ON TWO THEOREMS OF PHRAGMÉN-LINDELÖF FOR LINEAR ELLIPTIC AND PARABOLIC DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

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1. Introduction. In Part I of this paper our main interest is to generalize to elliptic equations the following theorem of Phragmén-Lindelöf:

THEOREM 0. *If* $f(z) \to a$ *as* $z \to \infty$ *along two straight lines, and* $f(z)$ *is regular and bounded in the angle between them, then* $f(z) \to a$ *uniformly in the whole angle as* $z \to \infty$.

A generalization of the classic Phragmén-Lindelöf theorem to elliptic equations was given by Gilbarg [1] and Hopf [4]. A refined form of that classic theorem, due to the Nevanlinnas [5], [6; 42–44] and Heins [3], was generalized to elliptic equations by Serrin [8].

In generalizing Theorem 0 we shall make an extensive use of the Gilbarg-Hopf results.

In Part II we generalize to parabolic equations both the classic Phragmén-Lindelöf Theorem and Theorem 0.

In § 2, Theorem 0 is proved for elliptic equations defined in any 2-dimensional domains (Theorems 1, 2). The case $n>2$ is treated in § 3, for domains contained in a half space. In § 4 we consider the behavior of solutions in an angular neighborhood of the origin, and we obtain results similar to those of §§ 2, 3. In §§ 5, 6, generalizations to parabolic equations are given: Theorems 7, 9 extend the classic Phragmén-Lindelöf Theorem and Theorems 8, 10 extend Theorem 0.

The results in Part I are somewhat analogous with Theorems 2, 3, 3′ of Gilbarg-Serrin's paper [2]. The similarity appears both in the type of conditions imposed on the coefficients of the elliptic operator and in the assertions. It is however important to note that our results cannot be obtained by the Gilbarg-Serrin methods, since Harnack Inequalities which play an essential role in their paper, do not hold uniformly in open domains.

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PART I

2. Consider the differential operator

$$Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} \qquad x=(x_1, \cdots, x_n) \tag{1}$$

defined in a domain D. In this and the following chapter D is supposed to be unbounded. We denote by ∂D the boundary of D, and by $\bar{D}$ the closure of D. We shall assume throughout Part I that L satisfies the following conditions ([1], [4]):

(i) $\sum_{i,j} |a_{ij}(x)|$ is bounded in D, and, for all $x \in D$, ξ_i real,

$$\sum_{i,j} a_{ij}(x)\xi_i\xi_j \geqq \alpha \sum_i \xi_i^2 \qquad \alpha > 0 ,$$

(ii) for all $x \in D$, $|x|=r$,

$$\sum_i |b_i(x)| \leqq p(r) , \tag{2}$$

where $p(r)$, defined for $0<r<\infty$, is monotone decreasing and

$$\int_0^\infty p(r)\, dr < \infty .$$

Define $a_{ij}(\infty)=\lim a_{ij}(x)$ as $|x| \to \infty$ $(x \in D)$, whenever the limit exists. The matrix $(a_{ij}(x))$ is said to be *Dini continuous* at infinity, if there exists a monotone decreasing function $\varphi(r)$ with $\int^\infty r^{-1}\varphi(r)\, dr < \infty$, such that for $x \in D$, $|x|=r$,

$$\sum_{i,j} |a_{ij}(x) - a_{ij}(\infty)| \leqq \varphi(r) .$$

Let $u(x)$ be defined in D and belong to $C^2(D)$. In Theorems 1–6 the function $u(x)$ is also assumed to be continuous in $\bar{D}$. Denote

$$m(r) = \inf_{x \in D,\, |x|=r} u(x) , \qquad \mu(r) = \sup_{x \in D,\, |x|=r} |u(x)| .$$

Let K_β denote the n-dimensional cone with angular opening β, $0<\beta \leqq 2\pi$, whose axis is the positive x_n-axis and whose vertex is at the origin.

LEMMA 1. *Suppose $D \subset K_\beta$, $n=2$. Assume that L satisfies* (i), (ii) *and that $(a_{ij}(x))$ is continuous at infinity with $a_{ij}(\infty)=\delta_{ij}$. If $Lu(x) \leqq 0$ in the open set $D_{r_0}=D \cap |x|>r_0$, $u(x) \geqq 0$ on ∂D_{r_0} and for some $\gamma' < \gamma = \pi/\beta$,*

$$\lim_{k\to\infty} r_k^{-\gamma'} m(r_k) = 0 \qquad (r_k \to \infty \text{ as } k \to \infty) ,$$

and if r_0 *is sufficiently large* (*depending only on* L, β *and* γ'), *then* $u(x) \geqq 0$ *in* D_{r_0}.

