THE FUNDAMENTAL GROUP OF A UNION OF SPACES

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1. Introduction. Given a connected[1] space X and a covering $\mathscr{C} \equiv \{X_\lambda \mid \lambda \in \Lambda\}$ of X by open sets, to what extent is the fundamental group G of X determined by the covering $\mathscr{C}$? When $\mathscr{C}$ consists of two connected sets X_1, X_2 with connected intersection X_{12}, then $G = \pi_1(X, x_0)$ is completely determined by $F_i = \pi_1(X_i, x_0)$, $i = 1, 2$ and $F_{12} = \pi_1(X_{12}, x_0)$ where $x_0 \in X_{12}$. For if $\theta_{i;12}$: $F_{12} \rightarrow F_i$, denotes the homomorphisms induced by the inclusion $X_{12} \subset X_i$, $i = 1, 2$ then G is obtained from the free product $F_1 \circ F_2$ by the identifications $\theta_{1,12}(a) = \theta_{2,12}(a)$ for all $a \in F_{12}$. Writing ξ_i: $F_i \rightarrow G$ for the homomorphisms induced by the inclusions $X_i \subset X$, $i = 1, 2$ then G is generated by the images of ξ_1, ξ_2 and $\xi_1\theta_{1,12} = \xi_2\theta_{2,12}$. Moreover, if φ_i: $F_i \rightarrow G'$ $i = 1, 2$ are homomorphisms such that $\varphi_1\theta_{1,12} = \varphi_2\theta_{2,12}$ then there is a homomorphism ψ: $G \rightarrow G'$ and $\varphi_i = \psi\xi_i$, $i = 1, 2$. G is completely determined by these properties. G is then the reduced free product of the system of groups and homomorphisms $\mathscr{F} = \{F_1, F_2, F_{12}, \theta_{1,12}\theta_{2,12}\}$. Using the notion of direct limit of a system of groups due to R. H. Fox [1, 3] an alternative statement is that Ξ: $\mathscr{F} \rightarrow G$ is a direct limit homomorphism where $\Xi \equiv \{\xi_1, \xi_2\}$. This result is usually known as the Van Kampen Theorem [4, 5]. A recent proof in terms of direct limits was given by Paul Olum [3].

More generally, if $\mathscr{C}$ consists of connected sets with a common point x_0 such that $X_{\lambda_1} \cap X_{\lambda_2} = X_{\lambda_1\lambda_2}$ is connected for any $\lambda_1, \lambda_2 \in \Lambda$ then, writing $G = \pi_1(X, x_0)$, $F_\lambda = \pi_1(X_\lambda, x_0)$, $F_{\lambda_1\lambda_2} = \pi_1(X_{\lambda_1\lambda_2}, x_0)$ the inclusions $X_{\lambda_1\lambda_2} \subset X_{\lambda i}$, $X_\lambda \subset X$, $X_{\lambda_1\lambda_2} \subset X$ induce canonical homomorphisms, $\theta_{\lambda_i,\lambda_1\lambda_2}$: $F_{\lambda_1\lambda_2} \rightarrow F_{\lambda_i}$, ξ_λ: $F_\lambda \rightarrow G$, $\xi_{\lambda_1\lambda_2}$: $F_{\lambda_1\lambda_2} \rightarrow G$ respectively where $i = 1, 2$; $\lambda, \lambda_1, \lambda_2 \in \Lambda$. Then

$$\mathscr{F} \equiv \{F_\lambda, F_{\lambda_1\lambda_2}, \theta_{\lambda_i,\lambda_1\lambda_2} \mid i = 1, 2; \lambda, \lambda_1, \lambda_2, \in \Lambda\}$$

is a system of groups and Ξ: $\mathscr{F} \rightarrow G$ is a direct limit homomorphism where $\Xi \equiv \{\xi_\lambda, \xi_{\lambda_1\lambda_2} \mid \lambda, \lambda_1, \lambda_2 \in \Lambda\}$. This is a slight generalization of a result proved by Richard H. Crowell [1].

The obstacles to further extending these results are twofold. In the first place, there need not be a point common to all the sets of $\mathscr{C}$ so that there are no canonical homomorphisms induced by inclusion. Thus, any path joining the base point x_λ of X_λ, say, to x_0, the base point of X (assuming for the moment that the X_λ's are connected) induces a homomorphism of $F_\lambda \equiv \pi_1(X_\lambda, x_\lambda)$ into G. Secondly, the images of the F_λ's, under all such induced homomorphisms do not, in general, generate

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[1] By "connected" we shall always mean "connected by arcs."

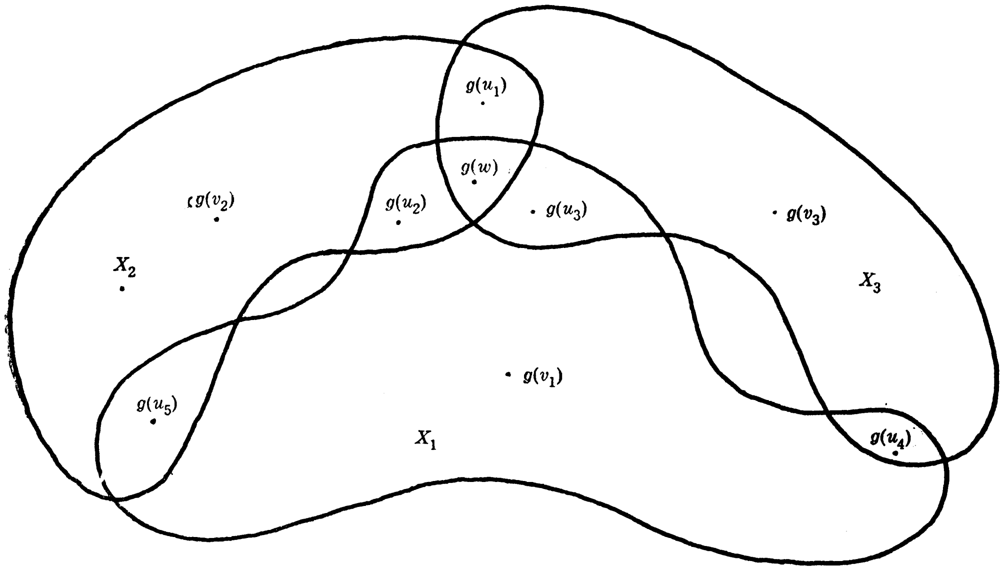
all of G but only a normal subgroup F of G. Both of these difficulties are overcome by the same device.

In the two theorems stated above, only the fundamental groups of sets of the covering and of intersections of sets of the covering entered into the description of G. But a covering is more than just a collection of sets. Thus the nerve of the covering can be regarded as a polyhedral approximation to the space. Unfortunately, this approximation is too coarse for our purposes. The covering $\mathscr{C}$, however, gives rise to another complex which more accurately reflects the intersection pattern of the covering. Taking a geometric realization Y of this complex, Y determines the elements of G not in F. More precisely $G/F \cong H = \pi_1(Y, y_0)$. Moreover, using Y we refine the notion of a system of groups, and define simplicial systems or S-systems, S-homomorphisms and S-limits. We then show that the covering gives rise to such an S-system and that G is the S-direct limit of this S-system. In the special cases considered above, this reduces to the above descriptions.

