algebraically closed, $\mathfrak{Y}$ and $\mathfrak{Z}$ are the modified extensions over K of the general elements of the homographic systems $\mathfrak{H}$, $\mathfrak{K}$ of $\mathfrak{h}$, $\mathfrak{z}$ respectively. According to Lemma 2.4, $\mathfrak{H}$ and $\mathfrak{K}$ contain two degenerate cycles $\mathfrak{h}'$, $\mathfrak{z}'$, and therefore they contain the homographic systems $\mathfrak{H}'$, $\mathfrak{K}'$ of $\mathfrak{h}'$, $\mathfrak{z}'$ respectively (Lemma 2.2). Two cycles $\mathfrak{h}''$, $\mathfrak{z}''$ of $\mathfrak{H}'$, $\mathfrak{K}'$, respectively, can be found in such a way that $\mathfrak{h}'' \cap \mathfrak{z}''$ exists; we have then that $\mathfrak{h}'' \cap \mathfrak{z}'' = \Lambda\{v\}^*$ for some $v \in M(K)$ (Theorem 3.2), and therefore

$$\operatorname{ord}(\mathfrak{h}'' \cap \mathfrak{z}'') = \operatorname{ord} \Lambda = \operatorname{ord}(\mathfrak{h} \cap \mathfrak{z}).$$

If $r = \operatorname{ord} \mathfrak{h}$, $s = \operatorname{ord} \mathfrak{z}$, we have

$$\mathfrak{h}'' = \sum_{i=1}^{r} 1\mathfrak{h}_i, \quad \mathfrak{z}'' = \sum_{i=1}^{s} 1\mathfrak{z}_i,$$

the $\mathfrak{h}_i$'s and $\mathfrak{z}_i$'s being linear varieties. Lemma 3.2 gives that for each i, j the intersection $1\mathfrak{h}_i \cap 1\mathfrak{z}_j$ is an irreducible cycle whose radical is a linear variety. Hence Theorem 3.1 implies that $\operatorname{ord}(\mathfrak{h} \cap \mathfrak{z}) = rs$, Q.E.D.

THEOREM 4.4 (CRITERION FOR SIMPLE INTERSECTIONS). *Let $\mathfrak{h}$, $\mathfrak{z}$ be irreducible cycles of S, of dimensions r, s respectively such that $r + s - n \geq 0$. Let V be a component of* rad $\mathfrak{h} \cap$ rad $\mathfrak{z}$. *Then the following four statements are equivalent*:

(1) *$i(V, \mathfrak{h} \cap \mathfrak{z}, S)$ exists and equals* 1;

(2) *let $\{X\}$ be the* h.g.p. *of S; let*

$$\{f_1(X), f_2(X), \cdots\} \quad \textit{and} \quad \{g_1(X), g_2(X), \cdots\}$$

be bases of $\wp(\text{rad } \mathfrak{h}/S)$ and $\wp(\text{rad } \mathfrak{z}/S)$ respectively. Let $\{x\}$ be the h.g.p. *of V. Then the Jacobian matrix $J(f(X), g(X); X, t)$ acquires the rank $2n - r - s$ when $\{X\}$ is replaced by $\{x\}$. Here $\{t\}$ is a p-independent basis of k over k^p if p is the characteristic of k*;

(3) *there are regular sets of parameters* $\{\zeta\}$, $\{\eta\}$ *of* $Q(\text{rad } \mathfrak{z}/S)$, $Q(\text{rad } \mathfrak{y}/S)$ *respectively such that* $\{\zeta, \eta\}$ *is a regular set of parameters of* $Q(V/S)$;

(4) $\wp(V/S)$ *is an isolated primary component of*

$$\wp(\text{rad } \mathfrak{y}/S) + \wp(\text{rad } \mathfrak{z}/S).$$

If ins $V = 1$, *then* $J(f(X), g(X); X, t)$ *in Statement* 2 *can be replaced by* $J(f(X), g(X); X)$.

Proof. Let $\mathfrak{Y}$, $\mathfrak{Z}$, Λ, K have the same meaning as in the proof of Theorem 4.2. Let S_1, S_2 be the projective spaces whose h.g.p. are $\{u\}$, $\{v\}$ respectively. Set

$$R = S_1 \times S_2, Z = D_{\mathfrak{Z},R}, \; Y = D_{\mathfrak{Y},R}, \; Z^* = D_{\mathfrak{Z}^*, S_2}, \; Y^* = D_{\mathfrak{Y}^*, S_1},$$

and let P, Q be points of S_1, S_2 such that

$$Y^*\{P\}^* = \mathfrak{y}, \; Z^*\{Q\}^* = \mathfrak{z};$$

set also $G = P \times Q$. Then the ideal whose basis is the set of the

$$F_i(X, u) = f_i(\cdots, \textstyle\sum_j u_{lj}\, u_{00}^{-1} X_j, \cdots)$$

has

$$\wp(\text{rad } Y/k[X, \cdots, u_{ij}\, u_{00}^{-1}, \cdots, \cdots, v_{ij}\, v_{00}^{-1}, \cdots])$$

as an isolated primary component, and the ideal whose basis is the set of the

$$G_i(X, v) = g_i(\cdots, \textstyle\sum_j v_{lj}\, v_{00}^{-1} X_j, \cdots)$$

has

$$\wp(\text{rad } Z/k[X, \cdots, u_{ij}\, u_{00}^{-1}, \cdots, \cdots, v_{ij}\, v_{00}^{-1}, \cdots])$$

