SUMMABILITY OF DERIVED CONJUGATE SERIES

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1. Introduction. In a recent paper ([3] it was shown that the summability of the successively derived Fourier series of a CP integrable function could be characterized by that of the Fourier series of another CP integrable function. It is the purpose of the present paper to give analogous theorems for the successively derived conjugate series of a Fourier series.

2. Definitions. The terminology used in [3] will be continued in this paper. In addition let us define:

$$\psi(t) = \psi(t, r, x) = \frac{1}{2}[f(x+t) + (-1)^{r-1}f(x-t)] \tag{1}$$

$$Q(t) = \sum_{i=0}^{[\frac{r-1}{2}]} \frac{\bar{a}_{r-1-2i}}{(r-1-2i)!} t^{r-1-2i} \tag{2}$$

$$g(t) = r!t^{-r}[\psi(t) - Q(t)] \tag{3}$$

The rth derived conjugate series of the Fourier series of $f(t)$ at $t = x$ will be denoted by $D_rCFSf(x)$, and the nth mean of order (α, β) of $D_rCFSf(x)$ by $\bar{S}^r_{\alpha,\beta}(f, x, n)$.

3. Lemmas.

LEMMA 1. *For $\alpha = 0, \beta > 1$ or $\alpha > 0, \beta \geqq 0$, and $r \geqq 0$,*

$$\bar{\lambda}^{(r)}_{1+\alpha,\beta}(x) = -\pi^{-1}r!(-x)^{r+1} + 0(|x|^{-1-\alpha}\log^{-\beta}|x|) + 0(|x|^{-r-2}) \text{ as } |x| \to \infty .$$

This is a result due to Bosanquet and Linfoot [2].

LEMMA 2. *For $\alpha > 0, \beta \geqq 0$ or $\alpha = 0, \beta > 0$ and*

$$r \geqq 0,\ x^r\bar{\lambda}^{(r)}_{1+\alpha+r,\beta}(x) = \sum_{i,j=0}^{r} B^r_{ij}(\alpha, \beta)\bar{\lambda}_{1+\alpha+r-i,\beta+j}(x) ,$$

where the B^r_{ij} are independent from x and have the properties:

(i) $B^r_{ij}(\alpha, 0) = 0$ *for* $j \geqq 1$;

(ii) $B^r_{r0}(\alpha, \beta) \neq 0$;

(iii) $\sum_{i,j=0}^{r} B^r_{ij}(\alpha, \beta) = (-1)^r r!$.

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The proofs of (i) and (ii) will be found in [3], Lemma 2, taking the imaginary parts of the equations there. Part (iii) follows immediately from the first part of the lemma and Lemma 1.

LEMMA 3. *For* $n > 0, \alpha = 0, \beta > 1$ *or* $\alpha > 0, \beta \geqq 0$, *and* $r \geqq 0$,

$$\left(\frac{d}{dt}\right)^r \left\{2B\pi^{-1}\sum_{\nu \leqq n}\left(1 - \frac{\nu}{n}\right)^{\alpha} \log^{-\beta}\left(\frac{C}{1 - \frac{\nu}{n}}\right)\sin \nu t\right\}$$

$$= 2n^{r+1}\sum_{k=-\infty}^{\infty} \bar{\lambda}^{(r)}_{1+\alpha,\beta}[n(t + 2k\pi)] \ .$$

Proof. Smith ([6], Lemma 6) has shown that for every odd, Lebesgue integrable function, $Z(t)$, of period 2π,

$$\bar{S}_{\alpha,\beta}(Z, 0, n) = -2n\int_0^{\infty} Z(t)\bar{\lambda}_{1+\alpha,\beta}(nt)dt \ .$$

Since the right side of this equation can be written

$$-2n\int_0^{\pi} Z(t) \sum_{k=-\infty}^{\infty} \bar{\lambda}_{1+\alpha,\beta}[n(t + 2k\pi)]dt$$

for every such $Z(t)$, the lemma is true for $r = 0$. For $r \geqq 1$ the interchange of $(d/dt)^r$ and $\sum_{-\infty}^{\infty}$ is justified by uniform convergence.

The following lemma is a direct consequence of Lemma 3:

LEMMA 4. *Let* $f(x) \in CP[-\pi, \pi]$ *and be of period* 2π. *For* $n > 0$ *and* $\alpha = 0, \beta > 1$ *or* $\alpha > 0, \beta \geqq 0$,

$$\bar{S}^r_{\alpha,\beta}(f, x, n) = 2(-n)^{r+1}\int_0^{\pi} \psi_r(t) \sum_{k=-\infty}^{\infty} \bar{\lambda}^{(r)}_{1+\alpha,\beta}[n(t + 2k\pi)]dt \ .$$

LEMMA 5. *For* $\alpha \geqq 0, \beta \geqq 0, n > 0$ *and* $r \geqq 0$,

$$n^{r+1}\int_0^{\infty} Q(t)\bar{\lambda}^{(r)}_{1+\alpha+r,\beta}(nt)dt = 0 \ ,$$

where $Q(t)$ *is defined by* (2).