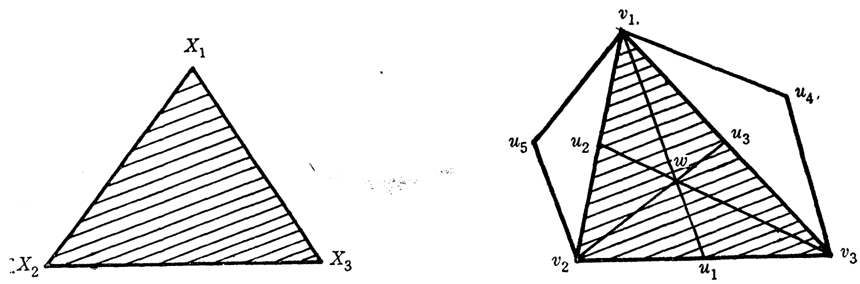
2. The intersection complex of a covering. Let $\mathscr{C} \equiv \{X_\lambda \mid \lambda \in \Lambda\}$ be a covering for the space X by open sets. No assumption is made about the connectivity of the sets of $\mathscr{C}$. The nerve of the covering $\mathscr{C}$, $\mathfrak{N}(\mathscr{C}) = \mathfrak{N}$ is the collection of finite subsets $\sigma = \{\lambda_0, \cdots, \lambda_k\} \subset \Lambda$ such that $X_\sigma = X_{\lambda_0} \cap \cdots \cap X_{\lambda_k}$ is nonempty. For each $\sigma \in \mathfrak{N}$ consider the set of components of X_σ and let $\{X_v \mid v \in \mathscr{V}\}$ be the collection of all such components for each $\sigma \in \mathfrak{N}$. That is, for each $v \in \mathscr{V}$ there is a unique $\mathfrak{p}(v) \in \mathfrak{N}$ and X_v is a component of $X_{\mathfrak{p}(v)}$. $\mathscr{V}$ is partially ordered by setting $v_1 < v_2$ whenever $X_{v_1} \supset X_{v_2}$ and $\mathfrak{p}(v_1)$ is a face of $\mathfrak{p}(v_2)$. The set of finite linearly ordered subsets of $\mathscr{V}$ is an abstract complex $\mathfrak{J}(\mathscr{V}) = \mathfrak{J}(\mathscr{C}) = \mathfrak{J}$, called the intersection complex of the covering. The function $\mathfrak{p}\colon \mathscr{V} \to \mathfrak{N}$ defines a simplicial map $\mathfrak{p}\colon \mathfrak{J} \to \mathfrak{N}'$ where $\mathfrak{N}'$ is the first barycentric subdivision of $\mathfrak{N}$. $\mathscr{V}$ can also be regarded as the set of vertices of a geometric realization Y of $\mathfrak{J}$. A vertex $v \in \mathscr{V}$ is a *k-vertex* if $\mathfrak{p}(v)$ is a k-simplex of $\mathfrak{N}$. $\mathscr{V}_k \subset \mathscr{V}$ is the set of j-vertices where $j \leqq k$ and $\mathfrak{J}_k$ (Y_k) the subcomplex of $\mathfrak{J}$ (subpolyhedron of Y) consisting of all simplexes with vertices in $\mathscr{V}_k$. The representation of a simplex of Y (Y_k, $\mathfrak{J}$, $\mathfrak{J}_k$) by its vertices will always be considered with the linear order. Thus if $v_0 v_1 \cdots v_r$ denotes an r-simplex then $v_0 < v_1 \cdots < v_r$.

Choose a fixed point $\mathfrak{g}(v) \in X_v$ for each $v \in \mathscr{V}$. For every 1-simplex $v_0 v_1$ of Y, $\mathfrak{g}(v_0)$ and $\mathfrak{g}(v_1)$ lie in the connected set X_{v_0} so that $\mathfrak{g}$ can be extended over $v_0 v_1$ into X_{v_0} and hence to a map $\mathfrak{g}\colon Y^1 \to X$ where Y^1 is the 1-skeleton of Y.

This is illustrated in figure 1, where $\mathscr{C} \equiv \{X_1, X_2, X_3\}$.



The space $X = X_1 \cup X_2 \cup X_3$
$\mathscr{C} \equiv \{X_1, X_2, X_3\}$



$\mathfrak{N}(\mathscr{C})$ $\mathfrak{J}(\mathscr{C})$

Figure 1.

3. Notation.

$\mathfrak{F}_v \equiv \pi_1(X_v, \mathfrak{g}(v)) \quad v \in \mathscr{V}$

$H \equiv \pi_1(Y, y_0),\ G \equiv \pi_1(X, x_0)$ where $y_0 \in \mathscr{Y}_0$ and $x_0 = \mathfrak{g}(y_0)$

$H_k^1 \equiv \pi_1(Y_k^1, y_0) \quad H^1 \equiv \pi_1(Y^1, y_0)$

$\xi\colon H^1 \to G,\ \xi_k\colon H_k^1 \to G$

denote the homomorphisms induced by $\mathfrak{g}$ and

$$K^1 \equiv \xi(H^1) \subset G \qquad K_k^1 \equiv \xi(H_k^1) \subset G\,.$$

4. The group F.

A path α in X is *admissible* if $\alpha(0) = x_0$ and $\alpha(1) = \mathfrak{g}(v)$ for some $v \in \mathscr{V}$. The path α then induces a homomorphism $F_v \to G$ which depends only upon the homotopy class[2] (α) of α. The homomorphism induced by the admissible paths α, written $(\alpha)_*$, is said to be *admissible* and F denotes the subgroup of G generated by the images of the admissible homomorphisms.

Any loop ω in X at x_0 induces an inner automorphism $(\omega)_*: G \to G$ depending only upon the homotopy class (ω) of ω. If α is any admissible path then so is $\omega\cdot\alpha$ and since $(\omega\cdot\alpha)_* = (\omega)_*(\alpha)_*$ it follows that F is invariant in G.

5. The homomorphism τ: $G \to H$.

LEMMA 1. *Any map $\mathfrak{f}$: $P \to X$ where P is a finite polyhedron, gives rise to a simplicial map $\mathfrak{f}^*$: $P^* \to Y$ where P^* is some simplicial subdivision of P.*

Proof. Let P' be a simplicial subdivision of P such that the covering by closed stars of vertices of P' refines the covering $\{\mathfrak{f}^{-1}(X_\lambda) \mid \lambda \in \Lambda\}$. For each vertex $p \in P'$ choose $\mathfrak{f}^*(p) \in \mathscr{V}_0$ such that $\mathfrak{f}(\overline{st}(p)) \subset X_{\mathfrak{f}^*(p)}$. Let $\sigma \equiv (p_0, \cdots, p_k)$ be any simplex of P' with barycentre $b(\sigma)$ and denote by $X_{\mathfrak{f}^*(b(\sigma))}$, $\mathfrak{f}^*(b(\sigma)) \in \mathscr{V}$ the component of $X_{\mathfrak{f}^*(p_0)} \cap \cdots \cap X_{\mathfrak{f}^*(p_k)}$ containing $\mathfrak{f}(\sigma)$. Then $\mathfrak{f}^*$ defines a simplicial map $\mathfrak{f}^*$: $P^* \to Y$ where P^* is the barycentric subdivision of P'.