as an isolated primary component. If assertion 1 is true, then only one component Λ' of Λ has the property that $L' = \text{rad } D_{\Lambda', R}$ contains $V \times G$; besides, rad Λ' has in Λ the multiplicity 1. Therefore, by Lemma 3.2, $\{F_1, F_2, \cdots, G_1, G_2, \cdots\}$ is the basis of an ideal of which

$$\wp(L'/k[X, \cdots, u_{ij}\, u_{00}^{-1}, \cdots, \cdots, v_{ij}\, v_{00}^{-1}, \cdots])$$

is an isolated primary component. Since

$$i(V, \mathfrak{y} \cap \mathfrak{z}, S) = e(V \times G/1L'; S, R)^*,$$

and since upon replacing the u_{ij}'s, v_{ij}'s by their values at G the $F_i(X, u)$, $G_i(X, v)$ are replaced by the $f_i(X)$, $g_i(X)$, Theorem 5.6 of [2] or its corollary implies that Statement 2 is true.

Assume now Statement 2 to be true; then Theorem 10 of [6] implies that Statement 3 is true. Finally, if Statement 3 is true, then $\mathfrak{h}$ and $\mathfrak{z}$ are complete intersections at V on S, and Lemma 3.2, together with Theorem 2.1 of [2], yields the result that Statement 1 is true. Statement 4 is clearly a consequence of Statement 3, and it implies Statement 2, Q.E.D.

COROLLARY. *With notations as in Theorem 4.4, if*

$$i(V, \mathfrak{h} \cap \mathfrak{z}, S) = 1,$$

then V is simple on rad $\mathfrak{h}$ *and* rad $\mathfrak{z}$.

Proof. This is a consequence of Statement 3 of the theorem and of a well-known result on regular local rings, Q.E.D.

5. **Intersection of cycles of an algebraic irreducible variety.** Let V be an irreducible variety over the field k, U a subvariety of V, S the *ambient space of* V. By this expression we mean to express the fact that if $\{X\}$ is the h.g.p. of S, then the h.g.p. $\{x\}$ of V is a homomorphic image of $\{X\}$; of course S is not the only projective space of which V is a subvariety. Let $\mathfrak{z}$ be an unmixed cycle of V. We say that $\mathfrak{z}$ is a *section of V at U* if there exists an unmixed cycle $\mathfrak{Z}$ of S such that $1V \cap \mathfrak{Z}$ and $\mathfrak{z}$ coincide locally at U. We shall develop in this section a theory of intersections of cycles of V which will be valid when the cycles are sections of V at some U; before we do so, however, it is important to show that this is the case under the customary conditions. Namely, we have:

THEOREM 5.1. *Let V be an irreducible variety over the algebraically closed field k, S the ambient space of V, U a nonempty irreducible subvariety of V, simple on V, $\mathfrak{z}$ an irreducible cycle of V; then there exists an irreducible cycle $\mathfrak{Z}$ of S such that $1V \cap \mathfrak{Z}$ coincides with $\mathfrak{z}$ locally at V.*

Proof. Since, by Theorem 3 of [6], each U simple on V contains a point P simple on V, the theorem will be proved in general if it is proved under the assumption that U is a point. Let $\{x\}$ be a n.h.g.p. of V for which U is a finite distance, $R = Q(U/V)$. If

$$m = \dim S,\ n = \dim V,\ r = \dim \mathfrak{z},$$

let $\{y_1, \cdots, y_n\}$ be a set of regular parameters of R contained in $k[x]$; then $y_1, \cdots, y_n$ are algebraically independent over k. Let F be the projective space over k whose n.h.g.p. is $\{y\}$, and set

$$U' = \wp(\wp(U/k[x]) \cap k[y]), \quad Z' = \wp(\wp(\mathrm{rad}\ \mathfrak{z}/k[x]) \cap k[y]).$$

Then U' is a point, and $\dim Z' \leq r$. The embedding of $k[y]$ in $k[x]$ gives an irreducible algebraic correspondence D between F and V, such that rad D is birationally equivalent to V in a birational correspondence which is regular[2] at finite distance; we shall therefore denote subvarieties of V and D which correspond to each other with the same symbol. Since $Q(U'/F)$ contains a set of regular parameters of R, from the corrollary to Lemma 1.1 we obtain

$$e(U/D;\ U',\ V)^* = 1.$$

Let Z be a component of $[D;\ Z',\ V]$ containing U; then Theorem 1.1 implies that $\dim Z = \dim Z'$; since among the Z's there is one which contains rad $\mathfrak{z}$, and which therefore has dimension $\geq r$, we conclude that $\dim Z' = r$, so that $\dim Z = r$ for each Z. Now, by Theorem 1.1, we have

$$1 = e(U/D;\ U',\ V)^* = e(U/\textstyle\sum_Z e(Z/D;\ Z',\ V)^*\ Z;\ U',\ V)^*.$$

