THE MOMENT PROBLEM AND WEAK CONVERGENCE IN L^2

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1. Introduction. Consider a sequence of functions $u_n(x)$ belonging to the real Hilbert Space $L^2(0, 1)$. Suppose the range of every $u_n(x)$ is contained in the bounded interval $[a, b]$. Then the $u_n(x)$ are uniformly bounded in the norm. The same is of course true for the functions $[u_n(x)]^i$, for any fixed positive integral exponent i. Since the unit sphere in $L^2(0, 1)$ is weakly compact we can find (by repeatedly constructing convergent subsequences and using the diagonal process) a new sequence of functions[1] $v^i(x)$ such that for an appropriate subsequence $u_{n_k}(x)$ of our original set,

$$[u_{n_k}(x)]^i \xrightarrow[k\to\infty]{} v^i(x)$$

weakly for all $i = 1, 2, \cdots$.

Now consider the converse problem. Given a closed subset of the line F, and a sequence of functions $v^i(x) \in L^2(0, 1)$; when does there exist an associated sequence of functions $u_n(x) \in L^2(0, 1)$ such that

(1) the range of $u_n(x)$ is included in F for all n and

(2) $[u_n(x)]^i \xrightarrow[n\to\infty]{} v^i(x)$ weakly for all i?

We shall show that a necessary and sufficient condition is that the $v^i(x)$ satisfy a positiveness Condition P:

Condition P. For every polynomial $p(t) = \sum_{i=0}^n a_i t^i$ nonnegative on the closed set F, the function $\sum_{i=0}^n a_i v^i(x) \geqq 0$ *p.p.* on $(0, 1)$. (We define $v^0(x) \equiv 1$).

Note that the interval $[a, b]$ has been replaced by the arbitrary closed set F. The result will be seen to be valid in $L^2(-\infty, \infty)$ provided that $v^{2i}(x) \in L(-\infty, \infty)$ for all $i > 0$. Finally we shall prove an analogous theorem for n-tuple sequences $v^{ij\cdots k}(x)$.

One trivial consequence of Condition P, of which we shall make use, is that $v^{2i}(x) \geqq 0$ *p.p.* for all i.

2. Construction of weakly convergent sequences. The following result is fundamental to what follows.

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[1] The index i for $v^i(x)$ is a superscript, not an exponent.

THEOREM 1. *For each positive integer n, let there be given n functions f_{ni}, $0 \leqq i \leqq n-1$, $\in L^2(0, 1)$ such that for every i and n*

$$\int_0^1 f_{ni}(x)dx = 0 . \tag{1}$$

Define $f_n(x)$ by

$$f_n(x) = f_{ni}(nx - i) \quad \text{for} \quad i/n \leqq x < (i+1)/n .$$

Suppose that for some constant M, $\|f_n\| < M$ for all n. Then $f_n(x) \xrightarrow[n\to\infty]{} 0$ weakly.

Proof. Let ϕ_{rs} be the characteristic function of the interval (r, s). Since the ϕ_{rs}, for all r and s with $0 < r < s < 1$, span $L^2(0, 1)$ it suffices to prove that $\lim_{n\to\infty}(f_n, \phi_{rs}) = 0$ for all ϕ_{rs}. Fix r and s. If n is an integer greater than $1/(s-r)$, there exist integers k_1 and k_2 with $s \geqq k_1/n \geqq k_2/n \geqq r$, and such that $(s - k_1/n) < 1/n$ and $(k_2/n - r) < 1/n$. Then

$$(f_n, \phi_{rs}) = \int_r^s f_n(x)dx = \int_{k_2/n}^{k_1/n} f_n(x)dx + \int_{k_1/n}^{s} f_n(x)dx + \int_r^{k_2/n} f_n(x)dx .$$

Each of the last two integrals is less in absolute value than $M(n)^{-1/2}$, and the first integral vanishes by hypothesis. Hence, $|(f_n, \phi_{rs})| < 2M(n)^{-1/2}$ or $\lim_{n\to\infty}(f_n, \phi_{rs}) = 0$. This completes the proof.

