ON THE COMPLEX ZEROS OF FUNCTIONS OF STURM-LIOUVILLE TYPE

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1. Let $Q(z)$ **be an analytic function** of the complex variable z in a region D. In the present paper only those solutions of

$$(1.1) \qquad W'' + Q(z)W = 0$$

which are distinct from the trivial solution ($\equiv 0$) shall be considered.

In this paper the following results shall be established.

THEOREM 1. *Suppose that the following conditions are satisfied:*

(a) *the circle* $|z| \leq R$ *is contained in* D,

(b) $W(z)$ *is a solution of* (1.1), $W(0) \neq 0$,

(c) $n(r)$ *is the number of zeros of* $W(z)$ *in* $|z| \leq r$, $r < R$.

Then $n(r)$ *satisfies the inequality*

$$(1.2) \qquad n(r) \leq (\log(Rr^{-1}))^{-1}[\log(1 + R|W'(0)|\,|W(0)|^{-1}) + (2\pi)^{-1}\int_0^{2\pi}\int_0^R (R-t)\,|Q(t\,e^{i\theta})|\,dtd\theta].$$

COROLLARY 1.1. *Suppose that the following conditions are satisfied:*

(a) $Q(z)$ *is a polynomial of degree* k,

(b) *conditions* (b) *and* (c) *of Theorem* 1 *hold.*

Then $W(z)$ *is an integral function of order at most* $k+2$. *Furthermore, as* $r \longrightarrow \infty$,

$$(1.3) \qquad n(r) = O(r^{k+2}).$$

Obviously the result of Theorem 1 is not good if r is close to R. Also it

Received August 30, 1952.
Pacific J. Math. 3 (1953), 837-843

does not apply to a solution which vanishes at the origin. The following theorem is free of these restrictions.

THEOREM 2. *Suppose that the following conditions are satisfied:*

(a) *S is a closed region contained in D,*

(b) *the boundary C of S is a closed contour,*

(c) *the maximum value of $|Q(z)|$ on C is M,*

(d) *S can be divided into n subregions such that each subregion has a diameter not greater than $\pi M^{-1/2}$; and for any two points z_1 and z_2 of a subregion, the linear segment $z_1 z_2$ lies in S (we agree that the common boundary of two subregions belongs to both subregions).*

Then

(e) *if $Q(z)$ is not a constant, the number of zeros of any solution $W(z)$ of (1.1) in S is not greater than n,*

(f) *more accurately, if $Q(z)$ is not a constant, each solution $W(z)$ of (1.1) has at most one zero in each subregion, and when it is known that $W(z)$ has some zero z_i which belongs to n_i ($n_i > 1$) different subregions, $i = 1, 2, \cdots, k$, its total number of zeros in S is not greater than $n + k - (n_1 + n_2 + \cdots + n_k)$,*

(g) *if some solution of (1.1) has more than one zero in some subregion, $Q(z)$ must be a constant and $|Q(z)| = M > 0$ in D.*

We may observe that if $Q(z)$ is not a constant, M must be positive, according to the principle of the maximum modulus. If $Q(z)$ is a constant, the problem is trivial as the distribution of the zeros is known.

2. **To prove Theorem 1**, we need the following known results.

LEMMA 1. *Suppose that the following conditions are satisfied:*

(a) *$f(x)$ and $g(x)$ are real-valued functions, continuous and nonnegative for $x \geq 0$,*

(b) *M is a positive constant,*

(c) $$f(x) \leq M + \int_0^x f(t)g(t)\,dt, \qquad x \geq 0.$$

Then we have

$$f(x) \leq M e^{\int_0^x g(t)dt} \qquad x \geq 0.$$

This lemma is due to R. Bellman. For a proof of it see [1] or [5].

LEMMA 2. *Suppose that the following conditions are satisfied:*

(a) $f(z)$ *is analytic for* $|z| \leq R$, $f(0) \neq 0$,

(b) *the moduli of the zeros of* $f(z)$ *in the circle* $|z| \leq R$ *are* $r_1, r_2, \cdots, r_k$ *arranged as a nondecreasing sequence (a zero of order* p *is counted* p *times).*

Then we have

$$\log [R^k (r_1 r_2 \cdots r_k)^{-1}] = (2\pi)^{-1} \int_0^{2\pi} \log |f(R e^{i\theta})| d\theta - \log |f(0)|.$$

Lemma 2 is known as Jensen's theorem (see [4]).

