A GENERALIZATION OF THE CENTRAL ELEMENTS OF A GROUP

EUGENE SCHENKMAN

1. Introduction. If a and g are elements of a group G, we shall denote by $a^{(1)}(g)$ or $a(g)$ the element $g^{-1}ag$, and then for $n = 2, 3, 4, \cdots$ define $a^{(n)}(g) = a(a^{(n-1)}(g))$.

If for some n and all $g \in G$, $a^{(n)}(g) = a$ then a will be called *weakly central of order n* or simply *weakly central*. Thus the center elements of G are weakly central of order 1.

As usual, let

$$[g, a] = a^{-1} g^{-1} ag = a^{-1} \cdot a(g);$$

then it can readily be verified by induction on n that

$$a^{-1} \cdot a^{(n)}(g) = a^{-1} \cdot [\overbrace{a \cdots [a}^{n-1 \text{ times}}, g] \cdots]^{-1} \cdot a \cdot [\overbrace{a \cdots [a}^{n-1 \text{ times}}, g] \cdots]$$

$$= [\overbrace{a \cdots [a}^{n \text{ times}}, g] \cdots].$$

Thus $a^{(n)}(g) = a$ is equivalent to

$$[\overbrace{a \cdots [a}^{n \text{ times}}, g] \cdots] = e,$$

where e is the identity of G. It follows that if a is an element of a normal nilpotent finite subgroup of G then a is weakly central. Another easy consequence of the definition is that if a is weakly central in G then a is its own normalizer in G if and only if $\{a\} = G$; here $\{a\}$ denotes the subgroup generated by a. It should also be noted that if a is weakly central in G, then $\overline{a}$ is weakly central in $\overline{G}$, where $\overline{a}$ is the image of a under a homomorphism which takes G onto $\overline{G}$.

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2. Theorems. We shall establish the following results.

THEOREM 1. *If in a locally finite group G all elements whose orders are powers of a certain prime p are weakly central then they comprise a normal subgroup of G.*

An immediate consequence is the following analogue to Engel's Theorem for Lie algebras.

COROLLARY 1. *If all the elements of a locally finite group are weakly central then the group is the direct product of p-groups.*

THEOREM 2. *If G is a locally finite solvable group then the weakly central elements comprise a normal subgroup of G which is the direct product of p-groups.*

This result can also be stated as follows for finite groups.

THEOREM 2*a*. *If G is a finite solvable group then an element is weakly central if and only if it is in the nil radical of the group. Here the nil radical refers to the largest of all the nilpotent normal subgroups—nilpotent in the sense that $H^n = e$, where $H^n = [H^{n-1}, H]$* (cf. [1, pp. 98-102]).

It has not been determined whether solvability must necessarily be assumed for Theorems 2 and 2*a* to be true.

3. Proof of Theorem 1. We shall first consider the case where G is finite, and use induction on the order of G. Let p be the prime such that all elements of G whose orders are a power of p are weakly central. We must show that if S_0 is a p-Sylow subgroup of G then S_0 is the only p-Sylow subgroup, and hence is normal in G. We do this by obtaining a contradiction in case S_0 is not normal in G. Let $S_1, \cdots, S_k$ be the conjugate Sylow subgroups of S_0, and suppose first that $S_i \cap S_0 = \{e\}$ for $i = 1, \cdots, k$. If $N \neq G$ is the normalizer of S_0, then every element y of G not in N transform S_0 into one of the S_i. But then for $e \neq a \in S_0$ we have $a(y) \notin S_0$ and consequently $a(y) \notin N$, since

$$N \cap S_i = S_0 \cap S_i = \{e\},$$

and hence, for all positive integers n, $a^{(n)}(y) \notin S_0$. It follows that $a^{(n)}(y) \neq a$ for all n, and a is not weakly central, contrary to hypothesis.