COROLLARY. *For each positive integer n, let there be given the functions $f_{ni}(x) \in L^2(0, 1)$ with $i = 0, \pm 1, \pm 2, \pm 3, \cdots$, such that for every i and n*

$$\int_0^1 f_{ni}(x)dx = 0 .$$

Define $f_n(x)$ by

$$f_n(x) = f_{ni}(nx - i) \quad \text{for} \quad i/n \leqq x < (i+1)/n .$$

Suppose that for all n, $f_n \in L^2(-\infty, \infty)$; and that there exists a number M such that $\|f_n\| < M$ for all n. Then $f_n(x) \xrightarrow[n\to\infty]{} 0$ weakly.

Suppose that $\psi(x)$ is a (not necessarily strictly) monotonically increasing bounded function, defined for $-\infty < x < \infty$. Let $\inf_x \psi(x) = A$ and $\sup_x \psi(x) = B$. Then we define the inverse function $\psi^{-1}(t)$ on the interval (A, B) as follows:

(a) If there exists an x such that $\psi(x) = t$, define $\psi^{-1}(t) = \sup_{\psi(x)=t} x$.

(b) If there exists no x with $\psi(x) = t$, ψ has a jump "past" t, i.e., there exists an x_0 such that $\psi(x_0^-) \leqq t$ and $\psi(x_0^+) \geqq t$. Define $\psi^{-1}(t) = x_0$ in this case.

Evidently $\psi^{-1}(t)$ is monotonically nondecreasing, is constant where ψ has a jump, and has a jump where ψ is constant.

It is well known (and easily verified) that for such functions $\psi(x)$, and for $f(x)$ continuous, that

$$\int_{-\infty}^{\infty} f(x)d\psi(x) = \int_A^B f(\psi^{-1}(t))dt \tag{2}$$

in the sense that if the former integral exists, and converges absolutely, the latter exists, and the two are equal.

We shall also say that x is a point of increase of the nondecreasing function $\psi(x)$, if for every neighborhood (a, b) of x, $\psi(b) > \psi(a)$.

In order to prove our main theorem we need a lemma.

LEMMA 1. *Let $v^i(x)\,(i \geqq 1)$ be a sequence of functions in $L(0, 1)$ satisfying Condition P. Then there exists a function $\rho(x)$ such that*

(a) *The range of $\rho(x)$ is included in F.*

(b) $[\rho(x)]^i \in L^2(0, 1)$ *for every* $i = 0, 1, 2, \cdots$.

(c) $\int_0^1 \{[\rho(x)]^i - v^i(x)\}dx = 0\ ,\qquad i = 0, 1, 2, \cdots\ .$

Proof. Let $b_i = \int_0^1 v^i(x)dx$. Since the $v^i(x)$ satisfy Condition P, the numbers b_i also do. Therefore, the b_i form a moment sequence on F[2], i.e., there exists a nondecreasing function $\psi(x)$ whose points of increase are included in F, such that

$$\int_{-\infty}^{\infty} x^i d\psi(x) = b_i = \int_0^1 v^i(x)dx \qquad \text{for } i = 0, 1, 2, \cdots\ .$$

In particular

$$\int_{-\infty}^{\infty} d\psi(x) = b_0 = 1$$

so that we may assume that $\inf \psi(x) = 0$ and $\sup \psi(x) = 1$. Define $\rho(x) = \psi^{-1}(x)$ so that $\rho(x)$ is defined on $(0, 1)$ and takes on values in F. Now making use of relation (2), we have

$$b_i = \int_{-\infty}^{\infty} x^i d\psi(x) = \int_0^1 [\rho(x)]^i dx = \int_0^1 v^i(x)dx\ .$$

Q.E.D.

COROLLARY. *By an obvious change in variable the result of the lemma remains valid with* $(0, 1)$ *replaced by an arbitrary finite interval* (r, s).

