MINIMAL SUPERADDITIVE EXTENSIONS OF SUPERADDITIVE FUNCTIONS

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Introduction. A real valued function f is said to be superadditive on an inverval $I = [0, a]$ if it satisfies the inequality $f(x + y) \geqq f(x) + f(y)$ whenever x, y and $x + y$ are in I. Such functions have been studied in detail by E. Hille and R. Phillips [1] and R. A. Rosenbaum [2]. In this paper we show that any superadditive function f on I has a minimal superadditive extension F to the non-negative real line E, and then proceed to show that F inherits much of its behavior from the behavior of f. We deal primarily with superadditive functions which are continuous and non-negative.

A simple example of a superadditive function on $[0, a]$ is furnished by a convex function f with $f(0) \leqq 0$. Also, if f is convex and $f(0) = 0$, then it is easy to verify that its minimal superadditive extension F is given by

$$F(x) = nf(a) + f(x - na)$$

for $na \leqq x < (n + 1)a$. In general, the minimal superadditive extension F is not easily computed. In the sequel we shall discuss two methods for obtaining F. For brevity we shall use the notation f^*F to mean "F is the minimal superadditive extension of f".

1. The decomposition method. DEFINITION. Let $x \in E$. The numbers $x^1, \cdots, x^n$ are said to form an *a-partition* for x if $x^1 + \cdots + x^n = x$ and for each $i = 1, \cdots, n$ we have $0 \leqq x^i \leqq a$.

THEOREM 1. *Let f be a superadditive function on $I = [0, a]$. Then the function F defined on E by the equation*

$$F(x) = \sup \Sigma f(u^i)\ ,$$

the supremum being taken over all a-partitions of x, is the minimal superadditive extension of f.

Proof. We will show that F is superadditive. The minimality of F will then follow from the fact that any superadditive extension $\hat{f}$ of f must satisfy $\hat{f}(x) \geqq \Sigma f(x^i)$ for all $x \in E$ and all a-partitions $x^1, \cdots, x^n$ of x. Let $x, y \in E$, $\varepsilon > 0$. Choose a-partitions $x^1, \cdots, x^m$ and $y^1, \cdots, y^n$ for

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x and y respectively such that $f(x^1) + \cdots + f(x^m) \geqq F(x) - \varepsilon/2$ and $f(y^1) + \cdots + f(y^n) \geqq F(y) - \varepsilon/2$. Then the numbers $x^1, \cdots, x^n, y^1, \cdots, y^n$ form an a-partition for $x + y$ and we have

$$F(x + y) \geqq f(x^1) + \cdots + f(x^m) + f(y^1) + \cdots + f(y^n) \\ \geqq F(x) + F(y) - \varepsilon .$$

Suppose we have an approximation for $F(x)$: that is, a number $\varepsilon > 0$ and an a-partition $x^1, \cdots, x^n$ for x such that $F(x) - \Sigma f(x^i) < \varepsilon$. If among the members of this a-partition there are two, say x^j and x^k such that $u = x^j + x^k \leqq a$, then since $f(u) \geqq f(x^j) + f(x^k)$, we have

$$F(x) - [f(u) + \sum_{i \neq j,k} f(x^i)] \leqq F(x) - \sum_1^n f(x^i) < \varepsilon .$$

In other words, replacing two numbers used in the approximation by *their sum* $u \leqq a$ *yields an approximation at least as good as the original.* It follows that if x satisfies the inequality $(M - 2)a/2 \leqq x \leqq (M - 1)a/2$, where M is a positive integer, then there exist arbitrarily good approximations for $F(x)$ using only M terms in the a-partition. If f is continuous, then a simple compactness argument results in the following theorem:

THEOREM 2. *Let f be a continuous superadditive function on* $[0, a]$, *and let F be its minimal superadditive extension. Let x satisfy the inequality* $(M - 2)a/2 \leqq x \leqq (M - 1)a/2$. *Then* $\exists$ *an a-partition* $x^1, \cdots, x^M$ *for x such that*

$$\Sigma f(x^i) = F(x) .$$

Such an a-partition for x will be called a *decomposition* of x, for which we shall use the notation $\langle x \rangle$ whenever convenient. We will denote by $N(x)$ a number so large that for any continuous superadditive function on $[0, a]$, $\exists$ a decomposition $\langle x \rangle$ of x with at most $N(x)$ members. It follows from the above that we can always let $N(x) = 2x/a + 2$, for example.