3. Now we shall prove Theorem 1. Along a fixed ray radiating out from the origin, $z = r \exp(i\theta)$, equation (1.1) becomes

$$\frac{d^2 W}{dr^2} + e^{2i\theta} Q(re^{i\theta}) W = 0. \tag{3.1}$$

Integrating (3.1) twice from 0 to r, we obtain

$$W(re^{i\theta}) = W(0) + W'(0) e^{i\theta} r - e^{2i\theta} \int_0^r \int_0^h Q(te^{i\theta}) W(te^{i\theta}) dt dh, \tag{3.2}$$

where $W'(0) \exp(i\theta)$ is the value of dW/dr at the origin. Integration by parts of the integral in (3.2) gives

$$W(re^{i\theta}) = W(0) + W'(0) e^{i\theta} r - e^{2i\theta} \int_0^r (r-t) Q(te^{i\theta}) W(te^{i\theta}) dt. \tag{3.3}$$

For $r \leq R$, (3.3) yields

$$|W(re^{i\theta})| \leq |W(0)| + |W'(0)| R + \int_0^r (R-t) |Q(te^{i\theta}) W(te^{i\theta})| dt. \tag{3.4}$$

Applying Lemma 1 to (3.4), we have

$$|W(Re^{i\theta})| \leq (|W(0)| + |W'(0)| R) e^{\int_0^R (R-t) |Q(te^{i\theta})| dt}. \tag{3.5}$$

Let the moduli of the zeros of $W(z)$ in the circle $|z| \leq r < R$ be $r_1, r_2, \cdots, r_k$, arranged as a nondecreasing sequence. Then an appeal to Lemma 2 gives

$$(3.6)\quad \log[R^k(r_1 r_2 \cdots r_k)^{-1}] \leq (2\pi)^{-1}\int_0^{2\pi} \log|W(Re^{i\theta})|\,d\theta - \log|W(0)|.$$

Clearly

$$(3.7)\quad \log[R^k(r_1 r_2 \cdots r_k)^{-1}] \geq \log[R^{n(r)} r^{-n(r)}]$$

$$= n(r)\log(Rr^{-1}), \qquad r < R,$$

where $n(r)$ is the number of zeros of $W(z)$ in $|z| \leq r$. On the other hand, (3.5) gives

$$(3.8)\quad \int_0^{2\pi} \log|W(Re^{i\theta})|\,d\theta \leq 2\pi \log[|W(0)| + |W'(0)|R]$$

$$+ \int_0^{2\pi}\int_0^R (R-t)\,|Q(te^{i\theta})|\,dt\,d\theta.$$

Combining (3.6), (3.7), and (3.8), we have

$$(3.9)\quad n(r)\log(Rr^{-1}) \leq \log[|W(0)| + |W'(0)|R] - \log|W(0)|$$

$$+ (2\pi)^{-1}\int_0^{2\pi}\int_0^R (R-t)\,|Q(te^{i\theta})|\,dt\,d\theta$$

for $r < R$. But (3.9) is equivalent to (1.2), so that this completes the proof of Theorem 1.

If $Q(z)$ is a polynomial of degree k, then $W(z)$ is analytic except at infinity and, from (3.5),

$$|W(Re^{i\theta})| = O\left(e^{A\cdot R^{k+2}}\right), \qquad R \longrightarrow \infty,$$

where A is a constant. Hence $W(z)$ is an integral function of order at most $k+2$. Finally if we set $R = 2r$ in (3.9), it is clear that

$$n(r) = O(r^{k+2}).$$

This proves Corollary 1.1.

4. To prove Theorem 2, we need the following known result. On the real axis, equation (1.1) becomes

$$(4.1) \qquad \frac{d^2W}{dx^2} + Q(x)W = 0,$$

where x is the real part of the complex variable z. Denote by $q_1(x)$ the real part of $Q(x)$.

LEMMA 3. *Let $W(x)$ be a solution of (4.1), $W(0) = 0$. Suppose that one of the following conditions is satisfied.*

(a) $\max q_1(x) = m > 0$ *in* $[0, a]$, $0 < a \leq \pi m^{-1/2}$, *and* $Q(x) \neq m$ *in* $[0, a]$,

(b) $q_1(x) \leq 0$ *in* $[0, a]$.

Then $W(x) \neq 0$ *in* $(0, a]$.

This lemma was proved in [3; Theorems 5.1, 5.2]. Part (b) is also covered by a theorem of Hille [2, p. 512 ff.]. Its proof remains valid even if $Q(x)$ is assumed only to be a continuous (complex-valued) function of a real variable x; consequently the lemma remains true under such an assumption on $Q(x)$.

We first prove (*f*) of Theorem 2.

Let S_i be one of the subregions of S with a diameter not greater than $\pi M^{-1/2}$. Suppose that $W(z)$ is a solution of (1.1) which vanishes at a point z_0, say, of S_i. Consider a fixed ray radiating out from z_0, $z - z_0 = r \exp(i\theta)$. Along this ray, equation (1.1) becomes

$$(4.2) \qquad \frac{d^2W}{dr^2} + e^{2i\theta} Q(z_0 + re^{i\theta})W = 0.$$

By virtue of the principle of the maximum modulus, we have

$$|e^{2i\theta} Q(z)| = |Q(z)| \leq M$$

for any point z of S on this ray. Hence on a segment of this ray between z_0 and any other point of S_i (by assumption, this segment lies in S) the maximum value m, say, of the real part of $\exp(2i\theta)Q(z)$ is not greater than M. If m is positive, then $\pi m^{-1/2} \geq \pi M^{-1/2}$. Since $Q(z)$ is not a constant, $\exp(2i\theta)Q(z) \neq m$ on this segment. By virtue of the fact that the diameter of S_i is not greater than $\pi M^{-1/2}$ and Lemma 3, it is clear that $W(z)$ does not vanish again on that part of the ray in S_i, regardless of the sign of m. Repeating this process for each ray radiating out from z_0, we see clearly that $W(z)$ cannot vanish again in S_i. Since S_i is an arbitrary subregion, $W(z)$ can vanish at most at one point of each subregion.