Accordingly we need only consider the case where $S_i \cap S_0 = \{e\}$ for some i. Let D be a maximal intersection of two different Sylow subgroups. Then the normalizer N_D of D in G must have more than one p-Sylow subgroup [2, Chap. IV, Theorem 7]. It follows by our induction assumption that N_D must equal G; for if N_D were properly contained in G it would have but one p-Sylow subgroup, contrary to the above. But now if $N_D = G$, then D is normal in G, and the order of G/D is less than that of G; consequently, again by the induction assumption, G/D has but one p-Sylow subgroup. On the other hand $N_D = G$ has more than one p-Sylow subgroup containing D, and therefore so also has G/D; this again leads us to a contradiction. Thus in this case S_0 must be normal in G as the theorem asserts.

REMARK The above proof shows that a weakly central element of prime power order must lie in the intersection of at least 2 p-Sylow subgroups if the number of p-Sylow subgroups is greater than one.

We return to the proof of Theorem 1 and consider the case where G is locally finite. This means that any finite set of elements of G generates a finite subgroup of G. Now we are assuming that the elements belonging to a certain prime p are weakly central, and wish to show that they comprise a normal subgroup of G. It is obvious that they form an invariant set, and hence they generate a normal subgroup of G. Furthermore, the product of any two elements whose orders are powers of p has also order a power of p because of the local finiteness of G and because the theorem is true for finite groups. It follows that the elements whose orders are powers of p actually comprise the group they generate. This completes the proof of Theorem 1.

4. Proof of Theorem $2a$. From a previous remark we know that if an element is in the nil radical then it is weakly central. We must show conversely that if an element is weakly central then it is in the nil radical. The proof will be made by induction on the order of G. If the order is one then the theorem is obviously true. We now assume the theorem true for groups whose orders are less than k, and let G be a group of order k. Let N be the nil radical, and g a weakly central element of G. If $\{g, N\} \neq G$ then gN is weakly central in G/N, and hence by the induction assumption gN is contained in a proper normal subgroup M/N of G/N; (if the nil radical of G/N is not a proper subgroup of G/N then G/N is nilpotent and the statement is true since every proper subgroup of a finite nilpotent group is contained in a proper normal subgroup). It follows that g and N are contained in a proper normal subgroup M of G, and therefore by the induction assumption g is in the nil radical N_M of M; but N_M is contained in N since the nil radical is a characteristic subgroup (cf. [1, p. 102]), and hence N_M is a normal nilpotent subgroup of G; therefore when $\{g, N\} \neq G$ then $g \in N$ as we wished to show.

We now consider the case where $\{g, N\} = G$. Let Z be the center of N; then Z is normal in G. Now if z is any element of Z such that $\{g, z\} \neq G$, then by the induction assumption $\{g, z\}$ is nilpotent and hence has a center Q. But $\{g, z\} \cap Z$ is normal in $\{g, z\}$ since Z is normal in G, and therefore

$$Q \cap \{g, z\} \cap Z \neq \{e\}$$

[2, Chap. IV, Theorem 14]. But then, if

$$H = Q \cap \{g, z\} \cap Z,$$

then H is in the center of G since $G = \{g, N\}$, and hence H is normal in G. It follows by the induction assumption that G/H is nilpotent, whence, for some k, $G^k \subseteq H$. But since H is in the center of G,

$$G^{k+1} = [G^k, G] \subseteq [H, G] = \{e\},$$

and therefore G is nilpotent; $G = N$, and $g \in N$ as was to be shown.

Accordingly we need now only consider the case where $\{g, z\} = G$ for every $z \in Z$. Since g is weakly central then $\{g\}$ cannot be its own normalizer in G; that is, $\{g\}$ is normal in R, where $R \neq \{g\}$. On the other hand, since $G = \{g, Z\}$, it follows that R or a subgroup of R is of the form $\{g, z\} = G$, so that $R = G$. Hence g is in a cyclic normal subgroup of G, and consequently is in the nil radical N as we wished to show. This completes the proof of Theorem $2a$.

5. **Proof of Theorem 2.** We first note that the product of two weakly central elements is weakly central since they generate a finite group in which Theorem $2a$ is applicable. Thus the weakly central elements comprise a subgroup which is obviously normal. It is the direct product of p-groups by Corollary 1.

References

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Louisiana State University