By $u(x) \geqq 0$ on ∂G we mean: $\liminf u(x) \geqq 0$ as x tends to ∂G ($x \in G$).

Proof. Following the Gilbarg-Hopf method, it is enough to prove the existence of functions $v_R(x)$, $r_0 < R < \infty$, with the following properties:

$$\begin{aligned} &v_R(x) \geqq 0 \quad \text{if} \quad |x| \leqq R\,, \quad x \in \partial D_{r_0} \\ &v_R(x) = 1 \quad \text{if} \quad |x| = R\,, \quad x \in D_{r_0}\,, \end{aligned} \tag{3}$$

$$Lv_R(x) \leqq 0 \quad \text{if} \quad |x| < R\,, \quad x \in D_{r_0}\,, \tag{4}$$

$$\text{for every } x \in D_{r_0}\,,\quad R^{\gamma'} v_R(x) \text{ is bounded as } R \to \infty\,. \tag{5}$$

Denote by $h(x'_1, x'_2)$ the harmonic function defined in the semicircle C': $x'^2_1 + x'^2_2 < 1$, $x'_2 > 0$, which takes the value 0 on the diameter and the value 1 on the rest of the boundary. The transformation $z' = z^\delta$, where $\gamma' < \delta < \gamma$, $z' = x'_2 + ix'_1$, $z = x_2 + ix_1$, maps $S = K_\beta \cap |x| < 1$ onto a domain $S' \subset C'$. The function $k(x_1, x_2) = h(x'_1, x'_2)$ is harmonic in S and takes boundary values $\geqq 0$ on the radii and the value 1 on the rest of the boundary. We shall find $v_R(x)$ in the form $v_R(x) = f_R\left(k\left(\frac{x}{R}\right)\right)$.

If we show, in addition to $Lf_R \leqq 0$, that

$$f_R(0) = 0\,,\quad f_R(1) = 1\,,\quad 0 \leqq f_R(k) \leqq 1 \quad \text{if} \quad 0 \leqq k \leqq 1\,, \quad \text{and} \tag{6}$$

$$f_R(k) = 0(k^{\gamma'/\delta}) \text{ uniformly in } R, \quad \text{as } k \to 0\,, \tag{7}$$

then (3), (4), (5) follow. Note, in proving (5), that $R^\delta k\left(\frac{x}{R}\right)$ is bounded as $R \to \infty$. The construction of f_R proceeds as in Hopf's proof [4], except for the facts that property d) p. 421 and the inequality

$$\sum_{i,j} \frac{|h''_{ij}(x)|}{|h'(x)|^2} < C \qquad 0 < |x| < 1 \tag{8}$$

do not hold for the corresponding k.

The image of K_β under the mapping $z' = z^\delta$ is a 2-dimensional cone $K'_{\pi-\varepsilon}$ ($\varepsilon > 0$) with opening $\pi - \varepsilon$ and $S' \subset K'_{\pi-\varepsilon}$. From Hopf's proof it is clear that instead of satisfying d), it is enough for k to satisfy:

d′) along each equipotential arc $k(x) = \text{const.}$,

$$|k'(x)| = \left(\sum \left(\frac{\partial k(x)}{\partial x_i}\right)^2\right)^{1/2} \geqq H \frac{\partial k}{\partial x_2}$$

on the axis of x_2 (say at $\tilde{x}$), $H > 0$. Since the equipotential arcs of $k(x)$ is S correspond to equipotental arcs of $h(x')$ in S', we have

$$|k'(x)|=|h'(x')|\left|\frac{dz'}{dz}\right|\geqq\frac{\partial h(\tilde{x}')}{\partial x_2'}\delta|x|^{\delta-1}=\frac{\partial h(\tilde{x}')}{\partial x_2'}\delta|x'|^{(\delta-1)/\delta}$$

$$\geqq H\frac{\partial h(\tilde{x}')}{\partial x_2'}\delta|\tilde{x}'|^{(\delta-1)/\delta}=H\frac{\partial k(\tilde{x})}{\partial x_2},$$

where $\tilde{x}'$ is the image of $\tilde{x}$ and $H>0$. Here, in the case $\delta<1$, we used the inequality $|x'|<H_1|\tilde{x}'|$ $(H_1>0)$, noting that $S'\subset K'_{\pi-\varepsilon}$.