Since Z' is simple on F, according to a remark preceding Theorem 5.5 of [2] we have that each $e(Z/D;\ Z',\ V)^*$ is an integer; we cannot state that $e(U/1Z;\ U',\ V)^*$ exists for each Z; however, according to Lemma 1.1, we may operate in the following way: Replace, in Lemma 1.1, D, D^*, V, F, G, k respectively by

$$\textstyle\sum_Z e(Z/D;\ Z',\ V)^*\ Z,\ U,\ V,\ Z',\ U',\ k,$$

and select correspondingly Z^*, G', D', U_1^*, U_2^*, $\cdots$ to replace F', G', D', D_1^*, D_2^*, $\cdots$ in Lemma 1.1; impose upon Z^* the additional condition that $\{1D_j';\ V,\ G'\}^*$ exists for each component variety D_j' of D'; for each Z, set

$$\alpha(Z) = \textstyle\sum_j e(U_j^*/D_l';\ V,\ G')^*\ \mathrm{ord}\,(1\,U_j^*)\,[G'],$$

where l is such that

$$(1D_l')\{Z^*\}^* = (1Z)\{Z'\}^*.$$

Since $\{1D_l';\ V,\ G'\}^*$ exists, and $h(U_j^*) = 1$, we deduce that $\alpha(Z)$ is an integer. The α of Lemma 1.1 is given by

$$\alpha = \textstyle\sum_Z e(Z/D;\ Z',\ V)^*\ \alpha(Z),$$

and therefore. since ord $U = 1$ in this case, Lemma 1.1 itself gives that

[2] T is regular at U if for each $U' = T(U)$ it is true that $Q(U/V) = Q(U'/V')$; in this case U' is unique.

$$1 = e(U/\sum_Z e(Z/D;\, Z',\, V)^* Z;\, V,\, U')^* = \sum_Z e(Z/D;\, Z',\, V)^* \alpha(Z).$$

Since we have seen that each $e(Z/D;\, Z',\, V)^*$ and each $\alpha(Z)$ is an integer, it follows that there is exactly one Z, namely rad $\mathfrak{z}$, and that $e(Z/D;\, Z',\, V)^* = 1$.

Now, the set $\{y\}$ can be identified with a subset of $k[X]$, $\{X\}$ being the n.h.g.p. of S which corresponds to $\{x\}$. Set

$$\mathfrak{Z} = 1\wp(\wp(Z'/k[y])\, k[X]),$$

so that $\mathfrak{Z}$ is an irreducible cycle of S of dimension $r + m - n$. The fact that the only Z is rad $\mathfrak{z}$ means that rad $\mathfrak{z}$ is the only component of $V \cap \text{rad } \mathfrak{Z}$ containing U; since

$$r = n + \dim \mathfrak{Z} - m,$$

we also have that rad $\mathfrak{z}$ is a component variety of $1V \cap \mathfrak{Z}$. Finally, since

$$e(Z/D;\, Z',\, V)^* = 1,$$

a regular set of parameters of $Q(Z'/F)$ is a regular set of parameters of $Q(Z/V)$, and this means that $\wp(\text{rad } \mathfrak{z}/k[X])$ is an isolated primary component of

$$\wp(V/k[X]) + \wp(\text{rad } \mathfrak{Z}/k[X]).$$

This, in turn, by Statement 4 of Theorem 4.4 shows that $i(\text{rad } \mathfrak{z},\, 1V \cap \mathfrak{Z},\, S) = 1$, Q.E.D.

Let V be an irreducible n-dimensional variety over the (arbitrary) field k, and let $\mathfrak{y}$, $\mathfrak{z}$ be unmixed cycles of V of dimension r, s respectively; if U is an irreducible subvariety of rad $\mathfrak{y} \cap$ rad $\mathfrak{z}$, we say that U is a *component variety of* $(\mathfrak{y} \cap \mathfrak{z},\, V)$ if $\dim U = r + s - n$. If $\mathfrak{z}$ is a section of V at U, let $\mathfrak{Z}$ be an unmixed cycle of the m-dimensional ambient space S of V, such that $\mathfrak{z}$ coincides locally at U with $\mathfrak{Z} \cap 1V$. If U is also a subvariety of rad $\mathfrak{y}$, then it is a subvariety of rad $\mathfrak{y} \cap$ rad $\mathfrak{Z}$. Since

$$\dim \mathfrak{Z} = s + m - n,$$

by Theorem 2.2 we have

$$\dim U \geq r + s - n.$$

Assume U to have exactly the dimension $r + s - n$, so that it is a component variety of $(\mathfrak{y} \cap \mathfrak{z},\, V)$ and of $\mathfrak{y} \cap \mathfrak{Z}$. Assume also $\mathfrak{y}$ to be a section of V at U, and let $\mathfrak{Y}$ be related to $\mathfrak{y}$ as $\mathfrak{Z}$ is to $\mathfrak{z}$. The number $i(U,\, \mathfrak{y} \cap \mathfrak{Z},\, S)$ exists, and by Theorem 3.4 it equals

$$i(U, \mathfrak{Y} \cap 1V \cap \mathfrak{Z}, S) = i(U, \mathfrak{Y} \cap \mathfrak{z}, S).$$

This proves that

$$i(U, \mathfrak{y} \cap \mathfrak{Z}, S) = i(U, \mathfrak{Y} \cap \mathfrak{z}, S)$$

does not depend on the choice of $\mathfrak{Y}$, $\mathfrak{Z}$, but depends only on U, $\mathfrak{y}$, $\mathfrak{z}$, V; accordingly, it will be denoted by $i(U, \mathfrak{y} \cap \mathfrak{z}, V)$. We shall put $i(U, \mathfrak{y} \cap \mathfrak{z}, V) = 0$ if $\dim U = r + s - n$ but $U \not\subseteq \operatorname{rad} \mathfrak{y} \cap \operatorname{rad} \mathfrak{z}$. A generalization of the meaning of this symbol will be given after Theorem 5.9; the remark following Theorem 5.9 contains comments on the validity of most results of this section for the generalized symbol. Theorem 3.3 yields:

$$i(U, \mathfrak{y} \cap \mathfrak{z}, V) = i(U, \mathfrak{y} \cap \mathfrak{Z}, S) = i(U, \mathfrak{Z} \cap \mathfrak{y}, S) = i(U, \mathfrak{z} \cap \mathfrak{y}, V);$$

that is, we have the following result:

THEOREM 5.2 (COMMUTATIVITY LAW). *If one of the symbols*

$$i(U, \mathfrak{y} \cap \mathfrak{z}, V), i(U, \mathfrak{z} \cap \mathfrak{y}, V)$$

has a meaning, the other also has a meaning, and their values are equal.