On the other hand, if $W(z)$ has a zero z_i which belongs to n_i $(n_i > 1)$ different subregions, then $W(z)$ cannot vanish again in any of these n_i subregions, as the foregoing proof shows. If it is known that there are k such zeros z_i, each z_i belonging to n_i subregions, $i = 1, 2, \cdots, k$, it is clear that the total number of zeros of $W(z)$ in S is not greater than $n + k - (n_1 + n_2 + \cdots + n_k)$.

To prove (g), let $W(z)$ be a solution of (1.1) having two zeros, say z_0 and z_1, in some subregion S_i. Let the argument of $z_1 - z_0$ be θ. Then along the linear segment $z_0 z_1$, equation (1.1) becomes (4.2). According to Lemma 3, the maximum value m of the real part of $\exp(2i\theta)Q(z)$ on the linear segment $z_0 z_1$ must be positive. Further, since

$$(4.3) \qquad |z_1 - z_0| \le \pi M^{-1/2} \le \pi m^{-1/2},$$

z_0 and z_1 can both be the zeros of $W(z)$ only if

$$(4.4) \qquad e^{2i\theta}Q(z) \equiv m$$

on the linear segment $z_0 z_1$, by Lemma 3 again. But if (4.4) is true, the general solution of (4.2) is $A \sin(m^{1/2}r + B)$, A and B being constants. If a solution of (4.2) has two zeros, the distance between them must not be less than $\pi m^{-1/2}$. In other words, the equality signs in (4.3) must hold. That is, $M = m$. From (4.4), we have $\exp(2i\theta)Q(z) \equiv M$ on the linear segment $z_0 z_1$. Since $Q(z)$ is an analytic function and constant on the linear segment $z_0 z_1$, $Q(z)$ is a constant in D. Obviously $|Q(z)| = M$; and since m is positive, so is M. This proves (g).

Clearly (e) follows from (f), and this completes the proof of Theorem 2.

5. Added in proof. The author is indebted to a referee for calling his attention to the fact that, in connection with Corollary 1.1, an entire function which satisfies a linear differential equation with coefficients which are rational functions of z is always of finite rational order and of perfectly regular growth. (See G. Valiron, *Lectures on the theory of integral functions*, Toulouse, 1923, p. 106 ff.)

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THE UNIVERSITY OF MISSOURI AND
THE CATHOLIC UNIVERSITY OF AMERICA

ACKNOWLEDGMENT

The editors gratefully acknowledge the services of the following persons not members of the Editorial Staff who have been consulted concerning the preparation of the third volume of this Journal:

R.P. Agnew, C.B. Allendoerfer, T.M. Apostol, Richard Arens, Nachman Aronszajn, R. Baer, R.E. Bellman, Stefan Bergman, Garrett Birkhoff, L.M. Blumenthal, R.P. Boas, Jr., H.E. Bray, F.E. Browder, R.C. Buck, C. Chevalley, I.S. Cohen, H.V. Craig, J.L. Doob, R.J. Duffin, Nelson Dunford, W.F. Eberlein, Arthur Erdélyi, H. Federer, Werner Fenchel, N.J. Fine, Harley Flanders, G.E. Forsythe, I.S. Gal, P.R. Garabedian, Wallace Givens, L.M. Graves, J.W. Green, W. Gustin, M. Hall, T.E. Harris, Olaf Helmer, I.N. Herstein, J.D. Hill, E. Hille, P. Hodge, R.C. James, R.D. James, F. John, S. Johnson, Jan Kalicki, Irving Kaplansky, W. Karush, M.S. Knebelman, R.M. Lakness, D.H. Lehmer, N. Levinson, Lee Lorch, M.M. Loève, R.C. Lyndon, G.R. MacLane, J.C.C. McKinsey, N.H. McCoy, Wilhelm Magnus, H.B. Mann, W.R. Mann, E. Michael, C.B. Morrey, Jr., T. Motzkin, Ivan Niven, L.J. Paige, H. Rademacher, R.M. Redheffer, W.T. Reid, J. Riordan, M.S. Robertson, J. Robinson, W.E. Roth, H.S. Ruse, Peter Scherk, W. Seidel, H.N. Shapiro, H. Shniad, D.C. Spencer, N.E. Steenrod, J.J. Stoker, D. Swift, Gabor Szegö, E.W. Titt, A.W. Tucker, F.A. Valentine, H.S. Vandiver, R. Wagner, J.L. Walsh, S.S. Walters, Morgan Ward, H.F. Weinberger, A. Weinstein, A.L. Whiteman, L.R. Wilcox, Frantisek Wolf, J. Wolfowitz, D. Zelinsky, M. Zorn, A. Zygmund.