The estimation of $\sum a_{ij}(x)k''_{ij}(\xi)$ in Lk (see [**4**; p. 423]) has to be modified, since (8) does not hold for k. Defining

$$(9)\qquad \varepsilon_{ij}(x)=a_{ij}(x)-\delta_{ij}, \qquad \varepsilon(r)=\sup_{x\in D,\ |x|=r}\sum|\varepsilon_{ij}(x)|,$$

and using the harmonicity of k, we get

$$I=|k'(\xi)|^{-2}|\sum a_{ij}(x)k''_{ij}(\xi)|\leqq AC+\sum|\varepsilon_{ij}(x)|\frac{|\delta-1|\,|\xi|^{\delta-2}}{\delta|h'(\xi')|\,|\xi|^{2(\delta-1)}}$$

$$\leqq AC+\frac{B\varepsilon(r)}{2|\xi|^{\delta}},$$

where A and B are constants, and $|\xi|<1$.

Using the inequality $2|\xi'|\geqq h(\xi')$ ([**1**; p. 414]), we obtain

$$I\leqq AC+B\varepsilon(r)k^{-1}.$$

Define r_0 to be such that if $r>r_0$ then $B\varepsilon(r)<1-\gamma'/\delta$. Then, the last inequality for I shows that Hopf's method can be applied to prove that $Lf_R\leqq 0$, provided that f_R satisfy:

$$(10)\qquad \frac{f''(k)}{f'(k)}=-AC-\frac{1-\gamma'/\delta}{k}-\frac{P(x_2)}{H(\partial k(\tilde{x})/\partial x_2)}, \qquad f'(k)>0,$$

where $\tilde{x}=(0, x_2)$ (k is a monotone function of x_2).
Solving (10) we obtain,

$$(11)\qquad f_R'(k)=Ek^{\gamma'/\delta-1}\exp(-ACk-P(x_2)), \qquad f_R(0)=0,$$

where

$$E^{-1}=\int_0^1 k^{\gamma'/\delta-1}\exp(-AC-P(x_2))\,dk, \qquad P(s)=H^{-1}\int_0^s p(t)\,dt.$$

The verification of (6), (7) is immediate and the proof is thereby completed.

LEMMA 2. *Suppose $D\subset K_\beta$, $n=2$. Assume that L satisfies* (i), (ii) *and that $(a_{ij}(x))$ is continuous at infinity with $a_{ij}(\infty)=\delta_{ij}$. If r_0 is sufficiently large, then there exists a function $w(x)$, defined in D_{r_0}, and having the following properties:*

(a) $w(x)\geqq 0$ *if* $x\in\partial D_{r_0}$,

(b) $w(x)=1$ *if* $x\in D$, $|x|=r_0$,

(c) $Lw(x)\leqq 0$ *if* $x\in D_{r_0}$, *and*

(d) $w(x)\to 0$ *uniformly in* D_{r_0} *as* $|x|\to\infty$.

Proof. To prove the lemma, define $\tilde{v}(x')=\dfrac{2}{\pi}\vartheta(x')$, where $\vartheta(x')$ is the polar angle of the point x' with $(-r_0', r_0')$ as a pole. Define also $v(x)=\tilde{v}(x')$, where x' is the image of x under the mapping $z'=z^{\gamma}$, where $\gamma=\pi/\beta$, $z'=x_2'+ix_1'$, $z=x_2+ix_1$. We try to find w in the form $w=f(v)$.

(c) implies that

$$(12)\qquad f''(v)\sum_{i,j}a_{ij}(x)\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}+f'(v)\Big(\sum_{i,j}a_{ij}(x)\frac{\partial^2 v}{\partial x_i\partial x_j}+\sum_i b_i(x)\frac{\partial v}{\partial x_i}\Big)\leqq 0\,.$$

Using the harmonicity of $v(x)$ we conclude, after some calculations (see [1; p. 414]), that (12) is a consequence of the inequalities:

$$(13)\qquad \frac{f''(v)}{f'(v)}<-A_1\varepsilon(|x|)\frac{|z'|}{r_0'}-A_2|x|p(|x|)\frac{|z'|}{r_0'}\,,\qquad f'(v)>0\,,$$

where A_1, A_2 are proper constants and $\varepsilon(r)$ is defined by (g).

Taking r_0 to be such that $2A_1\varepsilon(r)+2A_2rp(r)<1-\delta$ $(0<\delta<1)$ if $r>r_0$ (note that $rp(r)\to 0$), and using the elementary inequalities

$$|z'|\leqq r_0'\operatorname{ctg}\vartheta'/2\leqq 2r_0'/\tilde{v}(x')\,,$$

we conclude that if $f(v)$ satisfies:

$$(14)\qquad f''(v)/f'(v)=-(1-\delta)/v\,,\qquad f'(v)>0\,,$$

then (13) follows. Solving (14) we find that the function $f(v)=v^{\delta}$ satisfies (a)–(d).