It follows from Lemma 1 that any loop ω in X at x_0 defines a (not unique) edge-path loop ω' in Y at y_0. If ω is null-homotopic in X, then applying Lemma 1 with $P = I \times I$, ω' is also null-homotopic in Y. In particular, any two edge-path loops in Y, ω', ω'' defined in this way from the same loop ω in X, are homotopic. Thus, to every $g \in G$ there corresponds a unique element $\tau(g) \in H$ and the function τ: $G \to H$ so defined is easily seen to be a homomorphism.

A *lassoe* (resp. an *admissible lassoe*) in X is a loop at x_0 of the form $\alpha\cdot\omega\cdot\alpha^{-1}$ where α is a path (resp. an admissible path) and ω a loop in some X_v with $\mathfrak{g}(v) = \alpha(1)$.[3] Since the homotopy class $(\alpha\cdot\omega\cdot\alpha^{-1})$ of an admissible lassoe $\alpha\cdot\omega\cdot\alpha^{-1}$ is just the image under the admissible homomorphism $(\alpha)_*$ of the class $(\omega) \in F_v$, F can be considered as the subgroup of G generated by the homotopy classes of admissible lassoes.

For any admissible lassoe $\alpha\cdot\omega\cdot\alpha^{-1}$, the edge-path α' in Y defined by α according to Lemma 1 can be chosen so that $\alpha'(0) = y_0$, $\alpha'(1) \in \mathscr{V}$ and $\mathfrak{g}(\alpha'(1)) = \alpha(1)$, so that $\alpha'\alpha'^{-1}$ can be taken as an edge-path loop in y_0 representing $\tau(\alpha\cdot\omega\cdot\alpha^{-1})$. Hence $\tau(\alpha\cdot\omega\cdot\alpha^{-1}) = (\alpha'\cdot\alpha'^{-1}) = 1$ and $F \subset$ kernel τ.

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[2] We consider only homotopies of paths leaving the end points fixed.

[3] Since conjunction of paths is not associative the symbol $\alpha\cdot\omega\cdot\alpha^{-1}$ is ambiguous. We convene that $\alpha_1\alpha_2\cdots\alpha_k$ will denote the path α such that $\alpha(t) = \alpha_i(kt - i + 1)$ for $i - 1 \leqq kt \leqq i$, $i = 1, \cdots, k$.

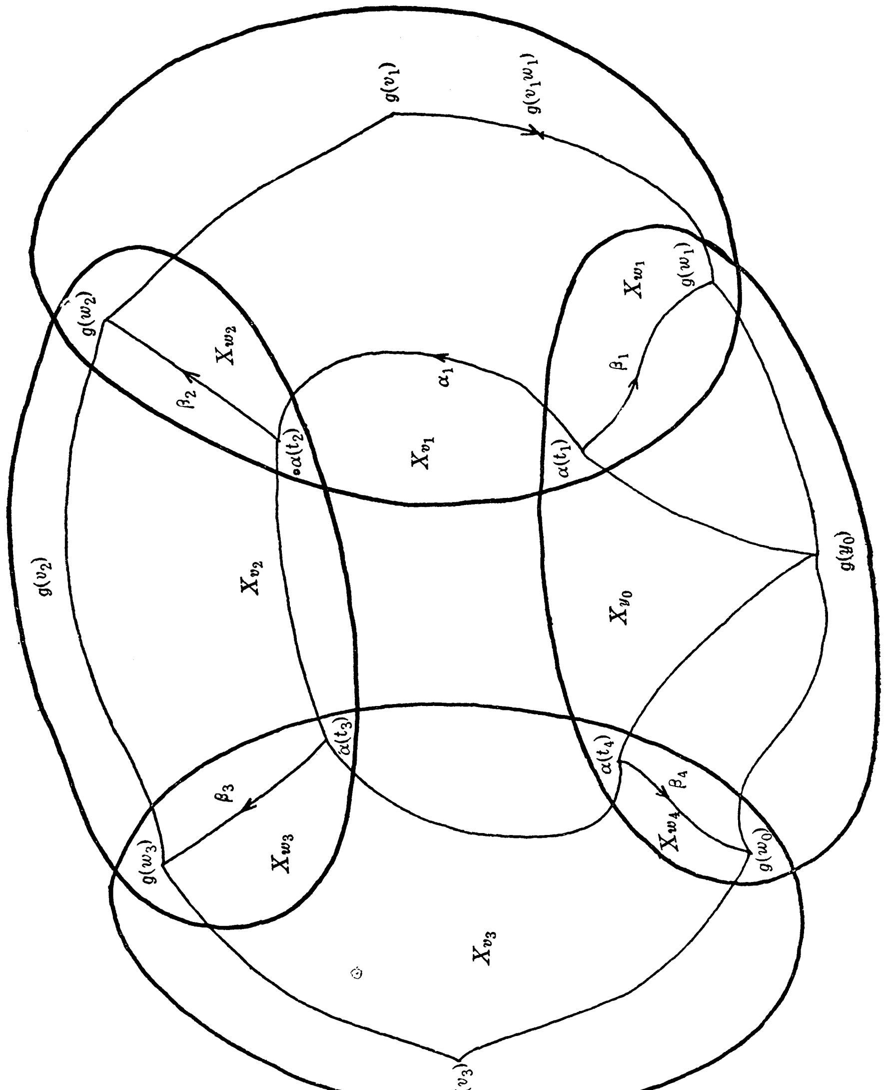

Figure 2.

A simple lassoe in Y is a loop of the form $\beta \cdot v_0v_1 \cdot v_1v_2 \cdot v_0v_2^{-1} \cdot \beta^{-1}$ where $v_0v_1v_2$ is a 2-simplex of Y and β an edge path in Y with $\beta(0) = y_0$, $\beta(1) = v_0$. Then if j: $H^1 \rightarrow H$ is the homomorphism induced by the inclusion $Y^1 \subset Y$, j is an epimorphism with kernel L the subgroup of H^1 generated by the simple lassoes (cf. [4. pp. 158–162]).