Henceforth we shall be concerned primarily with continuous nonnegative superadditive functions for which we shall use the abbreviation *csa*. It is readily verified that such functions are non-decreasing and vanish at the origin.

2. Combinations of extensions. One might expect that if the members of a family f of *csa* functions are combined in a linear fashion to give another *csa* function h, then combining the members of the family $\tilde{f}$ of minimal superadditive extensions of functions in f in the same way would give rise to a function H which is the minimal superadditive extension of h. However this is not always the case. Consider, for example, the functions f and g defined on $[0, 3]$ as follows: $f(0) = 0$, $f(1) = 0, f(2) = 0, f(3) = 1$ and $g(0) = 0$, $g(1) = 0$, $g(2) = 2$, $g(3) = 3$, f and g linear on $[n, n+1]$, $n = 0, 1, 2$. Simple computations show that whereas $(F + G)(4) = 5$ and $FG(4) = 4$, the minimal superadditive extensions of $f + g$ and fg take on the values 4 and 3 respectively at $x = 4$. The minimal superadditive extension of a sum (product) of superadditive functions is thus not necessarily the sum (product) of the minimal superadditive extensions. However, some processes do commute with taking minimal superadditive extensions.

THEOREM 3. *Let $\{f_n\}$ be a sequence of csa functions converging to the continuous function f on $I = [0, a]$. Let $f_n{}^* F_n$. Then f is csa and $f^* \lim_{n\to\infty} F_n$.*

Proof. That f is superadditive and non-negative is clear. Since for each positive integer n the function f_n is non-decreasing, the convergence of $\{f_n\}$ to f is uniform on I. Given $\varepsilon > 0$ and $x \in E$, let M be such that $n \geqq M \Rightarrow \max_{t \in I} |f_n(t) - f(t)| < \varepsilon/N(x)$ where $N(x)$ is a number chosen as in § 1. Let $k > M$ and let $\langle x^k \rangle \equiv x_k^1, \cdots, x_k^{N(x)}$ and $\langle x \rangle \equiv x^1, \cdots, x^{N(x)}$ be decompositions for x relative to F_k and F respectively. We have

$$F(x) = \sum_{i=1}^{N(x)} f(x^i) \geqq \sum_{i=1}^{N(x)} f(x_k^i)$$

and

$$F_k(x) = \sum_{i=1}^{N(x)} f_k(x_k^i) \geqq \sum_{i=1}^{N(x)} f_k(x^i) \ .$$

It follows from these two inequalities that

$$F(x) - F_k(x) \,| < \varepsilon \ ,$$

for $n \geqq M$.

3. **Behavior of the minimal superadditive extension.** It seems reasonable to expect that the minimal superadditive extension F of a *csa* function f will enjoy many of the properties of f. To a certain extent this is true and we are able to predict much about the behavior of F by examining the behavior of f.

THEOREM 4. *Let f be csa on $[0, a]$. If $f^* F$, then F is csa on E.*

Proof. Clearly F is non-negative. To prove that F is continuous let $\varepsilon > 0$ and choose $\delta < a/2 \ni$ if $u, v \leqq a$ and $|u - v| < \delta$ then $|f(u) - f(v)| < \varepsilon$. Now let x and y be points of E for which $|y - x| < \delta$, say $y = x + h$. Let $\langle y \rangle = y^1, \cdots, y^N$ be a decomposition for y with, say, $y^1, \geqq a/2$. We have

$$F(y) = \sum_1^N f(y_i) \text{ and } F(x) \geqq \sum_2^N f(y^i) + f(y^1 - h) .$$

Hence $0 \leqq F(y) - F(x) \leqq f(y^1) - f(y^1 - h) < \varepsilon$.