THEOREM 1. *Suppose* $D\subset K_\beta$, $n=2$, *and assume that* L *satisfies* (i), (ii) *and that* $(a_{ij}(x))$ *is continuous at infinity with* $a_{ij}(\infty)=\delta_{ij}$. *If* $Lu(x)=0$ *in* D, *and, for some* η,

$$(15)\qquad \lim_{r\to\infty}\frac{\mu(r)}{r^{\pi/\beta-\eta}}=0\qquad (\eta>0 \text{ if } \beta\neq\pi\,,\quad \eta=0 \text{ if } \beta=\pi)\,,$$

and if $u(x)\to 0$ *on* ∂D *as* $|x|\to\infty$, *then* $u(x)\to 0$ *uniformly in* D *as* $|x|\to\infty$.

Proof. Given $\varepsilon>0$, there exists $r_0>0$ such that $-\varepsilon<u(x)<\varepsilon$ for

$x\in\partial D$, $|x|\geqq r_0$. Denoting $M_0=\max_{|x|=r_0}|u(x)|$, we can apply Lemma 1 (in the case $\beta=\pi$ we apply the Gilbarg-Hopf theorem) to the function $v(x)=u(x)+M_0w(x)+\varepsilon$ in the open set D_{r_0}. We get $v(x)\geqq 0$ in D_{r_0}. Taking r_1 to be such that $M_0w(x)<\varepsilon$ in D_{r_1}, we conclude that $u(x)>-2\varepsilon$ in D_{r_1}. Similarly we get $u(x)<2\varepsilon$ in D_{r_1} and the theorem is proved.

REMARK. Using a proper linear transformation we conclude that the assumption $a_{ij}(\infty)=\delta_{ij}$, can be dismissed if in (15) β is replaced by β', where β' is the angular opening of the image of K_β under the linear transformation. The continuity assumption of the $a_{ij}(x)$ at infinity can be replaced by the weaker assumption that the oscillation of the $a_{ij}(x)$ near infinity is sufficiently small.

We can reduce the case $0<\beta\leqq 2\pi$ to the case $\beta=\pi$ by the conformal mapping $z'=z^{\pi/\beta}$, where $z=x_2+ix_1$, $z'=x_2'+ix_1'$. Applying Theorem 1, we get the following theorem after some calculation.

THEOREM 2. *Let* $D\subset K_\beta$, $n=2$, *and assume that* L *satisfies* (i), (ii), *that* $(a_{ij}(x))$ *is Dini continuous at infinity with* $a_{ij}(\infty)=\delta_{ij}$, *and that* $r^{1-\gamma}p(r)$ $(\gamma=\pi/\beta)$ *is monotone decreasing. If* $Lu(x)=0$ *in* D, *and*

$$\lim_{r\to\infty}\frac{\mu(r)}{r^{\pi/\beta}}=0\ , \tag{16}$$

and if $u(x)\to 0$ *on* ∂D *as* $|x|\to\infty$, *then* $u(x)\to 0$ *uniformly in* D *as* $|x|\to\infty$.

As in Theorem 1, the restriction $a_{ij}(\infty)=\delta_{ij}$ can be dismissed, but then in (16) and in $r^{1-\gamma}p(r)$, β should be replaced by β'.

In analogue with Theorem 2, one can formulate an extension of the Gilbarg-Hopf theorem to the case $0<\beta\leqq 2\pi$. Serrin's results [8] can also be extended to domains $D\subset K_\beta$ $(0<\beta\leqq 2\pi)$ such that the image of D under the mapping $z'=z^{\pi/\beta}$ contains a half plane $x_2'>c$. In particular we have the following.

If $Lu\leqq 0$ *in* D *and* $u\geqq 0$ *on* ∂D, *then* $\lim_{r\to\infty} r^{-\pi/\beta}m(r)$ *exists and is* $\leqq 0$.

3. In this section we consider the case $n\geqq 3$.

LEMMA 3. *Suppose* $D\subset K_\beta$, $\frac{\pi}{3}\leqq\beta<\pi$, $n\geqq 3$. *Assume that* L *satisfies* (i), (ii) *and that* $(a_{ij}(x))$ *is continuous at infinity with* $a_{ij}(\infty)=\delta_{ij}$. *If* $Lu(x)\leqq 0$ *in* D_{r_0}, $u(x)\geqq 0$ *on* ∂D_{r_0}, *and, for some* $\gamma'<\gamma=\pi/\beta$,

$$\lim_{k\to\infty} r_k^{-\gamma'}m(r_k)=0 \qquad (r_k\to\infty \ \textit{as}\ k\to\infty)\ ,$$

and if r_0 *is sufficiently large, then* $u(x)\geqq 0$ *in* D_{r_0}.