THEOREM 1. *The sequence* $0 \longrightarrow F \xrightarrow{i} G \xrightarrow{\tau} H \longrightarrow 0$ *is exact, where i is the inclusion monomorphism.*

Proof. It is easy to see that $\tau\xi = j$ whence we have exactness at

H. To prove exactness at G we need only show that $F \supset$ kernel τ. However, since $\xi(L) \subset F$ it is sufficient to show that for any $a \in G$, $a' \in H^1$ such that $\tau(a) = j(a')$, $a\xi(a')^{-1} \in F$. Let $\alpha \in a$ and choose a representative of a' of the form $v_0 w_1 \cdot v_1 w_1^{-1} \cdots v_k w_k^{-1}$ where $v_i \in \mathscr{V}_0$, $w_i \in \mathscr{V}_1 - \mathscr{V}_0$, $i = 1, \cdots, k$ and $v_0 = v_k = y_0$. Hence we can find real numbers $t_0 = 0 < t_1 < \cdots < t_k < t_{k+1} = 1$ with $\alpha(t_i) \in X_{w_i}$ and $\alpha(t) \in X_{v_{i-1}}$ for $t_{i-1} \leqq t \leqq t_i$, $i = 1, \cdots, k$. Choosing a path β_i in X_{w_i} with $\beta_i(0) = \alpha(t_i)$, $\beta_i(1) = \mathrm{g}(w_i)$, let

$$\gamma_i = \mathrm{g}(y_0 w_1) \cdot \mathrm{g}(v_1 w_1)^{-1} \cdots \mathrm{g}(v_i w_i)^{-1} \cdot \mathrm{g}(v_i w_i) \cdot \beta_i^{-1} \cdot \alpha_i \cdot \beta_{i+1} \cdot \mathrm{g}(v_i w_{i+1})^{-1} \cdots \mathrm{g}(w_1 y_0)^{-1}$$

where $\alpha_i(t) = \alpha(t_{i-1}(1-t) + tt_i)$ $0 \leqq t \leqq 1$, $i = 1, \cdots, k$. Then

$$a\xi(a')^{-1} = (\gamma_1) \cdots (\gamma_k) \in F \qquad \text{(cf. Figure 2).}$$

COROLLARY. $G = FK^1 = FK_k^1$.

6. S-system and S-direct limits. An alternative statement of Theorem 1 is that G is a group extension of F by H. Whereas H is defined in terms of the covering $\mathscr{C}$, F is defined only as a subgroup of G. It is desirable to obtain a description of G solely in terms of $\mathscr{C}$. To this end we introduce S-systems.

Let $\mathscr{F} \equiv \{F_v \mid v \in \mathscr{V}\}$ be a collection of groups indexed by the partially ordered set $\mathscr{V}$. The set of finite linearly ordered subsets of $\mathscr{V}$, $\mathfrak{Z}(\mathscr{V})$ is an abstract complex over $\mathscr{V}$. We will always write the vertices of any simplex in the linear order. Thus $\mathscr{F}$ can be regarded as a set of groups indexed by the vertices of a simplicial complex, hence we call such an $\mathscr{F}$ a *simplicial set of groups*, or more simply an *S-set*. $\mathscr{F}$ is connected if the complex $\mathfrak{Z}(\mathscr{V})$ is connected, that is, if the geometric realization Y of $\mathfrak{Z}(\mathscr{V})$ is connected. In the sequel we consider only connected S-sets.

Choose a fixed vertex y_0 in Y and let $F_u \equiv \pi_1(Y^1, y_0)$. We define a 1-cochain η of Y with coefficients in the (in general non-abelian) group F_u. It is not our intention to develop the theory of cohomology with non-abelian coefficients (cf. [2, 3]). We remark only that two 1-cochains η, η' are cohomologous if there is a 0-cochain ρ such that, for every 1-simplex $v_0 v_1$ of Y, $\eta'(v_0 v_1) = \rho(v_0)\eta(v_0 v_1)\rho(v_1)^{-1}$. To define η, we first choose a fixed edge path ω_v in Y, for each $v \in \mathscr{V}$, with $\omega_v(0) = y_0$, $\omega_v(1) = v$. Then $\eta(v_0 v_1) \in F_u$, is defined by the edge-path loop $\omega_{v_0} \cdot v_0 v_1 \cdot \omega_{v_1}^{-1}$. Although η clearly depends upon the choices of the paths ω_v, different choices lead to cohomologous 1-cochains. This cohomology class will be called the *standard class* of $\mathscr{F}$ and any cochain in the standard class, a *standard cochain*.[4]

[4] We remark that the standard chain is not in general, a cocycle of Y. It does, however, define a cocycle in Y^1 so that the standard class defines an element of $H^1(Y^1, F_u)$.

An S-system is an S-set $\{F_v \mid v \in \mathscr{V}\}$ together with a standard cochain η and two functions ζ, θ where

(i) ζ assigns to every 2-simplex $v_0v_1v_2$ of Y, $\zeta(v_0v_1v_2) \in F_{v_0}$

(ii) θ assigns to each 1-simplex of Y a homomorphism $\theta_{v_0v_1}$: $F_{v_1} \to F_{v_0}$.

To avoid special cases, we write $F_u = \pi_1(Y^1, y_0)$, $\mathscr{V}' = \mathscr{V} \cup \{u\}$ and denote an S-system by $\mathscr{F} \equiv \langle\{F_v \mid v \in \mathscr{V}'\}, \zeta, \theta, \eta\rangle$.

Let $\mathscr{F} \equiv \langle\{F_v \mid v \in \mathscr{V}'\}, \zeta, \theta, \eta\rangle$ be an S-system and G a group. An *S-homomorphism* Φ: $\mathscr{F} \to G$ of $\mathscr{F}$ into G is a function which assigns to each $v \in \mathscr{V}'$ a homomorphism φ_v: $F_v \to G$ such that

$$\text{(i)} \qquad \varphi_u(\eta(v_0v_1))_*\varphi_{v_1} = \varphi_{v_0}\theta_{v_0v_1}$$

$$\text{(ii)} \qquad \varphi_u(\eta(v_0v_1)\eta(v_1v_2)\eta(v_0v_2)^{-1}) = \varphi_{v_0}(\zeta(v_0v_1v_2))$$

where v_0v_1 is any 1-simplex and $v_0v_1v_2$ any 2-simplex of Y. The smallest subgroup of G containing the images of φ_v, $v \in \mathscr{V}'$ is called the image of Φ. If the image is all of G, Φ is onto. We write $\boldsymbol{A} \in \mathscr{F}$ whenever $\boldsymbol{A} = \langle a_1, \cdots, a_k\rangle$, $a_i \in F_{v_i}$, $v_i \in \mathscr{V}'$, $i = 1, 2, \cdots k$ and

$$\Phi(\boldsymbol{A}) = \varphi_{v_1}(a_1)\varphi_{v_2}(a_2) \cdots \varphi_{v_k}(a_k) \in G .$$

An S-homomorphism Φ: $\mathscr{F} \to G$ is an *S-direct limit* if Φ is onto and every S-homomorphism Φ': $\mathscr{F} \to G'$ is covered by Φ: $\mathscr{F} \to G$. That is, there is a homomorphism ψ: $G \to G'$ and $\varphi'_v = \psi\varphi_v$ for all $v \in \mathscr{V}'$. We write this as $\Phi' = \psi\Phi$. In this case, G is said to be an *S-direct limit of* $\mathscr{F}$.