The number $i(U, \mathfrak{y} \cap \mathfrak{z}, V)$ is called the *intersection multiplicity of* $\mathfrak{y}$ *and* $\mathfrak{z}$ *at* U *on* V. Assume that $\mathfrak{y}$, $\mathfrak{z}$ are such that each component U_j of $\operatorname{rad} \mathfrak{y} \cap \operatorname{rad} \mathfrak{z}$ is a component variety of $(\mathfrak{y} \cap \mathfrak{z}, V)$, and that $i(U_j, \mathfrak{y} \cap \mathfrak{z}, V)$ is defined for each j; in this case we shall set

$$(\mathfrak{y} \cap \mathfrak{z}, V) = \textstyle\sum_j i(U_j, \mathfrak{y} \cap \mathfrak{z}, V)\, U_j;$$

the cycle $(\mathfrak{y} \cap \mathfrak{z}, V)$ is called the *intersection of* $\mathfrak{y}$ *and* $\mathfrak{z}$ *on* V. The locutions "to be part of $(\mathfrak{y} \cap \mathfrak{z}, V)$", "to coincide locally at $\cdots$ with $(\mathfrak{y} \cap \mathfrak{z}, V)$", "to exist locally at $\cdots$", and "the local part of $(\mathfrak{y} \cap \mathfrak{z}, V)$ at $\cdots$" shall have a meaning even if $(\mathfrak{y} \cap \mathfrak{z}, V)$ does not exist, in exactly the same way as the similar locutions in § 3 have a meaning even if $\mathfrak{y} \cap \mathfrak{z}$ does not exist. Obviously, in the special case in which $V = S$, the symbols $i(U, \mathfrak{y} \cap \mathfrak{z}, S)$ as defined here or in § 3 have the same meaning; accordingly, the symbol $\mathfrak{y} \cap \mathfrak{z}$ of § 3 shall be denoted from now on by $(\mathfrak{y} \cap \mathfrak{z}, S)$.

From Theorem 4.2 we obtain:

THEOREM 5.3. *Let* V *be an irreducible variety over* k, $\mathfrak{y}$ *and* $\mathfrak{z}$ *two unmixed cycles of* V *such that*

$$\dim \mathfrak{y} + \dim \mathfrak{z} - \dim V \geq 0,$$

and let $\mathfrak{x}$ *be a part of* $(\mathfrak{y} \cap \mathfrak{z}, V)$. *Let* k' *be an extension of* k; V' *the extension of* V *over* k'; $\mathfrak{x}'$, $\mathfrak{y}'$, $\mathfrak{z}'$ *the modified extensions of* $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$ *respectively over* k'. *Assume* V' *to be irreducible. Then* $\operatorname{ins} V\,(\operatorname{ins} V')^{-1}\mathfrak{x}'$ *is part of* $(\mathfrak{y}' \cap \mathfrak{z}', V')$.

From the definition, and from Theorem 3.1, we obtain:

THEOREM 5.4 (DISTRIBUTIVITY LAW). *If* U, $\mathfrak{y}_j$, $\mathfrak{z}_l$, V *are such that* $i(U, \mathfrak{y}_j \cap \mathfrak{z}_l, V)$ *has a meaning for* $j, l = 1, 2$, *and if*

$$\dim \mathfrak{y}_1 = \dim \mathfrak{y}_2, \dim \mathfrak{z}_1 = \dim \mathfrak{z}_2,$$

then

$$i(U, (\mathfrak{y}_1 + \mathfrak{y}_2) \cap (\mathfrak{z}_1 + \mathfrak{z}_2), V)$$

has a meaning and equals

$$\sum_{j, l=1}^{2} i(U, \mathfrak{y}_j \cap \mathfrak{z}_l, V).$$

THEOREM 5.5 (ASSOCIATIVITY LAW). *Let* $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$ *be three unmixed cycles of the n-dimensional irreducible variety* V *over* k, *of dimensions* r, s, t *respectively. Let* U *be a component of* $\operatorname{rad} \mathfrak{x} \cap \operatorname{rad} \mathfrak{y} \cap \operatorname{rad} \mathfrak{z}$ *of dimensions* $r + s + t - 2n$; *assume* $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$ *to be sections of* V *at* U; *let* $\mathfrak{x}'$, $\mathfrak{y}'$ *be the local parts, at* U, *of* $(\mathfrak{x} \cap \mathfrak{y}, V)$, $(\mathfrak{y} \cap \mathfrak{z}, V)$ *respectively. Then* $i(U, \mathfrak{x}' \cap \mathfrak{z}, V)$ *and* $i(U, \mathfrak{x} \cap \mathfrak{z}', V)$ *exist and are equal.* Their common value is denoted by $i(U, \mathfrak{x} \cap \mathfrak{y} \cap \mathfrak{z}, V)$, and a similar notation is used when more than three cycles are involved.