3. The principal existence theorem. The main result is given in

THEOREM 2. *Let $v^i(x)$ be a sequence of functions belonging to $L^2(0, 1)$, and satisfying Condition P. Then there exists a sequence of functions $u_n(x)$ such that*

(a) *The range of $u_n(x)$ is contained in F for every n.*

(b) *$[u_n(x)]^i \in L^2(0, 1)$ for all i and n.*

(c) *$[u_n(x)]^i \xrightarrow[n\to\infty]{} v^i(x)$ weakly for all i.*

Proof. Consider the restriction of the $v^i(x)$ to the interval $(j/n, (j+1)/n)$, $0 \leqq j \leqq n-1$. Momentarily fix j and n. By appealing to the corollary of Lemma 1 we can construct functions $\rho_{nj}(x)$ defined on $(j/n, (j+1)/n)$ such that

(1) The range of $\rho_{nj}(x)$ is contained in F,

(2) $\left[\rho_{nj}\left(\frac{x+j}{n}\right)\right]^i \in L^2(0, 1)$ for all i,

(3) $\int_{j/n}^{(j+1)/n} \{[\rho_{nj}(x)]^i - v^i(x)\} dx = 0$ for all $i = 1, 2, \cdots$.

This may be done for every j, $0 \leqq j \leqq n-1$, and every n. Fix i for the remainder of the argument. We now appeal to Theorem 1. Namely we define the functions $f_{nj}(x)$ on $(0, 1)$ by

$$f_{nj}(x) = \left[\rho_{nj}\left(\frac{x+j}{n}\right)\right]^i - v^i\left(\frac{x+j}{n}\right), \qquad 0 \leqq j \leqq n-1$$

and the function $f_n(x)$ on $(0, 1)$ by

$$f_n(x) = [\rho_{nj}(x)]^i - v^i(x) \quad \text{for} \quad j/n \leqq x < (j+1)/n .$$

We must show that $||f_n|| < M$ for some $M < \infty$. But

$$\begin{aligned} ||f_n|| &\leqq \left\{\sum_{j=0}^{n-1} \int_{j/n}^{(j+1)/n} [\rho_{nj}(x)]^{2i} dx\right\}^{1/2} + ||v^i|| \\ &= \left\{\int_0^1 v^{2i}(x) dx\right\}^{1/2} + ||v^i|| \\ &\leqq ||v^{2i}||^{1/2} + ||v^i|| . \end{aligned}$$

Thus, by Theorem 1, $f_n(x) \xrightarrow[n\to\infty]{} 0$ weakly. If we define $u_n(x)$ by

$$u_n(x) = \rho_{nj}(x) \quad \text{for} \quad j/n \leqq x < (j+1)/n$$

then, the range of $u_n(x)$ is contained in F; $[u_n(x)]^i = f_n(x) + v^i(x)$ belongs to $L^2(0, 1)$, and

$$[u_n(x)]^i - v^i(x) \xrightarrow[n\to\infty]{} 0 \text{ weakly} .$$

Since i was arbitrary we have proved our theorem.

COROLLARY. *The conclusion of Theorem 2 remains valid in*

$L^2(-\infty, \infty)$ *if an additional hypothesis is made, namely that* $v^{2i}(x) \in L(-\infty, \infty)$ *for all* $i > 0$.

Proof. Consider the restriction of the $v^i(x)$ to the interval $(j/n, (j+1)/n)$ where j is any integer, positive, negative, or zero. We can construct functions $\rho_{nj}(x)$ as above, and for fixed i, define the function $f_n(x)$ by

$$f_n(x) = [\rho_{nj}(x)]^i - v^i(x)\ , \quad j/n \leqq x < (j+1)/n\ , \quad j = 0, \pm 1, \pm 2, \cdots .$$

Once we have shown that $\|f_n\| < M$ for all n and some $M < \infty$, we can appeal to the corollary of Theorem 1, define $u_n(x)$ as above, and obtain the desired result. But

$$\begin{aligned} \|f_n\| &\leqq \left\{\sum_{j=-\infty}^{\infty} \int_{j/n}^{(j+1)/n} [\rho_{nj}(x)]^{2i} dx\right\}^{1/2} + \|v^i\| \\ &= \left\{\int_{-\infty}^{\infty} v^{2i}(x)\,dx\right\}^{1/2} + \|v^i\| . \end{aligned}$$

Since $v^{2i}(x) \in L(-\infty, \infty)$ by hypothesis, the proof is complete.