Proof. If $r = 0$, then $Q(t) = 0$. For $r \geqq 1$ and $i = 0, 1, \cdots [r-1/2]$, the truth of the lemma follows from the equation:

$$\int_0^{\infty} x^{r-1-2i}\bar{\lambda}^{(r)}_{1+\alpha+r,\beta}(x)dx = 0 \ ,$$

which is easily verified by means of $r - 1 - 2i$ integrations by parts.

The final two lemmas of this section give the appropriate representation of the nth mean of $D_rCFSf(x)$.

LEMMA 6. *Let $f(x) \in C_\lambda P[-\pi, \pi]$ and be of period 2π. Let m, $0 \leqq m \leqq \lambda + 1$, be an integer for which $\Psi_m(t) \in L[0, \pi]$. Then, for $\alpha = m$, $\beta > 1$ or $\alpha > m$, $\beta \geqq 0$ and $r \geqq 0$,*

$$\bar{S}^r_{\alpha+r,\beta}(f, x, n) = 2(-n)^{r+1}\int_0^\pi [\psi(t) - Q(t)]\bar{\lambda}^{(r)}_{1+\alpha+r,\beta}(nt)dt + C_r + o(1)$$

as $n \to \infty$, where

$$C_r = 2\pi^{-1}(-1)^{r+1}\int_0^\pi \psi(t)\left(\frac{d}{dt}\right)^r\left[\frac{1}{2}ctn\frac{1}{2}t - t^{-1}\right]dt + 2r!\pi^{-1}\int_\pi^\infty t^{-r-1}Q(t)dt . \tag{4}$$

Proof. It follows from Lemmas 4 and 5 that

$$\begin{aligned}\bar{S}^r_{\alpha+r,\beta}(f, x, n) &= 2(-n)^{r+1}\int_0^\pi [\psi(t) - Q(t)]\bar{\lambda}^{(r)}_{1+\alpha+r,\beta}(nt)dt \\ &\quad + 2(-n)^{r+1}\int_0^\pi \psi(t)\sum_{-\infty}^{\infty}{}'\bar{\lambda}^{(r)}_{1+\alpha+r,\beta}[n(t + 2k\pi)]\,dt \\ &\quad + -2(-n)^{r+1}\int_\pi^\infty Q(t)\bar{\lambda}^{(r)}_{1+\alpha+r,\beta}(nt)dt \\ &= I_1 + I_2 + I_3 .\end{aligned} \tag{5}$$

Since the degree of $Q(t)$ is $r - 1$, Lemma 1 shows that

$$I_3 = 2r!\pi^{-1}\int_\pi^\infty t^{-r-1}Q(t)dt + o(1) . \tag{6}$$

Let us define:

$$J(n, t) = 2(-n)^{r+1}\sum_{-\infty}^{\infty}{}'\{\bar{\lambda}^{(r)}_{1+\alpha+r,\beta}[n(t + 2k\pi)] - (-1)^r r!\pi^{-1}[n(t + 2k\pi)]^{-r-1}\} .$$

Again appealing to Lemma 1, we see that $\lim_{n\to\infty} (\partial/\partial t)^j J(n, t) = 0$ uniformly for $t \in [0, \pi]$ and $j = 0, 1, \cdots, m$.

With the aid of the well-known cotangent expansion I_2 may be written:

$$I_2 = \int_0^\pi \psi(t)J(n, t)dt + (-1)^{r+1}2\pi^{-1}\int_0^\pi \psi(t)\left(\frac{d}{dt}\right)^r \left[\frac{1}{2}ctn\frac{1}{2}t - t^{-1}\right]dt . \tag{7}$$

But after m integrations by parts, it is seen that

$$\int_0^\pi \psi(t)J(n, t)dt = o(1) . \tag{8}$$

The lemma now follows from equations (5), (6), (7), and (8).

A particular, but useful, case of Lemma 6 is

LEMMA 7. *Let $f(x) \in C_\lambda P[-\pi, \pi]$ and be of period 2π. If $g(t) \in C_\mu P[0, \pi]$, where $g(t)$ is defined by* (3), *then*

$$\bar{S}_{\alpha,\beta}(g, 0, n) = -2n\int_0^\pi g(t)\bar{\lambda}_{1+\alpha,\beta}(nt)dt$$
$$-2\pi^{-1}\int_0^\pi g(t)\Big(\frac{1}{2}ctn\frac{1}{2}t - t^{-1}\Big)dt + o(1)$$

for $\alpha = 1+\xi, \beta > 1$ or $\alpha > 1+\xi, \beta \geqq 0$, where $\xi = \min[\mu, \max(r, \lambda)]$.