Proof. Let $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ be unmixed cycles of S (the ambient space of V) such that $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$ coincide locally at U with $(\mathfrak{X} \cap 1V, S)$, $(\mathfrak{Y} \cap 1V, S)$, $(\mathfrak{Z} \cap 1V, S)$ respectively. Then $(\mathfrak{X} \cap \mathfrak{Y}, S)$ and $(\mathfrak{Y} \cap \mathfrak{Z}, S)$ exist locally at U; let $\mathfrak{X}'$, $\mathfrak{Z}'$ be the local parts of $(\mathfrak{X} \cap \mathfrak{Y}, S)$, $(\mathfrak{Y} \cap \mathfrak{Z}, S)$, respectively, at U. Theorem 3.4 implies that

$$i(U, \mathfrak{X}' \cap \mathfrak{z}, S) = i(U, \mathfrak{x} \cap \mathfrak{Z}', S);$$

on the other hand, again by Theorem 3.4, $(\mathfrak{X}' \cap 1V, S)$ coincides locally at U with $(\mathfrak{X} \cap \mathfrak{y}, S)$, and therefore with $(\mathfrak{x} \cap \mathfrak{y}, V)$ and with $\mathfrak{x}'$; this proves that

$$i(U, \mathfrak{X}' \cap \mathfrak{z}, S) = i(U, \mathfrak{x}' \cap \mathfrak{z}, V).$$

In the same way we obtain

$$i(U, \mathfrak{x} \cap \mathfrak{Z}', S) = i(U, \mathfrak{x} \cap \mathfrak{z}', V), \text{ Q.E.D.}$$

THEOREM 5.6 (TRANSITIVITY LAW). *Let V be an irreducible variety over k, W an irreducible subvariety of V. Let $\mathfrak{y}$, $\mathfrak{z}$ be unmixed cycles of W, and U a component variety of* $(\mathfrak{y} \cap \mathfrak{z}, W)$. *Let $\mathfrak{Y}$, $\mathfrak{Z}$ be unmixed cycles of V such that* $(\mathfrak{Y} \cap 1W, V)$, $(\mathfrak{Z} \cap 1W, V)$ *exist locally at U and coincide at U with $\mathfrak{y}$, $\mathfrak{z}$ respectively. Then* $(\mathfrak{Y} \cap \mathfrak{Z}, V)$ *exists locally at U; let $\mathfrak{X}$ be the local part of* $(\mathfrak{Y} \cap \mathfrak{Z}, V)$ *at U. Then* $(\mathfrak{X} \cap 1W, V)$ *and* $(\mathfrak{y} \cap \mathfrak{z}, W)$ *both locally exist and coincide at U.*

Proof. Let $\mathfrak{Y}^*$, $\mathfrak{Z}^*$ be unmixed cycles of the ambient space S of V such that $(\mathfrak{Y}^* \cap 1V, S)$, $(\mathfrak{Z}^* \cap 1V, S)$ coincide locally at U with $\mathfrak{Y}$, $\mathfrak{Z}$ respectively. Then $(\mathfrak{Y}^* \cap 1W, S)$, $(\mathfrak{Z}^* \cap 1W, S)$ coincide locally at U with $\mathfrak{y}$, $\mathfrak{z}$ respectively by definition. Let $\mathfrak{X}^*$ be the local part of $(\mathfrak{Y}^* \cap \mathfrak{Z}^*, S)$ at U; $\mathfrak{X}^*$ exists because the dimensions fulfill the correct relations. Then $(\mathfrak{X}^* \cap 1V, S)$ coincides locally at U with $\mathfrak{X}$, so that $(\mathfrak{X}^* \cap 1W, S)$ coincides locally at U with $(\mathfrak{X} \cap 1W, V)$. On the other hand, $(\mathfrak{y} \cap \mathfrak{z}, W)$ coincides locally at U with $(\mathfrak{Y}^* \cap \mathfrak{Z}^* \cap 1W, S)$, Q.E.D.

Theorem 5.6 also shows that in the definition of $i(U, \mathfrak{y} \cap \mathfrak{z}, V)$, the ambient space S could be replaced by any space S containing V as a subvariety.

THEOREM 5.7 (LAW OF THE CONSERVATION OF THE NUMBER). *Let A be an irreducible variety over k, and K an algebraic function field over k; let $\mathfrak{B}$ be an irreducible algebraic correspondence between K and A, and let $\mathfrak{X}$, $\mathfrak{Y}$ be unmixed cycles of* rad $\mathfrak{B}$. *Let $v \in M(K)$ be such that $K_v = k$, and that $\mathfrak{B}\{v\}^*$ is irreducible, say $\mathfrak{B}\{v\}^* = 1V$, where V is an irreducible variety over k. Set*

$$\mathfrak{x} = \mathfrak{X}\{v\}^*, \quad \mathfrak{y} = \mathfrak{Y}\{v\}^*,$$

so that $\mathfrak{x}$, $\mathfrak{y}$ are unmixed cycles of V; let U be a component variety of $(\mathfrak{x} \cap \mathfrak{y}, V)$. *Among the components of* rad $\mathfrak{X} \cap$ rad $\mathfrak{Y}$, *let $\mathfrak{U}_j$ $(j = 1, 2, \cdots)$ be those such that* rad $(1\mathfrak{U}_j)\{v\}^*$ *contains U; then*

$$\dim \mathfrak{U}_j = \dim \mathfrak{X} + \dim \mathfrak{Y} - \dim \mathfrak{B}$$

for each j. Assume $\mathfrak{X}$, $\mathfrak{Y}$ to be sections of rad $\mathfrak{B}$ *at $\mathfrak{U}_1 \cup \mathfrak{U}_2 \cup \cdots$. Then* (1) $\alpha_j = i(\mathfrak{U}_j, \mathfrak{X} \cap \mathfrak{Y}, \text{rad } \mathfrak{B})$ *exists for each j, so that $\mathfrak{U} = \sum_j \alpha_j \mathfrak{U}_j$ exists,* (2) *U is a component variety of each $(1\mathfrak{U}_j)\{v\}^*$, and* (3) *$i(U, \mathfrak{x} \cap \mathfrak{y}, V)$ exists and equals the multiplicity of U in $\mathfrak{U}\{v\}^*$.*