We shall now summarize Theorem 2 and its corollary, together with a converse, in one result:

THEOREM 3. *Given a sequence of functions* $v^i(x)\,(i = 1, 2, \cdots)$ *in* $L^2(c, d)$, $-\infty \leqq c < d \leqq \infty$. *Necessary and sufficient conditions that there exist a sequence of functions* $u_n(x)$ *such that*

(1) $[u_n(x)]^i \in L^2(c, d)$ *for all* $i > 0$ *and* n;

(2) $[u_n(x)]^i \xrightarrow[n\to\infty]{} v^i(x)$ *weakly for all* $i > 0$; *and*

(3) *the range of* $u_n(x)$ *is contained in* F *for every* n,

are that the $v^i(x)$ *satisfy Condition P, and that* $v^{2i}(x) \in L(c, d)$ *for all* $i > 0$.

Proof. The sufficiency has already been shown. To prove the necessity note that the weak limit of nonnegative functions is nonnegative *p.p.* Also, if c and d are finite, $v^{2i} \in L^2(c, d)$ implies that $v^{2i} \in L(c, d)$. If $c = 0$ and $d = \infty$ we must prove that $v^{2i} \in L(0, \infty)$. Now $[u_n(x)]^{2i} \xrightarrow[n\to\infty]{} v^{2i}$ weakly by hypothesis (2). $[u_n(x)]^{2i} \in L(0, \infty)$ by hypothesis (1), so that v^{2i} is the weak limit of functions in $L(0, \infty)$. By hypothesis (2)

$$0 \leqq \int_0^N v^{2i}(x)dx = \lim_{n\to\infty} \int_0^N [u_n(x)]^{2i} dx \leqq \limsup_{n\to\infty} \|[u_n(x)]^i\|^2 .$$

Again by hypothesis (2), the $\|[u_n(x)]^i\|$ are bounded for fixed i, so that

$$\int_0^\infty v^{2i}(x)dx < \infty$$

or $v^{2i}(x) \in L(0, \infty)$. A similar proof exists if $c = -\infty$. This completes the proof.

4. Generalizations to multiple sequences. We now proceed to multiple sequences of functions $v^{ij\cdots k}(x) \in L^2(0, 1)$ defined for $i, j, \cdots, k = 0, 1, \cdots$. In order to simplify the notation we shall restrict ourselves to double sequence $v^{ij}(x)$, but the generalization to higher order sequences will be self evident.

We have a two-dimensional analog of Condition P:

Condition Q. For every polynomial $p(t, \tau) = \sum_{i,j=0}^{n} a_{ij} t^i \tau^j$ nonnegative in the closed set F, the function $\sum_{i,j=0}^{n} a_{ij} v^{ij}(x) \geqq 0$ $p.p.$ in $(0, 1)$ where $v^{\circ\circ}(x) \equiv \mathbf{1}$.

Before proving an analog of Theorem 3 we shall prove a lemma, based on a result of Halmos and von Neumann [1, § 2]. This is a two-dimensional version of Lemma 1.