In a similar manner one can establish the following theorem, which is stated without proof.

THEOREM 5. *Let f be csa on $[0, a]$. If f^*F, then the following statements hold:*

(a) *If f satisfies a Lipschitz condition with coefficient M, then so does F;*

(b) *If $\langle y \rangle = y^1, \cdots, y^M$ is a decomposition for y and f is differentiable at y^i and y^j, then $f'(y^i) = f'(y^j)$. If, in addition, F is differentiable at y, then $F'(y) = f'(y^i)$.*

One might expect that the differentiability of f on $[0, a]$ would imply the differentiability of F, except possibly at integral multiples of a. Although this turns out not to be the case, we do have the following theorem:

THEOREM 6. *Let f be a csa function on the interval $[0, a]$, with f' continuous on $(0, a)$. For x not an integral multiple of a, let X be the set of points of $[0, a]$ which can be used in a decomposition for x. Then F has a right hand derivative $F_+(x)$ and a left hand derivative $F_-(x)$ at x with*

$$F_+(x) = \sup_{t \in X} f'(t) \equiv S$$

and

$$F_-(x) = \inf_{t \in X} f'(t) \equiv I .$$

Proof. We will prove only the upper equality. The lower can be proved in a similar manner. It suffices to show $D^+F(x) = D_+F(x) = S$ where D^+F and D_+F are the upper and lower right hand derivatives of F. Suppose $\exists \varepsilon > 0 \ni D^+F(x) > S + 2\varepsilon$. Then a sequence $\{h_i\}$ of numbers approaching 0 such that

$$F(x) < F(x + h_i) - (S + \varepsilon)h_i \tag{1}$$

for $i = 1, 2, \cdots$. For each positive integer i, let $(u^i, v^i, \cdots, w^i)$ be a decomposition for $x + h_i$. Without loss of generality, we assume that the sequence $(u^i, v^i, \cdots, w^i)$ converges to, say, $(u, v, \cdots, w)$; otherwise we consider a convergent subsequence. Since x is not an integral multiple of a, one of the numbers $u, v, \cdots, w$ is not equal to 0 or a. Denote such a one by u. From (1) we have

$$(2) \qquad F(x) < f(u^i) + f(v^i) + \cdots + f(w^i) - (S + \varepsilon)h_i \ .$$

Choose $N_1 \ni i > N_1$ implies that

$$(3) \qquad f(u^i) < f(u^i - h_i) + [f'(u^i - h_i) + \varepsilon/2]h_i \ .$$

That N_1 can be so chosen follows from the continuity of f'. In fact, let δ be such that $|u - v| < \delta \Rightarrow |f'(u) - f'(v)| < \varepsilon/4$. Now choose N_1 such that $i > N_1 \Rightarrow u - \delta < u^i - h_i < u^i < u + \delta$. If $y \in [u^i - h_i, u^i]$, with $i > N_1$, then $f'(u^i - h_i) + \varepsilon/2 > f'(y)$. Hence (3) is a valid inequality. For $i > N_1$ we have from (2) and (3),

$$(4) \quad F(x) < f(u^i - h_i) + f(v^i) + \cdots + f(w^i) + [f'(u^i - h_i) - (S + \varepsilon/2)]h_i.$$

Now the sequence $(u^i - h_i, v^i, \cdots, w^i)$ converges to $(u, v, \cdots, w)$ and $u + v + \cdots + w = x$. Thus, since

$$f(u^i) + f(v^i) + \cdots + f(w^i) = F(x + h_i) \geq F(x) \ ,$$

and F is a superadditive function, we have

$$f(u) + f(v) + \cdots + f(w) = F(x)$$

and $u \in X$. Therefore $f'(u) \leqq S$. By the continuity of f', $\lim_{i\to\infty} f'(u^i - h_i) = f'(u)$. Hence $\exists$ a positive number N_2 such that $i > N_2 \Rightarrow f'(u^i - h_i) < S + \varepsilon/2$. Let $i = \max(N_1, N_2)$. For this i we have from (4),

$$F(x) < f(u^i - h_i) + f(v^i) + \cdots + f(w^i).$$

This is impossible, for $u^i - h_i + v^i + \cdots + w^i = x$ for each $i = 1, 2, \cdots$ and F is superadditive. We have shown $D^+F(x) \leqq S$.