Proof. The proof proceeds as in Lemma 1, if (following Hopf [4]), we define

$$K(x)=k(\rho, x_n)\ , \qquad \rho=\sqrt{x_1^2+\cdots+x_{n-1}^2}=\sqrt{r^2-x_n^2}\ , \qquad 0<r<1\ ,$$

where k is the function defined in the proof of Lemma 1. The only essential difference will be in estimating $\sum a_{ij}(x)K''_{ij}(\xi)$. Clearly,

$$\sum K''_{ii}(x)=(n-2)\frac{1}{\rho}\frac{\partial k}{\partial \rho}\ ,$$

and

$$\sum |K''_{ij}(x)|\leqq A_3\sum |k''_{ij}|+A_4\frac{1}{\rho}\left|\frac{\partial k}{\partial \rho}\right| \qquad (|x|<1\ ,\ A_3>0\ ,\ A_4>0)\ .$$

If we show that

$$J\equiv\frac{1}{\rho}\frac{\partial k}{\partial \rho}\Big/|k'|^2\leqq B_1 \qquad \text{and} \quad |J|\leqq B_1+\frac{B_2}{k}\ , \tag{17}$$

where B_1 and B_2 are positive constants, then we can proceed as in the proof of Lemma 1, and the proof of Lemma 3 will be completed.

To prove the first part of (17), we write J in the form

$$J=\frac{|z|^{\delta-1}\sin\delta\vartheta}{\sin\vartheta}\frac{1}{\rho'}\frac{|z|^{\delta-1}\cos(\delta-1)\vartheta}{|h'(z')|^2\delta|z|^{2(\delta-1)}}\frac{\partial h}{\partial\rho'}$$
$$-\frac{1}{|z|\sin\vartheta}\frac{|z|^{\delta-1}\sin(\delta-1)\vartheta}{|h'(z')|^2\delta|z|^{2(\delta-1)}}\frac{\partial h}{\partial x'_n}=J_1+J_2$$

where J_1 is the first term and $z'=z^\delta$, $z=x_n+i\rho$, $z'=x'_n+i\rho'$, $\rho=|z|\sin\vartheta$, etc.. Since $\dfrac{1}{\rho'}\dfrac{\partial h}{\partial \rho'}$ is bounded near $\rho'=0$, and since $|h'(z')|$ is bounded from below by a positive constant, we get $|J_1|\leqq B_1$.

Since $\dfrac{\partial h(z')}{\partial x'_n}\geqq 0$ and $\dfrac{\sin(\delta-1)\vartheta}{\sin\vartheta}\geqq 0$ if $1<\delta<3$ (since $1<\gamma\leqq 3$ we can take $1<\delta<3$), it follows that $J_2\leqq 0$ and consequently, $J\leqq B_1$.

The second part of (17) follows from noting that $|J_2|\leqq\dfrac{B_2}{2|z|^\delta}\leqq\dfrac{B_2}{k}$.

LEMMA 4. *Lemma 2 is true also in the case* $n\geqq 3$.

Proof. The function $t(x)=r_0^{n-2}|x|^{2-n}$ satisfies (a), (b) and (d). We shall find $w(x)$ in the form $f(t)$. Condition (c) implies that

$$(18)\qquad f''(t)\sum_{i,j} a_{ij}(x)\frac{(n-2)^2 x_i x_j r_0^{n-2}}{|x|^{2n}} + f'(t)\Big(\sum_{i,j} a_{ij}(x)\frac{n(n-2)x_i x_j}{|x|^{n+2}} - \sum_i a_{ij}(x)\frac{n-2}{|x|^n} - \sum_i b_i(x)\frac{(n-2)x_i}{|x|^n}\Big) \leqq 0 .$$

By our assumptions, $\sum |a_{ij}(x) - \delta_{ij}| \leqq \varepsilon(|x|) \to 0$ as $|x| \to 0$. Using the harmonicity of $|x|^{2-n}$, we find that if $f(t)$ satisfies

$$(19)\qquad f''(t)/f'(t) < -(B_1\varepsilon(|x|) + B_2|x|p(|x|))/t , \qquad f'(t)>0 ,$$

where B_1 and B_2 are proper constants, then (18) follows. Now, if r_0 is such that $B_1\varepsilon(r)+B_2 rp(r) < 1-\delta$ $(0<\delta<1)$ for $r>r_0$, and if

$$(20)\qquad f''(t)/f'(t) = -(1-\delta)t^{-1} , \qquad f'(t)>0 ,$$

then (19) follows. Solving (20) we get the function $f(t)=t^{\delta}$, which satisfies (a)–(d).

With Lemmas 2 and 3 at hand, we can use the argument used in proving Theorem 1 and thus get the following.