LEMMA 2. *Let $v^{ij}(t)$ be a double sequence of functions in $L(0, 1)$ satisfying Condition Q. Then there exist two functions $\rho(t)$ and $\lambda(t)$ such that*

(a) *The curve given by $x = \rho(t)$, $y = \lambda(t)$ is contained in the subset F of the plane.*

(b) *The functions $\{[\rho(t)]^i \cdot [\lambda(t)]^j\}$ belong to $L^2(0, 1)$ for all i and j.*

(c) $\int_0^1 \{[\rho(t)]^i [\lambda(t)]^j - v^{ij}(t)\} dt = 0$ *for all i and j.*

Proof. Let $b_{ij} = \int_0^1 v^{ij}(t) dt$. Since the $v^{ij}(t)$ satisfy Condition Q, the numbers b_{ij} also do. Hence the b_{ij} form a moment sequence on F[2], i.e., there exists a measure ψ, defined for all Borel sets of the plane E_2, such that

(1) $\int_{E_2} x^i y^j d\psi = b_{ij}$ for all i and $j \geqq 0$.

(2) If $(x, y) \notin F$, there exists a neighborhood N of (x, y), with $\psi(N) = 0$.

If the measure space $\{F, \mathscr{B}, \psi\}$, where $\mathscr{B}$ is the class of all Borel subsets of F, has atoms (see [1] for definition of an atom), every atom may be shown to consist of a point, plus a set of ψ measure zero. These "atomic points" are either finite or denumerably infinite in number. Denote them by P_i, and let $P = \bigcup_i \{P_i\}$. Clearly $P \subset F$. If we define the measure $\bar{\psi}$ by $\bar{\psi}(A) = \psi(A) - \psi(A \cap P)$, $\bar{\psi}$ is non-atomic. Say $\psi(P) = \sum_i \psi(P_i) = p$.

From relation (1) with $i = j = 0$, we have $\psi(F) = \psi(E_2) = b_{\circ\circ} = 1$, so that $\bar{\psi}(F) = 1 - p$. There is a one-to-one mapping $\bar{\phi}$ from almost all of the interval $(0, 1 - p)$ onto almost all of F, such that B_1 is a Borel subset of $(0, 1 - p)$ if and only if $\bar{\phi}(B_1)$ is in $\mathscr{B}$, and then $\bar{\psi}(\bar{\phi}(B_1)) =$

$m(B_1)$ where m is the ordinary Lebesgue measure [1, Theorem 2]. We can easily construct a map $\hat{\phi}$ from $(1-p, 1)$ onto P, such that $m(\hat{\phi}^{-1}(P_i)) = \psi(P_i)$. If we define $\phi = \bar{\phi} \cup \hat{\phi}$, Then ϕ has the following properties: ϕ maps almost all of $(0, 1)$ onto almost all of F, such that if $A \subset F$ and $A \in \mathscr{B}$, $\phi^{-1}(A)$ is a Borel set, and $m(\phi^{-1}(A)) = \psi(A)$. Let $\rho(t)$ be the projection of $\phi(t)$ on the x-axis, and $\lambda(t)$ the projection on the y-axis. Then it follows that $\rho(t)$ and $\lambda(t)$ satisfy conditions (a), (b), and (c).

COROLLARY. *The result of the lemma is valid if* $(0, 1)$ *is replaced by an arbitrary finite interval* (r, s).

THEOREM 4. *Given a double sequence of functions* $v^{ij}(t)$ $i, j = 0, 1, 2, \cdots$ (except i and j both zero) *in* $L^2(c, d)$; $-\infty \leqq c < d \leqq \infty$. *Necessary and sufficient conditions that there exist two sequences of functions* $u_n(t)$, $w_n(t)$ *belonging to* $L^2(c, d)$ *such that* (a) *the curve in the plane defined by* $x = u_n(t)$, $y = w_n(t)$ *for* $c \leqq t \leqq d$, *is contained in the closed set* F; *and* (b) *for every* i *and* j (except i and j both zero) (1) $[u_n(t)]^i[w_n(t)]^j \in L^2(c, d)$ *for all* n *and* (2) $[u_n(t)]^i[w_n(t)]^j \xrightarrow[n\to\infty]{} v^{ij}$ *weakly*; *are that* (1) *the* $v^{ij}(t)$ *satisfy Condition* Q, *and* (2) $v^{2i,2j} \in L(c, d)$ *for all* i *and* j (not both zero).

Proof. The proof is very similar to that of Theorems 2 and 3, and is therefore omitted.

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