Proof. We need to prove only the last statement, since the others are an immediate consequence of the relations between the dimensions. Let $\mathfrak{S}$ be the ambient space of $\mathfrak{B}$, $\mathfrak{X}'$ an unmixed cycle of $\mathfrak{S}$ such that $(\mathfrak{X}' \cap \mathfrak{B}, \mathfrak{S})$ and $\mathfrak{X}$ coincide locally at $\mathfrak{U}_1 \cup \mathfrak{U}_2 \cup \cdots$. Then $\mathfrak{U}$ coincides locally at each $\mathfrak{U}_j$ with $(\mathfrak{X}' \cap \mathfrak{Y}, \mathfrak{S})$ by definition. Let $S = \mathfrak{S}\{v\}^*$ be the ambient space of V; the application of Theorem 3.2 to the two algebraic correspondences $\mathfrak{X}'$, $\mathfrak{B}$ between K and S proves that $(\mathfrak{X}'\{v\}^* \cap 1V, S)$ coincides locally with $\mathfrak{x}$ at U. The same theorem, applied to $\mathfrak{X}'$ and $\mathfrak{Y}$, yields the result that $(\mathfrak{X}'\{v\}^* \cap \mathfrak{y}, S)$ coincides locally at U with $\mathfrak{U}\{v\}^*$; therefore $\mathfrak{U}\{v\}^*$ coincides locally at U with $(\mathfrak{x} \cap \mathfrak{y}, V)$, Q.E.D.

THEOREM 5.8. *Let V be an n-dimensional irreducible variety over k, $\mathfrak{y}$ an r-dimensional irreducible cycle of V, U an irreducible subvariety of* rad $\mathfrak{y}$, *$\mathfrak{z}$ an s-dimensional cycle of V which is a complete intersection at U on V, and such that*

$$r + s - n = \dim U.$$

Assume $\mathfrak{y}$ to be a section of V at U. Let $\{\zeta\}$ be a set of representatives of $\mathfrak{z}$ at U on V, and set

$$\mathfrak{p} = \mathfrak{P}(\mathrm{rad}\ \mathfrak{y}/V) \cap Q(U/V).$$

If σ is the homomorphic mapping of $Q(U/V)$ whose kernel is $\mathfrak{p}$, then $\{\sigma\zeta\}$ is a set of parameters of $Q(U/\mathrm{rad}\ \mathfrak{y})$, and $i(U, \mathfrak{y} \cap \mathfrak{z}, V)$ exists and equals

$$e(Q(U/\mathrm{rad}\ \mathfrak{y});\ \sigma\zeta).$$

Proof. If $\{x\}$ is a n.h.g.p. of V for which U is at finite distance, we may assume $\zeta_j \in k[x]$ for each j. Let $\{X\}$ be the correspondent n.h.g.p. of the ambient space S of V, τ the homomorphic mapping of $k[X]$ onto $k[x]$ such that $\tau X_j = x_j$. Let z_j $(j = 1, 2, \cdots)$ be elements of $k[X]$ such that $\tau z_j = \zeta_j$; then the set $\{z\}$ is a subset of a set of parameters of $Q(U/S)$, and, if $m = \dim S$, there exists a cycle $\mathfrak{Z}$ of S, of dimension $s + m - n$, such that $\mathfrak{Z}$ is a complete intersection at U on S, and has $\{z\}$ as a set of representatives at U on S. Therefore, by Lemma 3.2, $\mathfrak{z}$ coincides locally at U with $(\mathfrak{Z} \cap 1V, S)$, so that $i(U, \mathfrak{y} \cap \mathfrak{z}, V)$ exists and coincides with $i(U, \mathfrak{y} \cap \mathfrak{Z}, S)$ locally at U; this, in turn, by Lemma 3.2, equals $e(Q(U/\mathrm{rad}\ \mathfrak{y}); \sigma\tau z)$, Q.E.D.

THEOREM 5.9 (RELATIVE INVARIANCE OF THE INTERSECTION MULTIPLICITY). *Let V, V' be irreducible varieties over k, T a birational correspondence between V and V'; let $\mathfrak{y}$, $\mathfrak{z}$ be unmixed cycles of V, U a component variety of $(\mathfrak{y} \cap \mathfrak{z}, V)$ such that $i(U, \mathfrak{y} \cap \mathfrak{z}, V)$ exists. Assume T to be regular*[2] *at U, so that T is also regular at each component variety $\mathfrak{y}_j$ of $\mathfrak{y}$ containing U and at each component variety $\mathfrak{z}_l$ of $\mathfrak{z}$ containing U. Let a_j, b_l be the multiplicities of $\mathfrak{y}_j$, $\mathfrak{z}_l$, respectively, in $\mathfrak{y}$, $\mathfrak{z}$; set*

$$U' = T(U),\ \mathfrak{y}'_j = T(\mathfrak{y}_j),\ \mathfrak{z}'_l = T(\mathfrak{z}_l),\ \mathfrak{y}' = \sum_j a_j\, \mathfrak{y}'_j,\ \mathfrak{z}' = \sum_l b_l\, \mathfrak{z}'_l.$$

Then if $i(U', \mathfrak{y}' \cap \mathfrak{z}', V')$ *exists it equals* $i(U, \mathfrak{y} \cap \mathfrak{z}, V)$.