THEOREM 3. *Suppose* $D \subset K_\beta$, $\frac{\pi}{3} \leqq \beta \leqq \pi$, $n \geqq 3$. *Assume that* L *satisfies* (i), (ii) *and that* $a_{ij}(x))$ *is continuous at infinity with* $a_{ij}(\infty) = \delta_{ij}$. *If* $Lu(x) = 0$ *in* D, *and for some* η,

$$\lim_{r\to\infty} \frac{\mu(r)}{r^{\pi/\beta - \eta}} = 0 \qquad (\eta>0 \text{ if } \beta \neq \pi ,\ \eta = 0 \text{ if } \beta = \pi) ,$$

and if $u(x) \to 0$ *on* ∂D *as* $|x| \to \infty$, *then* $u(x) \to 0$ *uniformly in* D *as* $|x| \to \infty$.

REMARKS. (a) The remark which follows Theorem 1, applies also to Theorem 3.

(b) If we assume in Theorem 3, that $u(x) = 0(r^{2-n+\delta})$, $\delta>0$ on ∂D then the same holds in D. This follows by applying the maximum principle to functions of the form $u(x) \pm Ar^{2-n+\delta} \pm \varepsilon$, where A is a proper fixed constant and $\varepsilon > 0$ (compare [2; 324–325]).

4. Let D belong to the half space $x_n > 0$ and denote by C_r the open set $D \cap |x| < r$. We shall consider the behavior of solutions near $x=0$; it is therefore assumed that $0 \in \overline{D}$.

We first observe that the construction of $w(x)$ in Lemma 4, can be easily modified to derive functions $w_r(x)$ defined in $C'_r = C_{r_0} \cap |x| > r$ for all $0 < r < r_0$, and having the following properties:

(a) $w_r(x) \geqq 0$ if $x \in \partial C'_r$,

(b) $w_r(x) = 1$ if $x \in C_{r_0}$, $|x| = r$,

(c) $Lw_r(x) \leqq 0$ in C'_r, and

(d) there exists δ $(0<\delta<1)$ depending on r_0 ($\delta \to 1$ as $r_0 \to 0$), such that

$$\lim_{r\to 0} r^{\delta(2-n)} w_r(x) = 0 \qquad \text{if} \quad x \in C_{r_0} ;$$

here, r_0 is assumed to be sufficiently small, and, $(a_{ij}(x))$ is assumed to be continuous at $x=0$ with $a_{ij}(0)=\delta_{ij}$.

With the aid of $w_r(x)$ we can prove an analogue of the Gilbarg-Hopf theorem.

If $Lu \leqq 0$ *in* C_{r_0}, $u \geqq 0$ *on* ∂C_{r_0} *and*

$$\lim_{r\to 0} r^{\delta(n-2)} m(r) = 0 \qquad (0<\delta<1) ,$$

and if r_0 *is sufficiently small (depending on* δ*), then* $u \geqq 0$ *in* C_{r_0}.

We can now use the method used in proving Theorem 1, noting that the role that $w(x)$ played in that proof is now given to the function $f_{r_0}\left(h\left(\frac{x}{r_0}\right)\right)$ of Gilbarg-Hopf. The following theorem is thus proved.

THEOREM 4. *Let* D *belong to the half space* $x_n>0$, $n \geqq 3$. *Assume that* L *satisfies* (i), (ii) *and that* $(a_{ij}(x))$ *is continuous at* $x=0$. *If* $Lu(x) = 0$ *in* D, *and, for some positive* ε,

$$\lim_{r\to 0} r^{n-2-\varepsilon} \mu(r) = 0 ,$$

and if $u(x) \to 0$ *on* ∂D *as* $|x| \to 0$, *then* $u(x) \to 0$ *uniformly in* D *as* $|x| \to 0$.

The continuity assumption on the $a_{ij}(x)$ at $x=0$, can be weakened.

The case $n=2$ can be treated in a similar manner. Note that now, instead of modifying Lemma 4, we rather modify Lemma 2 and thus obtain $w_r(x)$ in the form $\left(\frac{2}{\pi}\vartheta(x'_1, x'_2)\right)^{\delta}$, where (x'_1, x'_2) is the image of (x_1, x_2) under the mapping $z'=z^{\pi/\beta}$. We have the following.

THEOREM 5. *Let* $D \subset K_\beta$, $n=2$, *and assume that* L *satisfies* (i), (ii) *and that* $(a_{ij}(x))$ *is continuous at* $x=0$ *with* $a_{ij}(0)=\delta_{ij}$. *If* $Lu(x)=0$ *in* D, *and, for some positive* ε,

$$\lim_{r\to 0} r^{\pi/\beta-\varepsilon} \mu(r) = 0 ,$$

and if $u(x) \to 0$ *on* ∂D *as* $|x| \to 0$, *then* $u(x) \to 0$ *uniformly in* D *as* $|x| \to 0$.