The hypothese of Lemma 6 are fulfilled, because $t^r g(t) \in C_\lambda P[0, \pi]$ implies $G_{1+\xi}(t) \in L[0, \pi]$ by Lemma 6 of [3].

4. Theorems.

THEOREM 1. *Let $f(x) \in C_\lambda P[-\pi, \pi]$ and be of period 2π. If there exist constants $\bar{a}_{r-1-2i}$, $i = 0, 1, \cdots [r-1/2]$, such that*

(i) *$g(t) \in C_\mu P[0, \pi]$ for some integer μ;*

(ii) *$CFSg(0) = s(\alpha, \beta)$ for $\alpha = 1+\xi, \beta > 1$ or $\alpha > 1+\xi, \beta \geqq 0$, where $\xi = \min[\mu, \max(r, \lambda)]$;*

then $D_r CFSf(x) = S(\alpha + r, \beta)$, $s = \pi^{-1}\int_0^\pi g(t)ctn(1/2)tdt$ and

$$S = -2\pi^{-1}\int_0^\pi t^{-1}g(t)dt + C_r\,,$$

where C_r is defined by equation (4).

THEOREM 2. *Let $f(x) \in C_\lambda P[-\pi, \pi]$ and be of period 2π. If $D_r CFSf(x) = S(\alpha + r, \beta)$ for $\alpha = 1+\lambda, \beta > 1$ or $\alpha > 1+\lambda, \beta \geqq 0$, then there exist constants $\bar{a}_{r-1-2i}$, $i = 0, 1, \cdots [r-1/2]$, such that*

(i) *$g(t) \in C_\mu P[0, \pi]$ for some integer μ:*

(ii) *$CFSg(0) = s(\alpha', \beta')$, where*

$\alpha' = 1+\xi, \beta' > 1$ if $1+\lambda \leqq \alpha < 1+\xi$ or $\alpha = 1+\xi, \beta \leqq 1$ $\alpha' = \alpha$, $\beta' = \beta$ if $\alpha = 1+\xi, \beta > 1$ or $\alpha > 1+\xi, \beta \geqq 0$, and ξ, s and S have the values given in Theorem 1.

Before passing to the proofs of these theorems, let us observe that the existence of the constants $\bar{a}_{r-1-2i}$ implies their uniqueness from the definition of $g(t)$. In fact, it can be shown that the $\bar{a}_{r-1-2i}$ are given by

$$D_{r-1-2i}FSf(x) = \bar{a}_{r-1-2i}(C), \qquad i = 0, 1, \cdots \Big[\frac{r-1}{2}\Big].$$ [1]

[1] Bosanquet ([**1**], Theorem 1) has shown this for $f(x)$ Lebesgue integrable and (C) replaced by Abel summability.

In addition it can be shown that when $f(x) \in L$, the sum, S, of $D_r CFSf(x)$ may be written

$$S = -2\pi^{-1}\int_{\to o(C)}^{\infty} t^{-1}g(t)dt \; .^{2}$$

Proof of Theorem 1. That $s = -\pi^{-1}\int_0^{\pi} g(t)ctn(1/2)tdt$ follows from the consistency of (α, β) summability and a result due to Sargent ([4], Theorem 3). Therefore, both $g(t)ctn(1/2)t$ and $t^{-1}g(t)$ are CP integrable over $[0, \pi]$.

From Lemma 7 we have

$$(9) \qquad \bar{S}_{\alpha,\beta}(g, 0, n) - s = -2n\int_0^{\pi} g(t)[\bar{\lambda}_{1+\alpha,\beta}(nt) - (\pi nt)^{-1}]dt + o(1) \; .$$

The left side of (9) is $o(1)$ by hypothesis. By consistency equation (9) remains valid if α is replaced by $\alpha + r - i$ and β by $\beta + j, i, j = 0, 1, \cdots r$. Therefore,

$$-2n\int_0^{\pi} g(t) \sum_{i,j=0}^{r} B_{ij}^{r}(\alpha, \beta)[\bar{\lambda}_{1+\alpha+r-i,\beta+j}(nt) - (\pi nt)^{-1}]dt = o(1) \; .$$

With the aid of Lemmas 2 and 6, the last equation becomes

$$\bar{S}_{\alpha+r,\beta}^{r}(f, x, n) = -2\pi^{-1}\int_0^{\pi} t^{-1}g(t)dt + C_r + o(1) \; .$$

This completes the proof of Theorem 1.