It remains to show $D_+F(x) \geqq S$. Let $\varepsilon > 0$, and let $(u, v, \cdots, w)$ be a decomposition for x such that $u \neq a$, and $f'(u) > S - \varepsilon/4$. Choose $\delta > 0 \ni h < \delta \Rightarrow f(u + h) > f(u) + (S - \varepsilon/2)h_i$. For $h < \delta$,

$$F(x + h) \geq f(u + h) + f(v) + \cdots + f(w) > F(x) + (S - \varepsilon/2)h \ .$$

The first and third members of this inequality give

$$\frac{F(x + h) - F(x)}{h} > S + \varepsilon/2 \ .$$

Since ε was arbitrary, $D_+F(x) \geqq S$, and the proof of the theorem is complete.

We now proceed to obtain a linear upper bound for F. If f is *csa* on $[0, a]$, then the function g defined by $g(x) = f(x)/x$ is continuous on $[0, a]$ and satisfies $g(nx) \geqq g(x)$, $n = 1, 2, \cdots$, whenever $nx \leqq a$. It follows that g attains a maximum at some point of $(0, a]$.

THEOREM 7. *Let f be csa on $[0, a]$, f^*F, and let g be defined as* *above. Let t be a point of $(0, a]$ at which g attains its maximum M. Then*

(a) $F(x)/x \leq M$ *for all* $x > 0$,

(b) $F(x)/x = M$ *if* x *is an integral multiple of* t,

(c) $\lim_{x\to\infty} F(x)/x = M$,

(d) $\max_{x\in[0,a]} [Mx - f(x)] = \max_{x\in E} [Mx - F(x)]$,

(e) $\lim_{x\to\infty} [F(x) - Mx] = 0$ *if* f *is differentiable at* t.

Proof. The proofs of (a), (b), (c) and (d) are straightforward and will be omitted. Let us then turn to (e). For each $x \in E$, write x in the form $x = nt + y$, where n is an integer and $0 \leqq y < t$. Define a function F^* by $F^*(nt + y) = nf(t + y/n)$, $n = 1, 2, \cdots$. Clearly $F^*(x) \leqq F(x) \leqq Mx$ for all $x \in E$. We will show that $\lim_{x\to\infty} [Mx - F^*(x)] = 0$. By the definition of F^* we have

$$Mx - F^*(x) = M(nt + y) - nf(t + y/n) .$$

Noting that $f(t) = Mt$, we see that the right hand member of this last equation can be written in the form

$$y\left[M - \frac{f(t + y/n) - f(t)}{y/n}\right] \tag{1}$$

Now let $x \to \infty$. Then y is bounded between 0 and t and $n \to \infty$. The expression (1) approaches 0, since $f'(t) = M$.

We observe that the function F^* of the preceding theorem is asymptotic to F with $F^* \leqq F$. Hence $F(x)$ is bounded between $F^*(x)$ and Mx, two functions which are easy to calculate, and whose difference is small when x is large.

4. The polygonal method. The minimal superadditive extension of a *csa* function may also be obtained as the limit of a sequence of polygonal functions. A function p is said to be *polygonal* if p is continuous and piecewise linear. The point $x \in [0, a]$ is called a *vertex* of p if $(x, p(x))$ is a vertex of the polygon forming the graph of p.