PROPOSITION 1. There is an S-direct limit for every S-system, unique up to isomorphism.

Proof. Let $\mathscr{F} \equiv \langle\{F_v \mid v \in V'\}, \zeta, \theta, \eta\rangle$ be an S-system and let G be the group derived from the free product Γ of the groups $\{F_v \mid v \in \mathscr{V}'\}$ thby ə identifications

$$\text{(i)} \qquad \varphi'_u(\eta(v_0v_1))_*\varphi'_{v_1}(a) = \varphi'_{v_0}\theta_{v_0v_1}(a)$$

$$\text{(ii)} \qquad \varphi'_u\eta(v_0v_1)\eta(v_1v_2)\eta(v_0v_2)^{-1}) = \varphi'_{v_0}(\zeta(v_0v_1v_2))$$

for every $a \in F_{v_1}$, $v_1 \in V$, any 1-simplex v_0v_1 and any 2-simplex $v_0v_1v_2$ of Y, where φ'_v: $F_v \to \Gamma$, $v \in \mathscr{V}'$ is the homomorphism induced by the canonical imbedding of F_v in Γ. Then Φ: $\mathscr{F} \to G$ is an S-direct limit where φ_v is the composition of φ'_v with the natural projection $\Gamma \to G$.

The uniqueness follows easily.

To what extent does the S-direct limit of an S-system $\mathscr{F} \equiv \langle\{F_v \mid v \in \mathscr{V}'\}, \zeta, \theta, \eta\rangle$ depend upon the particular standard cochain η? This is answered in the following

PROPOSITION 2. Let $\mathscr{F} \equiv \langle\{F_v \mid v \in \mathscr{V}\}, \zeta, \theta, \eta\rangle$ and $\mathscr{F}' \equiv \langle\{F_v \mid v \in \mathscr{V}\}, \zeta, \theta, \eta'\rangle$ be two S-systems, and G any group. There is a one-to-one correspondence between the S-homomorphisms $\Phi\colon \mathscr{F} \to G$ and $\Phi'\colon \mathscr{F}' \to G$ and corresponding S-homomorphisms have the same image in G.

Proof. There is a 0-cochain ρ of Y with coefficients in F_u such that $\eta'(v_0v_1) = \rho(v_0)\eta(v_0v_1)\rho(v_1)^{-1}$ for every 1-simplex v_0v_1 of Y_1. Then if

$\Phi\colon \mathscr{F} \to G$ is a given S-homomorphism of $\mathscr{F}$ into G

$\Phi'\colon \mathscr{F}' \to G$ is defined by $\varphi'_u = \varphi_u$ and $\varphi'_v = \varphi_u(\rho(v))_*\varphi_v$ for all $v \in \mathscr{V}$.

PROPOSITION 3. G is an S-direct limit of $\mathscr{F}$ if and only if G is an S-direct limit of $\mathscr{F}'$. Hence the S-direct limit of $\mathscr{F}$ depends only upon the standard class of the S-set $\{F_v \mid v \in \mathscr{V}\}$.

More generally, two S-systems $\mathscr{F} \equiv \langle\{F_v \mid v \in \mathscr{V}\}, \zeta, \theta, \eta\rangle$ and $\mathscr{F}' \equiv \langle\{F'_w \mid w \in \mathscr{W}\}, \zeta', \theta', \eta'\rangle$ are *equivalent* if

(i) $\mathscr{V} = \mathscr{W}$ as partially ordered sets. Hence $\mathfrak{Z}(\mathscr{V}) = \mathfrak{Z}(\mathscr{W})$ and η' is cohomologous to η, so that we may assume $\eta = \eta'$ without loss of generality.

(ii) There is an isomorphism $\nu_v\colon F'_v \to F_v$ for each $v \in \mathscr{V} = \mathscr{W}$

(iii) There is a function ρ which assigns to each 1-simplex v_0v_1 of Y, $\rho(v_0v_1) \in F_{v_0}$ such that

$$\text{(a)} \qquad \theta'_{v_0v_1} = \nu_{v_0}^{-1}\rho(v_0v_1)_*\theta_{v_0v_1}\nu_{v_1}$$

$$\text{(b)} \qquad \zeta'(v_0v_1v_2) = \nu_{v_0}^{-1}(\rho(v_0v_1)\theta_{v_0v_1}(\rho(v_1v_2))\zeta(v_0v_1v_2)\rho(v_0v_2)^{-1}) .$$

In this case we write $\mathscr{F} \sim \mathscr{F}'$ and say that $\langle\nu, \rho\rangle$ defines the equivalence. This is clearly an equivalence relation.

PROPOSITION 4. If $\mathscr{F}$ and $\mathscr{F}'$ are equivalent then every S-homomorphism $\Phi\colon \mathscr{F} \to G$ determines an S-homomorphisms $\Phi'\colon \mathscr{F}' \to G$ with the same image, and vice versa.

Proof. Let $\langle\nu, \rho\rangle$ define the equivalence between

$$\mathscr{F} \equiv \langle\{F_v \mid v \in \mathscr{V}\}, \zeta, \theta, \eta\rangle$$

and

$$\mathscr{F}' \equiv \langle\{F'_v \mid v \in \mathscr{V}\}, \zeta', \theta', \eta\rangle$$

where in view of Proposition 1, we take $\eta = \eta'$. Let ρ' be defined on the 1-simplexes of Y by

$$\rho'(v_0v_1) = \varphi_{v_0}(\rho(v_0v_1))\varphi_u(\eta(v_0v_1)) \in G.$$

Regarding v_0v_1 as an edge-path of Y and setting

$$\rho'(v_0v_1^{-1}) = \rho'(v_0v_1)^{-1},\ \rho'(v_0v_1\cdot v_1v_2) = \rho'(v_0v_2)\rho'(v_1v_2)$$

ρ' is then a function from the edge-paths of Y into G and in particular, gives rise to a homomorphism φ'_u: $F_u \to G$. This together with $\varphi'_v = \rho'(\omega_v)_* \varphi_v \nu'_v$ defines an S-homomorphism Φ': $\mathscr{F}' \to G$. Since $\rho'(F_u) \subset$ image Φ, image $\Phi' \subset$ image Φ. The desired conclusion then follows easily.

PROPOSITION 5. Equivalent S-system have isomorphic S-direct limits.

Let $\mathscr{V}$ be a partially ordered set. A chain in $\mathscr{V}$ is a finite sequence $v_0, \cdots, v_k$ in $\mathscr{V}$ such that $v_0 < v_1 < \cdots < v_k$. The chain $v_0, \cdots, v_k$ is maximal if, for any $v \in \mathscr{V}$ we have that

(a) $v_{i-1} \leqq v \leqq v_i$ implies $v_{i-1} = v$ or $v = v_i$, $i = 1, \cdots, k$;

(b) $v \leqq v_0$ implies $v = v_0$ and

(c) $v_k \leqq v$ implies $v_k = v$.

Two chains are contiguous if they are both subchains of the same chain.