Proof of Theorem 2. Due to the length of this proof and its similarity to the proof of Theorem 2, ([3]), only a brief outline of the proof will be given.

Putting $Q(t) = 0, \beta = 0$ and $p > \alpha + r$ in Lemma 6 and integrating the right-hand side of the resulting equation $\lambda + 1$ times, one can show that

$$D_{r+\lambda+1}CFS(\Psi_{\lambda+1}, 0, n) \text{ is summable } (C, p) \; .$$

A result due to Bosanquet ([1], Theorem 1) and the stepwise procedure employed in the proof of Theorem 2 ([3], equations 18 through 22) lead to the conclusion: $t^{-r-1}[\psi_r(t) - Q(t)] \in CP[0, \pi]$ for an appropriate polynomial $Q(t)$, i.e., $t^{-1}g(t) \in CP[0, \pi]$. From this statement and a results due to Sargent ([4], Theorem 3), $g(t) \in C_\mu P[0, \pi]$ for some integer μ and $CFSg(0) = s(C)$, where $s = \pi^{-1}\int_0^{\pi} g(t)ctn(1/2)tdt$.[3]

[2] Ibid. The difference in sign is due to the distinction between allied and conjugate series.

[3] The CP integrability of $g(t)ctn(1/2)t$ is equivalent to that of $t^{-1}g(t)$.

That S, the $(\alpha + r, \beta)$ sum of $D_r CFSf(x)$, has the value

$$-2\pi^{-1}\int_0^\pi t^{-1}g(t)dt + C_r$$

follows immediately from Theorem 1 and the consistency of the summability scale.

Thus, it remains to prove only the order relations (α', β') in (ii) of the theorem. A straightforward calculation using the representations in Lemmas 6 and 7, the properties of the $B^r_{ij}(\alpha, \beta)$ in Lemma 2, and the consistency of the summability scale applied to $D_r CFSf(x)$, leads to the following equations:

$$\sum_{i,j=0}^{r} B^r_{ij}(\alpha' + k, \beta')\Big[\bar{S}_{\alpha'+k+r-i,\beta'+j}(g, 0, n) \\ -\pi^{-1}\int_0^\pi g(t)ctn\frac{1}{2}tdt\Big] = o(1) ,$$

for $k = 0, 1, 2, \cdots$.

The expression in brackets may be considered the nth mean of order $(\alpha' + k + r - i, \beta' + j)$ of a series formed from $CFSg(0)$ by altering the first term. Since this series is summable (C) to 0, then Lemma 8 [3] shows that $CFSg(0) = s(\alpha', \beta')$.

The following theorem gives a sufficient condition for the (α, β) summability of $CFSg(0)$ for $\beta \neq 0$. Since the proof follows the usual lines for Riesz summability, it is omitted.

THEOREM 3. *Let $g(t)$ be an odd function of period 2π. If $t^{-1}g(t) \in C_k P[0, \pi]$, where k is a non-negative integer, then*

$$CFSg(0) = -\pi^{-1}\int_0^\pi g(t)ctn\frac{1}{2}tdt(1 + k, \beta), \beta > 1 .$$

As an application of these theorems it can be shown that

$$D_r CFSf(0, m) = S(1 + m + 2r, \beta), \beta > 1 ,$$

where $f(x; m)$ is either $x^{-m}\sin x^{-1}$ or $x^{-m}\cos x^{-1}$, $m = 0, 1, 2, \cdots$.

The following results may be deduced from Theorems 1 and 2. It is assumed that $f(x) \in C_\lambda P[-\pi, \pi]$ and is of period 2π. The values of S and s, when either exists, and ξ are given in Theorem 1.

(A). If $g(t) \in C_\mu P[0, \pi]$, then for $\alpha = 1 + \xi, \beta > 1$ or $\alpha > 1 + \xi$, $\beta \geqq 0$, $D_r CFSf(x) = S(\alpha + r, \beta)$ if and only if $CFSg(0) = s(\alpha, \beta)$.

(B). For $\alpha = 1 + \max(r, \lambda), \beta > 1$ or $\alpha > 1 + \max(r, \lambda), \beta \geqq 0$, $D_r CFSf(x) = S(\alpha + r, \beta)$ if and only if $g(t) \in CP[0, \pi]$ and $CFSg(0) = s(\alpha, \beta)$.

These results generalize, to various degrees, results obtained by Takahashi and Wang [7] and Bosanquet [1].

A weak, but none the less interesting, form of these results is

(C). If $f(x) \in CP[-\pi, \pi]$ and is of period 2π, then in order that $D_r CFSf(x)$ be summable (C), it is necessary and sufficient that $g(t) \in CP[0, \pi]$ and $CFSg(0)$ be summable (C).

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