THEOREM 8. *Let p be polygonal on $[0, a]$ with equally spaced vertices. Then p is superadditive if and only if p is superadditive on its vertices.*

Proof. If p is superadditive, then p is clearly superadditive on its vertices. To prove the converse consider the function g defined on the set

$$D \equiv \{(x, y): 0 \leqq x, y \leqq a \text{ and } x + y \leqq a\}$$

by the equation $g(x, y) = p(x + y) - p(x) - p(y)$. It is easy to verify that g is planar on any triangle T of the form

$$T = \{(x, y)\colon u_1 \leqq x \leqq u_2;\ v_1 \leqq y \leqq v_2,\ x + y \leqq (\text{or} \geqq)\ u_2 + v_2\}\ ,$$

where (u_1, v_1) and (u_2, v_2) are pairs of successive vertices of p. Hence g attains its minimum on T at one of the points (u_i, v_i) and therefore its minimum on D at a point (u, v) where both u and v are vertices of p. Thus, if g is anywhere negative then g is negative at a point whose two coordinates are vertices of p. This proves the theorem.

Now let p be a polygonal function on $[0, a]$ with vertices at $0, v, 2v, \cdots, mv = a$. We define a function P on E as follows:

$$P(x) = p(x) \quad \text{for } x \leqq a$$
$$P(Mv) = \max_{K=1,\cdots,M-1} [P(kv) + P(Mv - kv)] \quad M \text{ an integer} \geqq m + 1$$

and

$$P \text{ linear on } [Mv, (M + 1)v],\ M = m, m + 1, \cdots.$$

P will be called the function associated with p. It is easy to see that if p is *csa*, then P is *csa*.

DEFINITION. A sequence $\{p_n\}$ of functions defined on $[0, a]$ is called a p-sequence if

(i) each p_n is a polygonal function

(ii) the vertices of p_n are $Ka/2^n$, $\quad K = 0, 1, \cdots, 2^n$

(iii) $P_n(Ka/2^m) = p_m(Ka/2^m)$ if $m \leqq n$.

In terms of this concept we have

THEOREM 9. *Let $\{p_n\}$ be a p-sequence coverging to the csa function f on $[0, a]$. For each positive integer n let P_n be the function associated with p_n. Then, if f^*F, $\{P_n\}$ converges to F on E.*

Proof. It suffices to show that P_n approaches F on $[0, 2a]$. Let $F^*(x) = \overline{\lim}_{n\to\infty} P_n(x)$. It is easy to check that F^* is superadditive. Let V_k be the set of vertices of P_k in $[a, 2a]$, and let $V = \bigcup_1^\infty V_k$. If $v \in V$, then $\lim_{n\to\infty} P_n(v)$ exists since the sequence $\{P_n(v)\}$ is ultimately non-decreasing and $P_n(v) \leqq F(v)$ for all n. We have $\lim_{n\to\infty} P_n(v) \leqq F(v)$. But since F^* is superadditive, we have $F^* \geqq F$. Hence $F^* = F$ on V. By standard arguments involving the continuity of F, the density of V in $[a, 2a]$, and the monotonicity of each P_n and F^*, it follows that $F \equiv F^*$ and that $F^* = \lim_{n\to\infty} P_n(x)$.

5. Superadditive functions in n-dimensions. It turns out that many of the results obtained in one dimension have their analogues in n-dimensions. The interval $I \equiv [0, a]$ is replaced by a fundamental region R defined by the inequalities $0 \leqq x_i \leqq a_i$, $i = 1, \cdots, n$, where the a_i are arbitrary positive numbers. The decomposition method works, just as it does on the line, and we can prove with little difficulty that to any superadditive function f on R there corresponds a minimal superadditive extension F to $E_n^+ \equiv \{(x_1, \cdots, x_n): 0 \leqq x_i, i = 1, \cdots, \cdots, n\}$. We can also prove a theorem corresponding to Theorem 5, the derivatives here being directional derivatives. In Theorem 7 a certain line $l(x) = Mx$ played an important role. In n-dimensions, for each direction θ we have a plane P_θ which plays the role of l in some direction, and when the function P, defined on the fundamental region R by the equation

$$P(z) = \inf_{\theta} P_\theta(z) ,$$

is extended to E_n^+ by homogeneity it is the minimal concave superadditive function which bounds F from above. It can be proved, at least in E_2^+, that

$$n \max_{z \in R} [P(z) - f(z)] \geqq \max_{z \in E_h^+} [P(z) - F(z)] .$$

Bibliography

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