The cycle $\mathfrak{y}'$ is called a *transform of* $\mathfrak{y}$ *at* U *in* (or *with respect to*) T.

Proof. By considering the composite variety of V and V', we may clearly reduce the proof to the following simpler case: There exists a n.h.g.p. $\{x\}$ of V for which U is a finite distance, and there exist elements $x'_1, x'_2, \cdots \in k(V)$ such that $\{x, x'\}$ is a n.h.g.p. of V' for which U' is a finite distance. In this case let S be the ambient space of V, and let $\{X\}$ be the n.h.g.p. of S corresponding to $\{x\}$; if S' is similarly related to V', we may assume that a n.h.g.p. of S' has the form $\{X, X'\}$ $\{X'\}$ being a set of indeterminates. The correspondence between V' and V is now visualized as a "projection" of V' on $S \subseteq S'$.

If A (resp. A') is an irreducible subvariety of S (resp. of V') containing U (resp. U'), whose n.h.g.p. is $\{\xi\}$ (resp. $\{\xi, \xi'\}$), we shall denote by A^* the irreducible subvariety of S' whose n.h.g.p. is $\{\xi, X'\}$; therefore we have

$$A^* \cap S = A \qquad (\text{resp. } A^* \cap V' = A').$$

This correspondence generates in an obvious way a correspondence

$$\mathfrak{Z} \longrightarrow \mathfrak{Z}^* \qquad (\text{resp. } \mathfrak{Z}' \longrightarrow \mathfrak{Z}^*)$$

among cycles. Now, let $\mathfrak{Z}$ be a cycle of S such that $(1V \cap \mathfrak{Z}, S)$ coincides locally at U with $\mathfrak{z}$; then

$$i(U, \mathfrak{y} \cap \mathfrak{z}, V) = i(U, \mathfrak{y} \cap \mathfrak{Z}, S)$$

by definition. Theorem 4.4 readily shows that $\mathfrak{y}$, $\mathfrak{Z}$, $1V$ coincide, respectively, with $(\mathfrak{y}^* \cap 1S, S')$, $(\mathfrak{Z}^* \cap 1S, S)$, $(1V^* \cap 1S, S')$ locally at U, and then an immediate application of Theorem 5.6 yields that

$$i(U, \mathfrak{y} \cap \mathfrak{Z}, S) = i(U^*, \mathfrak{y}^* \cap \mathfrak{Z}^*, S').$$

In like manner, we obtain that $\mathfrak{z}^*$ coincides locally at U^* with $(\mathfrak{Z}^* \cap 1V^*, S')$, and therefore also that

$$i(U^*, \mathfrak{y}^* \cap \mathfrak{Z}^*, S') = i(U^*, \mathfrak{y}^* \cap \mathfrak{z}^*, V^*).$$

We now wish to show that $1V'$ is a complete intersection on V^* at each irreducible subvariety A' of V' which contains U' (and which is therefore regular for the birational correspondence between V and V'). Let in fact A be the transform of A' in V; since

$$Q(A/V) = Q(A'/V'),$$

there exists a $p \in k[x] - \wp(A/k[x])$ such that $px'_j \in k[x]$ for each j. We also have $p \in k[x, X'] - \wp(A'/k[x, X'])$, and therefore

$$X'_j - x'_j = p^{-1}(pX'_j - x'_j) \in Q(A'/V^*)$$

for each j. The set $\{\cdots, X'_j - x'_j, \cdots\}$ is a regular set of parameters of $Q(V'/V^*)$, hence a subset of a set of parameters of $Q(A'/V^*)$, hence also a set of representatives of $1V'$ at A' on V^*, as announced.

This being established, we apply Theorem 5.8 to the varieties V^*, U' and and the irreducible cycles $1V'$, $1U^*$, obtaining the result that $i(U', 1U^* \cap 1V', V^*)$ exists and equals

$$e(Q(U'/U^*); \cdots, X'_j - \xi'_j, \cdots),$$

where we have denoted by $\{\xi, \xi'\}$ the n.h.g.p. of U'; but, as before,

$$\{\cdots, X'_j - \xi'_j, \cdots\}$$

is a regular set of parameters of $Q(U'/U^*)$, and therefore

$$(1U^* \cap 1V', V^*) = 1U'.$$

Likewise, we obtain that $(\mathfrak{y}^* \cap 1V', V^*)$ and $(\mathfrak{z}^* \cap 1V', V^*)$ coincide locally at U' with $\mathfrak{y}'$, $\mathfrak{z}'$ respectively. Now, Theorem 5.6 applied to V^*, V', $\mathfrak{y}'$, $\mathfrak{z}'$, U', $\mathfrak{y}^*$, $\mathfrak{z}^*$ yields the result that $(\mathfrak{y}' \cap \mathfrak{z}', V')$ exists locally at U' and coincides locally at U' with

$$(i(U^*, \mathfrak{y}^* \cap \mathfrak{z}^*, V^*)\, U^* \cap 1V', V^*).$$

In view of the previous equalities, this amounts to saying that

$$i(U', \mathfrak{y}' \cap \mathfrak{z}', V') = i(U, \mathfrak{y} \cap \mathfrak{z}, V), \text{ Q.E.D.}$$