Keeping these definitions in mind, let $\mathscr{W} \subset \mathscr{V}$. $\mathscr{W}$ is cofinal with $\mathscr{V}$ if

(i) for any $v \in \mathscr{V}$, $w \in \mathscr{W}$, whenever $v < w$ then $v \in \mathscr{W}$.

(ii) for every $v \in \mathscr{V}$ there is a $w \in \mathscr{W}$ with $w \leqq v$.

(iii) every maximal chain of $\mathscr{V}$ of length 2, (i.e. with only 2 elements) is contained in $\mathscr{W}$.

(iv) if $w \in \mathscr{W}$ and $v_1, v_2 \in \mathscr{V}$ and $w < v_1 < v_2$ then there are $w_1, w_2 \in \mathscr{W}$ such that $w < w_1 < w_2$. Moreover the two chains w, v_1, v_2 and w, w_1, w_2 are contiguous.

It follows that $\mathfrak{J}(\mathscr{W})$ is a subcomplex of $\mathfrak{J}(\mathscr{V})$ and that $\pi_1(\mathfrak{J}(\mathscr{V}), y_0) \cong \pi_1(\mathfrak{J}(\mathscr{W}), y_0)$[5], $y_0 \in W$. Let $\mathscr{F} \equiv \langle \{F_v \mid v \in \mathscr{V}'\}, \zeta, \theta, \eta \rangle$ be an S-system where η is based on the paths ω_v defined as follows: if $w \in \mathscr{W}$ then ω_w is a path in $\mathfrak{J}(\mathscr{W})$ and if $v \in \mathscr{V} - \mathscr{W}$ then $\omega_v = \omega_w \cdot wv$ where w is chosen so that $w < v$. Then $\mathscr{F}' \equiv \langle \{F_w \mid w \in \mathscr{W}'\}, \zeta', \theta', \eta' \rangle$ is *cofinal* with $\mathscr{F}$ where $\mathscr{W}' \equiv \mathscr{W} \cup \{u_1\}$, $F_{u_1} \equiv \pi_1(\mathfrak{J}(\mathscr{W})^1, y_0)$, ζ', θ', are the restrictions of ζ and θ to $\mathfrak{J}(\mathscr{W})$ and η' is defined by the paths ω_w, $w \in \mathscr{W}$.

PROPOSITION 6. $\mathscr{F}$ and $\mathscr{F}'$ have isomorphic S-direct limits

7. The S-system associated with $\mathscr{C}$ and $\mathfrak{g}$: $Y^1 \to X$. The collection of groups $\{F_v \mid v \in V\}$ is clearly an S-set (cf. § 3). Following the convention introduced in § 6 we set $F_u \equiv H^1$, $\mathscr{V}' \equiv \mathscr{V} \cup \{u\}$ and write $\{F_v \mid v \in \mathscr{V}'\}$. To each 1-simplex v_0v_1 of Y corresponds a path $\mathfrak{g}(v_0v_1)$ in X_{v_0} with $\mathfrak{g}(v_0v_1)(0) = \mathfrak{g}(v_0)$, $\mathfrak{g}(v_0v_1)(1) = \mathfrak{g}(v_1)$. Denote by $\theta_{v_0v_1}$: $F_{v_1} \to F_{v_0}$, the homomorphism induced by this path. Each 2-simplex $v_0v_1v_2$ of Y determines a loop $\mathfrak{g}(v_0v_1) \cdot \mathfrak{g}(v_1v_2) \cdot \mathfrak{g}(v_0v_2)^{-1}$ in X_{v_0} and $\zeta(v_0v_1v_2) \in F_{v_0}$ is the class of this loop. To define η, we first choose a fixed edge-path in Y_1^1, ω_{v_0}, for each $v_0 \in \mathscr{V}_0$, taking $\omega_{y_0} = y_0$. Then for each $v \in \mathscr{V} - \mathscr{V}_0$ choose a $v_0 \in \mathscr{V}_0$ such that $v > v_0$ and set $\omega_v = \omega_{v_0} \cdot v_0v$. Then $\mathscr{F} \equiv$

[5] We write $\pi_1(\mathfrak{J}(\mathscr{V}), y_0)$ for the fundamental group of a geometric realization of $\mathfrak{J}(\mathscr{V})$.

$\langle\{F_v \mid v \in V'\}, \zeta, \theta, \eta\rangle$ is an S-system. This S-system depends only upon $\mathfrak{g}$ and $\mathscr{C}$. The map $\mathfrak{g}$ is of course, not uniquely defined. However, if $\mathfrak{g}'$ is another such map, and $\mathscr{F}'$ the S-system associated with $\mathfrak{g}'$ and $\mathscr{C}$ then $\mathscr{F}$ is equivalent to $\mathscr{F}'$ under the equivalence $\langle\nu, \rho\rangle$ defined as follows: $\mathfrak{g}(v)$ and $\mathfrak{g}'(v)$ both lie in the connected set X_v. ν_v: $F'_v \to F_v$ is the isomorphism induced by some fixed path ν'_v with $\nu'_v(0) = \mathfrak{g}(v)$, $\nu'_v(1) = \mathfrak{g}'(v)$. $\rho(v_0v_1) \in F_{v_0}$ is the class determined by the loop $\nu'_{v_0} \cdot \mathfrak{g}'(v_0v_1) \cdot \nu'^{-1}_{v_1} \cdot \mathfrak{g}(v_0v_1)^{-1}$, for every 1-simplex v_0v_1 of Y.

An edge path in Y is taken by $\mathfrak{g}$: $Y^1 \to X$ into a path in X. Admissible paths (§ 4) arising in this way we call *simple paths*, and the induced admissible homomorphisms, *simple homomorphisms*. Moreover, if $\alpha \cdot \omega \cdot \alpha^{-1}$ is an admissible lassoe where α is simple then we say $\alpha \cdot \omega \cdot \alpha^{-1}$ is a *simple lassoe*. The homotopy class $(\alpha \cdot \omega \cdot \alpha^{-1}) \in G$ of a simple lassoe is just the image of $(\omega) \in F_v$ under the simple homomorphism $(\alpha)_*$ where $\alpha(1) = \mathfrak{g}(v)$.

LEMMA 1. *F is generated by the images of simple homomorphisms.*

Proof. We have already remarked that any admissible path α defines an edge path α' in Y with $\alpha'(0) = y_0$, $\alpha'(1) = v$, $\mathfrak{g}(v) = \alpha(1)$, so that $\mathfrak{g}(\alpha')$ is a simple path in X. In the course of proving Theorem 1 we showed that $(\alpha \cdot \mathfrak{g}(\alpha')^{-1}) = a \in F$ and that a can be represented as the product of simple lassoes. Hence $(\alpha)_* = a_* \mathfrak{g}(\alpha')_*$ so that for any $b \in F_v$, $(\alpha)_*(b) = a(\mathfrak{g}(\alpha')_*(b))a^{-1}$ can be represented as the product of simple lassoes and the desired conclusion follows.