Another way to treat the case $n=2$, is to reduce it to Theorem 1, using the mapping $z'=z^{-\pi/\beta}$. We thus get the following.

THEOREM 6. *Let* $D \subset K_\beta$, $n=2$, *and assume that* L *satisfies* (i), (ii) *and that* $(a_{ij}(x))$ *is Dini continuous at* $x=0$ *with* $a_{ij}(0)=\delta_{ij}$. *Assume further that* $r^{1+\pi/\beta}p(r)$ *is monotone increasing. If* $Lu(x)=0$ *in* D *and*

$$\lim_{r\to 0} r^{\pi/\beta}\mu(r)=0 ,$$

and if $u(x) \to 0$ *on* ∂D *as* $|x| \to 0$, *then* $u(x) \to 0$ *uniformly in* D *as* $|x| \to 0$.

By using the same mapping $z'=z^{-\pi/\beta}$, we can derive theorems analogous with the Gilbarg-Hopf ([1], [4]) and Serrin's ([8]) theorems, provided that L satisfies the assumptions of Theorem 6.

In the case $n \geqq 3$, $\beta \leqq \pi$, such theorems can also be obtained, by using the transformation $x_i'=x_i/|x|^n$ $(i=1, \cdots, n)$.

PART II

5. Let $x=(x_1, \cdots, x_n)$ **and denote** $X=(x, t)$, $|X|=(|x|^2+t^2)^{1/2}$. Consider the operator

$$Lu \equiv \sum_{i,j=1}^n a_{ij}(X)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(X)\frac{\partial u}{\partial x_i} - \frac{\partial u}{\partial t} , \tag{1}$$

defined in an unbounded domain D. We shall assume that L satisfies the following conditions:

(i) $\sum_{i,j}|a_{ij}(X)|$ is bounded in D, and, for all $X \in D$, ξ_i real,

$$\sum_{i,j} a_{ij}(X)\xi_i\xi_j \geqq \alpha \sum_i \xi_i^2 \qquad (\alpha>0) ,$$

(ii) for all $X \in D$, $|X|=R$,

$$|\sum_i x_i b_i(X)| \leqq p(R) , \tag{2}$$

where $p(R)$ $(0<R<\infty)$ is bounded and $p(R) \to 0$ as $R \to \infty$.

Beside the functions $m(R)$, $\mu(R)$ defined in Part I, we introduce the functions

$$m'(R)=\inf_{X\in T_R} u(X) , \qquad \mu'(R)=\sup_{X\in T_R}|u(X)| ,$$

where $T_R \equiv D \cap |x|^2+|t|=R$.

Let K_β denote the cone with angular opening β, whose axis is the positive t-axis and whose vertex is in the origin. In what follows, $u(X)$ is assumed to belong to $C^2(D)$. In Theorems 8, 10 $u(X)$ is also assumed to be continuous in $\overline{D}$.

THEOREM 7. *Let D belong to the half space $t>0$, and assume that L satisfies* (i), (ii). *If $u(X)\geqq 0$ on ∂D, $Lu(X)\leqq 0$ in D, and if*

$$\lim_{k\to\infty}\frac{m(R_k)}{R_k^2}=0 \qquad (R_k\to\infty \text{ as } k\to\infty)\,, \tag{3}$$

then $u(X)\geqq 0$ in D.

Proof. The function $v_R(X)=(|x|^2+(t+K)^2)/R^2$ $(K>0)$ has the following properties:

(a) $v_R(X)\geqq 0$ if $X\in\partial D$, $|X|\leqq R$,

(b) $v_R(X)\geqq 1$ if $X\in D$, $|X|=R$,

(c) $Lv_R(X)<0$ in $C_R=D\cap|X|<R$, if K is sufficiently large, and

(d) $R^2v_R(X)$ is bounded, for every X, as $R\to\infty$.

The function $\tilde{u}(X)=u(X)-\sigma(R)v_R(X)$, where $\sigma(R)=\min(0, m(R))$, is nonnegative on ∂C_R and $Lu(X)\leqq 0$ in C_R. Applying the (weak) minimum principle [7], we conclude that $\tilde{u}(X)\geqq 0$ in C_R. Taking $R=R_k\to\infty$ and using (3), we get $u(X)\geqq 0$.

REMARK. It is clear that the same proof holds under weaker assumptions on L: (ii) may be replaced by $\sum x_ib_i(X)\leqq H$, where H is a constant, and in (i), the boundedness of $\sum|a_{ij}(X)|$ in D may be replaced by the boundedness of $\sum a_{ii}(X)$ in D and the boundedness of $\sum|a_{ij}(X)|$ in each C_R.