Theorem 5.9 implies that $i(U, \mathfrak{y} \cap \mathfrak{z}, V)$ depends only on $Q(U/V)$, on the quotient rings in V of those component varieties of $\mathfrak{y}$, $\mathfrak{z}$ which contain U, and on the multiplicities of such component varieties in $\mathfrak{y}$, $\mathfrak{z}$ respectively. Accordingly, in the notations of Theorem 5.9, if $\mathfrak{y}'$, $\mathfrak{z}'$ are not both sections of V' at U', but $i(U, \mathfrak{y} \cap \mathfrak{z}, V)$ exists, we shall define $i(U', \mathfrak{y}' \cap \mathfrak{z}', V')$ to be equal to $i(U, \mathfrak{y} \cap \mathfrak{z}, V)$; Theorem 5.9 itself shows that this is a good difinition, that is, that it is independent of the choice of V'. This enables us to define $i(U, \mathfrak{y} \cap \mathfrak{z}, V)$ also when V is an irreducible pseudovariety (see [1]), since each irreducible pseudovariety is regularly equivalent to an irreducible variety. The question is now raised as to whether all the results of this section remain true for the present extended definition of the meaning of the symbol $i(U, \mathfrak{h} \cap \mathfrak{z}, V)$. The answer is as follows:

REMARK. Theorems 5.2 to 5.9 remain true after we replace the word "vaiety" by the word "pseudovariety", and the sentence "$\mathfrak{h}$ is a section of V at U" (or a logically equivalent one) by the sentence "there exists an irreducible variety V', birationally equivalent to V in a correspondence T which is regular at each component of U, such that a transform of $\mathfrak{h}$ at U in T is a section of V' at $T(U)$" (or by a logically equivalent one). The question is not even raised, however, when U is simple on V and the ground field is algebraically closed (see Theorem 5.1).

A comparison between Theorem 5.8 and the corollary to Lemma 1.1 shows the *a posteriori* connection between the theory of intersections and the theory of algebraic correspondences, namely:

THEOREM 5.10. *Let D be an unmixed algebraic correspondence between the irreducible variety F over k and the projective space S over k, and assume each component of D to operate on the whole F. Let G be an irreducible subvariety of F, D^* a component of $[D; S, G]$. Then if*

$$e(D^*/D; S, G)^* \text{ and } i(D^*, D \cap 1(S \times G), S \times F)$$

both exist, they are equal.

From Theorems 5.1, 4.1, and 4.4 we obtain:

THEOREM 5.11. *Let U be a simple irreducible subvariety of the irreducible variety V over the algebraically closed field k. If $\mathfrak{h}$, $\mathfrak{z}$ are irreducible cycles of V such that U is a component variety of $(\mathfrak{h} \cap \mathfrak{z}, V)$, then $i(U, \mathfrak{h} \cap \mathfrak{z}, V)$ exists and is a positive integer. A necessary and sufficient condition in order that*

$$i(U, \mathfrak{h} \cap \mathfrak{z}, V) = 1$$

is that $\wp(U/V)$ be an isolated primary component of

$$\wp(\text{rad } \mathfrak{h}/V) + \wp(\text{rad } \mathfrak{z}/V).$$

Let finally U, V be irreducible subvarieties of a projective space S over an algebraically closed field k; let S' be a "copy" of S over k, U' a copy of U in S', M a component variety of $(1U \cap 1V, S)$. Let Δ be the identical algebraic correspondence between S and S', and set

$$M_\Delta = [\Delta; M, S'],\ U_\Delta = [\Delta; U, S'],\ V_\Delta = [\Delta; V, S'].$$

From the results of the present section, the following equalities are easily established:

$$1(U' \times V) = (1(U' \times S) \cap 1(S' \times V), S \times S');$$

$$1U_\Delta = (\Delta \cap 1(U' \times S), S \times S'),\quad 1V_\Delta = (\Delta \cap 1(V \times S'), S \times S'),$$

$$i(M, 1U \cap 1V, S) = i(M_\Delta, 1U_\Delta \cap 1V_\Delta, \mathrm{rad}\,\Delta)$$

$$= i(M_\Delta, \Delta \cap 1(U' \times S) \cap 1(V \times S'), S \times S')$$

$$= i(M_\Delta, \Delta \cap 1(U' \times V), S \times S'),$$

and this, by Theorem 5.8, proves that our definition of intersection multiplicities coincides with the one given in [3] for the case of algebraic varieties, when the latter is defined.

References

1. I. Barsotti, *Algebraic correspondences between algebraic varieties*, Ann. of Math. (2) 52 (1950), 427-464; see also *Errata* to same, 53 (1951), 587.

2. ———, *Local properties of algebraic correspondences*, Trans. Amer, Math. Soc. 71 (1951), 349-378.

3. C. Chevalley, *Intersection of algebraic and algebroid varieties*, Trans. Amer. Math. Soc. 57 (1945), 1-85.

4. F. Severi, *Sulle molteplicità d'intersezione delle varietà algebriche ed analitiche e sopra una teoria geometrica dell'eliminazione*, Math Zeit. 52 (1950), 827-851.

5. A. Weil, *Foundations of algebraic geometry*, Amer. Math. Soc. Coll. Publ., vol. 29, (1946).

6. O. Zariski, *The concept of a simple point of an abstract algebraic variety*, Trans. Amer. Math. Soc. 62 (1947) 1-52.

THE UNIVERSITY OF PITTSBURGH