Let ξ_v denote the simple homomorphism induced by the path $\mathfrak{g}(\omega_v)$, $v \in V$ and write $\xi = \xi_u$: $F_u = H^1 \to G$. One verifies easily that these homomorphisms define an S-homomorphism

$$\Xi\colon \mathscr{F} \to G$$

THEOREM 2. *Ξ: $F \to G$ is an S-direct limit.*

Proof. For any simple path $\mathfrak{g}(\alpha)$, the simple homomorphism $\mathfrak{g}(\alpha)_* = \xi_u((\alpha \cdot \omega_v))_* \xi_v$ where $\alpha(1) = v$. It follows from this, Lemma 1 and the corollary to Theorem 1 that Ξ is onto.

Let Φ: $\mathscr{F} \to G'$ be any S-homomorphism. We must show Φ is covered by Ξ, that is, we must find a homomorphism ψ: $G \to G'$ such that $\Phi = \psi\Xi$. It is sufficient to show that whenever $\boldsymbol{A} \in \mathscr{F}$ with $\Xi(\boldsymbol{A}) = 1$, $\Phi(\boldsymbol{A}) = 1$, for then $\psi(\Xi(\boldsymbol{A})) = \Phi(\boldsymbol{A})$ would define ψ.

Consider any subdivision $0 = s_0 < s_1 < \cdots < s_n = 1$ of the unit interval and denote by γ the edge-path $s_0s_1 \cdot s_1s_2 \cdots s_{k-1}s_k$. Then every edge-path β in Y can be considered as $\mathfrak{b}'(\gamma)$, the path into which γ is taken by the simplicial map $\mathfrak{b}'$ of a suitable subdivision of I, and $\mathfrak{g}(\beta)$

as $\mathfrak{b}(\gamma) = \mathfrak{g}(\mathfrak{b}'(\gamma))$ where $\mathfrak{b}$ maps I into X. Similarly if ω is any loop in X_v, $\mathfrak{g}(\omega_v)\cdot\omega\cdot\mathfrak{g}(\omega_v)^{-1}$ can be regarded as $\mathfrak{b}(\gamma)$ where $\mathfrak{b}\colon I \to X$ and I is suitably subdivided. Hence, for any $a \in F_v$, $v \in \mathscr{V}'$, and more generally, for any $\boldsymbol{A} \in \mathscr{F}$, $\xi_v(a)$ or $\Xi(\boldsymbol{A})$ can be represented by a path of the form $\mathfrak{a}(\gamma)$ where $\mathfrak{a}$ is a map of the unit interval suitably subdivided, into X. Moreover, to each vertex s_i, there corresponds a vertex u_{s_i} of Y with $\mathfrak{a}(s_i) = \mathfrak{g}(u_{s_i})$ $i = 0, \cdots k$, where $u_{s_0} = u_{s_k} = y_0$. Identifying s_0 and s_k, we can regard $\mathfrak{a}$ as a map of a simplicial decomposition of the 1-sphere, S^1, into X.

Let $\boldsymbol{A} \in \mathscr{F}$ and let $\Xi(\boldsymbol{A})$ be represented, as above by $\mathfrak{a}(\gamma)$, where $\mathfrak{a}\colon S \to X$ and S is a simplicial subdivision of S^1. If $\Xi(\boldsymbol{A}) = 1$, $\mathfrak{a}(\gamma)$ is null-homotopic and so $\mathfrak{a}$ can be extended over the unit disk.

Choose a subdivision D of the unit disk, such that S is a subpolyhedron of D and (i) for any vertex t of $D - S$, $\mathfrak{a}(\overline{St}\,t) \subset S_{u_t}$ for some $u_t \in \mathscr{V}_0$ (ii) for any "boundary" 2-simplex, that is, a 2-simplex of the form ts_is_{i+1} where s_is_{i+1} is a simplex of S and t a vertex in $D - S$, $\mathfrak{a}(ts_is_{i+1}) \subset X_{u_{s_i}}$ if $u_{s_i} \leqq u_{s_{i+1}}$, otherwise $\mathfrak{a}(ts_is_{i+1}) \subset X_{u_{s_{i+1}}}$. We write X_{w_t} for the component of $X_{u_t} \cap X_{u_{t_1}} \cap \cdots \cap X_{u_{t_m}}$ containing $\mathfrak{a}(t)$ where t is a vertex of $D - S$ and $t_1, \cdots, t_m$ are all the remaining vertices of $\overline{St}\,t$, the closed star of t. Choose δ_t a path in X_{w_t} with $\delta_t(0) = \mathfrak{a}(t)$, $\delta_t(1) = \mathfrak{g}(w_t)$.

Let $X_{v_{t_0t_1}}$, $v_{t_0t_1} \in \mathscr{V}$ be the component of $X_{u_{t_0}} \cap X_{u_{t_1}}$ containing $\mathfrak{a}(t_0t_1)$. Then

$$\mathfrak{g}(v_{t_0t_1}w_{t_0})\cdot\delta_{t_0}^{-1}\cdot\mathfrak{a}(t_0t_1)\cdot\delta_{t_1}\cdot\mathfrak{g}(v_{t_0t_1}w_{t_1})^{-1}$$

is a loop in $X_{v_{t_0t_1}}$ and defines $b_{t_0t_1} \in F_{v_{t_0t_1}}$. Set

$$\alpha(t_0t_1) \equiv \langle \eta(v_{t_0t_1}w_{t_0})^{-1}, b_{t_0t_1}, \eta(v_{t_0t_1}, w_{t_1}\rangle$$

and

$$\alpha(t_0t_1\cdot t_1t_2) \equiv \langle \eta(v_{t_0t_1}w_{t_0})^{-1}, b_{t_0t_1}, \eta(v_{t_0t_1}w_{t_1}), \eta(v_{t_1t_2}w_{t_2})^{-1}, b_{t_1t_2}, \eta(v_{t_1t_2})\rangle\ .$$

α then assigns to each edge-path in D an element of $\mathscr{F}$. In particular, $\alpha(\gamma) = \boldsymbol{A}$.

The function α has the nice property that it takes any null homotopic edge-path of D into an element of $\mathscr{F}$ whose image under any S-homomorphism is 1. To see this, it suffices to show that $\Phi(\alpha(t_0t_1\cdot t_1t_2\cdot t_2t_0)) = 1$ for any 2-simplex $t_0t_1t_2$ of D. But this follows by a straight forward computation from the definition of α, the properties of an S-homomorphism and the fact that $\mathfrak{a}(t_0t_1\cdot t_1t_2\cdot t_2t_0)$ is a null homotopic loop in $X_{u_{t_0}}$. Since γ is null homotopic in D it follows that $\Phi(\alpha(\gamma)) = \Phi(\boldsymbol{A}) = 1$. This completes the proof.

REMARK. The set $\mathscr{V}_k$ is cofinal with $\mathscr{V}$ for $k \geqq 2$. Hence the resulting cofinal S-system has the same S-direct limit.