LEMMA 5. *Let D belong to the half space $t>0$, and assume that L satisfies* (i), (ii). *If R_0 is sufficiently large, then there exists a function $w(X)$ defined in $D_{R_0}=D\cap|X|>R_0$, and having the following properties:*

(a) *$w(X)\geqq 0$ if $X\in\partial D_{R_0}$,*

(b) *$w(X)\geqq 1$ if $X\in D$, $|X|=R_0$,*

(c) *$Lw(X)\leqq 0$ in D_{R_0}, and*

(d) *$w(X)\to 0$ uniformly in D_{R_0} as $|X|\to\infty$.*

Proof. Define

$$w(X)=\frac{C}{(t+1)^{\varepsilon}}\exp\left(\frac{-H|x|^2}{t+1}\right) \qquad (C>0,\ \varepsilon>0,\ H>0)\ .$$

Since $W(X)>0$ if $|X|=R_0$, $t\geqq 0$, we can choose C such that (b) is satisfied. Since (a) and (d) are also satisfied, it remains to verify (c).

$$Lw=w\left\{\sum a_{ij}\frac{4H^2x_ix_j}{(t+1)^2}-\sum a_{ii}\frac{2H}{t+1}-\sum x_ib_i\frac{2H}{t+1}+\frac{\varepsilon}{t+1}-\frac{H|x|^2}{(t+1)^2}\right\}\ ;$$

consequently, if

$$(4) \qquad 4H\sum a_{ij}x_ix_j\leqq|x|^2\ , \qquad 2H\sum a_{ii}+2H\sum x_ib_i\geqq\varepsilon\ ,$$

then $Lw\leqq 0$. Obviously we can choose H and ε such that (4) is satisfied.

With Theorem 7 and Lemma 5 at hand, we can now proceed as in the proof of Theorem 1 and get the following.

THEOREM 8. *Let D belong to the half space $t>0$, and assume that L satisfies* (i), (ii). *If $Lu(X)=0$ in D and*

$$(5) \qquad \lim_{R\to\infty}\frac{\mu(R)}{R^2}=0\ ,$$

and if $u(X)\to 0$ on ∂D as $|X|\to\infty$, then $u(X)\to 0$ uniformly in D as $|X|\to\infty$.

Theorems 7, 8 are not true for domains D in the half space $t<0$. As an example take D to be the whole half space $t<0$, and take $u(x,t)=t^{1/m}$, where m is an odd positive integer. Then

$$u=0 \quad \text{on} \quad t=0\ , \qquad Lu=-\frac{1}{m}t^{1/m-1}<0 \quad \text{if} \quad t<0\ ,$$

$$\lim_{R\to\infty}\frac{\mu(R)}{R^{\varepsilon}}=0 \quad \text{if} \quad \frac{1}{m}<\varepsilon\ ,$$

but $u(X)<0$ if $t<0$, and $\lim u(X)$ does not exist as $|X|\to\infty$, $t\leqq 0$.

6. THEOREM 9. *Let $D\subset K_{\beta}$, $0<\beta<2\pi$, and assume that L satisfies* (i), (ii). *If $Lu(X)\leqq 0$ in D, $u(X)\geqq 0$ on ∂D, and if*

$$(6) \qquad \lim_{k\to\infty}\frac{m'(R_k)}{R_k^2}=0 \qquad (R_k\to\infty\ \text{as}\ k\to\infty)\ ,$$

then $u(X)\geqq 0$ in D.

Taking $v_R(X)=2(|x|^2+Bt+C)/R^2$ (B and C are proper constants), we proceed as in the proof of Theorem 7. Details will be omitted. The remark that follows Theorem 7 applies also to Theorem 9.

Lemma 5 can also be generalized to the case $D\subset K_\beta$, $0<\beta<2\pi$. Indeed, the function $w(X)$ may be defined as follows:

$$w(X)=\begin{cases} \dfrac{C}{(t+R_0)^\varepsilon}\exp\left(-\dfrac{H|x|^2}{t+R_0}\right) & \text{if} \quad t>-R_0 \\ 0 \quad \text{if} \quad t\leqq -R_0 . \end{cases}$$

Proceeding as in § 5, we get the following theorem.

THEOREM 10. *Let $D\subset K_\beta$, $0<\beta<2\pi$, and assume that L satisfies* (i), (ii). *If $Lu(X)=0$ in D and*

$$\lim_{R\to\infty}\frac{\mu'(R)}{R^2}=0\ , \tag{7}$$

and if $u(X)\to 0$ on ∂D as $|X|\to\infty$, then $u(X)\to 0$ uniformly in D as $|X|\to\infty$.

Note that (7) can be replaced by the stronger assumption

$$\lim_{R\to\infty}\frac{\mu(R)}{R}=0\ . \tag{7'}$$

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