8. S-systems and systems. Consider an S-system $\mathscr{F} \equiv \langle \{F_v \mid v \in \mathscr{V}'\}, \zeta, \theta, \eta \rangle$ with Ξ: $\mathscr{F} \to G$ an S-direct limit. Let F be the subgroup of G generated by elements of the form $\xi_u(a)\xi_v(b)\xi_u(a)^{-1}$ for $a \in F_u$, $b \in F_v$, $v \in \mathscr{V}$. Then F is normal in G and $G = FK$ where $K = \xi_u(F_u)$. Moreover, if j: $H^1 \to H$ is the homomorphism defined by the inclusion of $Y^1 \subset Y$ then $\xi_u(\ker j) = F \cap K$. Hence $G/F = FK/F \cong K/(F \cap K) \cong H^1/(\ker j) \cong H$ so that G is an extension of F by H. (cf. Theorem 1). If $\zeta = 1$, that is, $\zeta(v_0 v_1 v_2) = 1$ for every 2-simplex of Y then $F \cap K = 1$ so that $K \cong H$ and $G = F \times H$ is the direct product extension. If further, $H = \{1\}$, then $\xi_u = 1$ and $G = F$ is generated by the images of the ξ_v, $v \in \mathscr{V}$.

R. H. Fox defines *a system* to be any collection $\mathscr{M}$ of groups and homomorphisms such that if μ: $M_\alpha \to M_\beta$ is in $\mathscr{M}$, then so are M_α and M_β. *A homomorphism* Φ: $\mathscr{M} \to N$ of the system $\mathscr{M}$ into the group N is a function assigning to each group M_α in $\mathscr{M}$ a homomorphism φ_α: $M_\alpha \to N$ such that for every μ: $M_\alpha \to M_\beta$ in $\mathscr{M}$, $\varphi_\alpha = \varphi_\beta \mu$. The image of Φ is the smallest subgroup of N containing the image of every φ_α in Φ, and if this is all of N then Φ is onto. Φ is a direct limit if it is onto and every homomorphism Φ': $\mathscr{M} \to N'$ is covered by Φ.

Thus if $\mathscr{F} \equiv \langle \{F_v \mid v \in \mathscr{V}'\}, \zeta, \theta, \eta \rangle$ is any S-system, the collection of groups $\{F_v \mid v \in \mathscr{V}'\}$ together with θ defines a system. If further, $\zeta = 1$ and $H = \{1\}$ then any S-homomorphism of $\mathscr{F}$ is a homomorphism of this system and the S-direct limit of $\mathscr{F}$ is the direct limit of this system. Conversely, let $\mathscr{M}$ be a system where the groups of $\mathscr{M}$ are indexed by $\mathscr{A}$. Setting $\beta < \alpha$ whenever there is a sequence of groups and homomorphisms of $\mathscr{M}$, $M_\alpha \xrightarrow{\mu_1} M_{\alpha_1} \longrightarrow \cdots \longrightarrow M_\beta$, defines a partial ordering for $\mathscr{A}$ and $\mathfrak{J}(\mathscr{A})$ is the complex of finite linearly ordered subsets of $\mathscr{A}$. This sequence need not, of course, be unique. However, for each 1-simplex $\beta\alpha$ of $\mathfrak{J}(\mathscr{A})$ choose a fixed such sequence and let $\theta_{\beta\alpha}$: $M_\alpha \to M_\beta$ be the composition of the homomorphisms occuring in this sequence. Hence $\langle \{M_\alpha \mid \alpha \in \mathscr{A}'\}, \zeta, \theta, \eta \rangle$ is an S-system where θ has been defined above and $\zeta = 1$. If further $H = \{1\}$ then S-homomorphisms and the S-direct limit of this S-system are simply the homomorphisms and the direct limit of the original system.

Thus an S-system is a system with some additional structure, (ζ and H) which plays a significant role in taking S-direct limits. If this additional structure is trivial ($\zeta = 1, H = \{1\}$) then this reduces to the direct limit of a system.

9. Further comments.

A. Consider again the S-system $\mathscr{F} \equiv \langle \{F_v \mid v \in \mathscr{V}'\}, \zeta, \theta, \eta \rangle$ associated with the covering $\mathscr{C}$ and the map $\mathfrak{g}$: $Y^1 \to X$ considered in § 7. The function ζ can be regarded as an obstruction to extending $\mathfrak{g}$ over Y^2.

If $\zeta = 1$ then g can be extended to g: $Y^2 \to X$ and defines a monomorphism of H into G. As in § 8 above, $G \cong F \times H$.

B. If $\mathscr{C} \equiv \{X_\lambda \mid \lambda \in \varLambda\}$ consists of connected sets such that $x_0 \in \bigcap_{\lambda \in \varLambda} X_\lambda$ and the intersection of any finite collection of sets of $\mathscr{C}$ is also in $\mathscr{C}$, then $H = \{1\}$, $\mathscr{V} = \varLambda$ and g: $Y^1 \to X$ can be taken as the constant map $g(Y^1) = x_0$. Hence $\zeta = 1$ and $\theta_{v_0 v_1}$: $F_{v_1} \to F_{v_0}$ is the homomorphism defined by the inclusion of $X_{v_1} \subset X_{v_0}$. It follows then, as in § 8 that the S-direct limit $\varXi$: $\mathscr{F} \to G$ is just the direct limit of the system $\langle \{F_v \mid v \in \mathscr{V}\}, \theta \rangle$. This is Crowell's result [1].

The following simple extension of this result follows from the remarks following Theorem 2 and Proposition 6 on S-systems: If $\mathscr{C} \equiv \{X_\lambda \mid \lambda \in \varLambda\}$ consists of connected sets such that $x_0 \in \bigcap_{\lambda \in \varLambda} X_\lambda$ and the intersection of any two sets of $\mathscr{C}$ is connected then G is the direct limit of the system consisting of the groups

$$\{\pi_1(X_\lambda, x_0), \pi_1(X_{\lambda_1} \cap X_{\lambda_2}, x_0) \mid \lambda, \lambda_1, \lambda_2 \in \varLambda\}$$

together with the natural inclusion homomorphisms. When $\mathscr{C} = \{X_1, X_2\}$, $X_1, X_2, X_1 \cap X_2$ connected, then this gives the so called *Van Kampen Theorem* [3, 4].

C. Actually Van Kampen proved a more general result. A slight modification of our methods gives Van Kampen's complete results.

D. If the groups $F_v \equiv \{1\}$, $v \in \mathscr{V}_0$ then $G \cong H$.

E. The fact that the covering $\mathscr{C}$ consisted of open sets was used only in Lemma 1 and Theorem 2 to permit the decomposition of a polyhedron mapped into X. Any condition on the covering $\mathscr{C}$, permitting this to be carried out would be sufficient. Thus if $\mathscr{C}$ is such that every point of X is interior to some set of $\mathscr{C}$, everything goes through. This question is fully discussed in the paper by Olum [3]. His comments are, with simple obvious modifications, applicable to the more general situation discussed in this paper.

F. A description in terms of generators and relations can easily be given but we omit it.

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