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# ON THE CONSTRUCTION OF CERTAIN BOUNDED CONTINUOUS FUNCTIONS

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We give an elementary method for constructing continuous functions fulfilling the hypothesis of Theorem 1 of the preceding paper. Such functions thus constitute counterexamples to the proposition and theorem discussed therein.

THEOREM. *Let* $\varphi(x)$ *be continuously differentiable on* $[0, \infty)$, *and suppose*

(i) $\varphi(0) = 0$

(ii) $\varphi'(x)$ *is nonnegative, and strictly increasing to* $\infty$ *on* $[0, \infty)$

(iii) $\varphi'(x)/\varphi(x) \to \infty, x \to \infty$.

*Put*

$$f(x) = \sum_1^\infty 2^{-m} \exp\left(\frac{2\pi i}{2^m}x\right), \quad x < 0 \tag{1}$$

$$f(x) = e^{i\varphi(x)}, \qquad x \geqq 0 . \tag{2}$$

*Then the bounded continuous function* $f(x)$ *has properties* 1, 2, *and* 3 *of Theorem* 1 *in the previous paper.*

*Proof.* That $0 \in sp\ f$ follows from (1) as in § 2 of the previous paper.

To establish property 3, let us show that

$$\frac{1}{T}\int_0^T f(x + a)dx \to 0$$

uniformly in $a$ as $T \to \infty$. If $I$ is any interval of length $T$, denote by $A$ the part of $I$ lying to the left of 0, and by $B$ that part lying to the right. We have, by (1),

$$\frac{1}{T}\int_A f(x)dx = \frac{|A|}{T}\left\{\frac{1}{|A|}\int_A \sum_1^\infty 2^{-m} \exp\left(\frac{2\pi i}{2^m}x\right)dx\right\} .$$

The quantity in brackets is always in absolute value $\leqq 1$, and tends to zero independently of the position of $A$ as $|A| \to \infty$ (this fact belongs to the rudiments of the theory of almost periodic functions, and can here be verified by direct calculation). Since $|A| \leqq T$, we have

$$\frac{1}{T}\int_A f(x)dx \to 0 \text{ independently of the position of } I \text{ as } T \to \infty . \tag{3}$$

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The integral $\int_B f(x)dx$ is bounded for all intervals $B$ of the form $[0, b]$. Indeed, if $b > 1$,

$$\int_0^b f(x)dx = \int_0^1 f(x)dx + \int_1^b f(x)dx \,.$$

Since $\varphi'(x) \geqq 0$ we can, by (2), make the substitution $\varphi(x) = \xi$ in the second integral on the right, getting for it the value

$$\int_1^b e^{i\varphi(x)}dx = \int_{\varphi(1)}^{\varphi(b)} e^{i\xi}\frac{d\xi}{\varphi'(x)} \,.$$

In view of (ii), this last is in absolute value $\leqq 4/\varphi'(1)$ by the second mean value theorem. It follows that $\int_B f(x)dx$ is bounded for *all* intervals $B$ lying to the right of the origin, whence

(4) $\frac{1}{T}\int_B f(x)dx \to 0$ independently of the position of $I$ as $T \to \infty$ .

From (3) and (4) we see that $1/T\int_I f(x)dx$ is small in absolute value for all intervals $I$ of length $T$, if only $T$ is sufficiently large, which is property 3.

It remains to verify property 2. We show that if $0 < X_1 < \cdots < X_M$ and the $A_k$ are complex numbers

$$\sup_{x>0}\left|\sum_{k=1}^{M} A_k e^{i\varphi(x+X_k)}\right| = \sum_1^M |A_k| \,. \tag{5}$$

So as not to lose the reader in details, we do this for the case $M = 2$; it will be clear how to extend the reasoning to any value of $M$.

Let $\varepsilon$ be given, $0 < \varepsilon < \pi/2$, and, choosing a positive determination of the argument, put, for $k = 1, 2, 3, \cdots$

$$a_k = \varphi^{-1}\left(2\pi k + \arg\frac{1}{A_1} - \varepsilon\right) - X_1 \tag{6}$$

$$b_k = \varphi^{-1}\left(2\pi k + \arg\frac{1}{A_1} + \varepsilon\right) - X_1 \,. \tag{7}$$

Clearly $a_k < b_k < a_{k+1}$, $a_k \to \infty$ as $k \to \infty$, and by (ii),

$$b_k - a_k \to 0,\ k \to \infty \,. \tag{8}$$

Also,

$$\mathscr{R}(A_1 e^{i\varphi(x+X_1)}) \geqq (1 - \varepsilon^2)\,|A_1| \quad \text{for} \quad a_k \leqq x \leqq b_k \,. \tag{9}$$

I claim that $\varphi(b_k + X_2) - \varphi(a_k + X_2) \to \infty$ as $k \to \infty$. If $c > 0$, by (ii):

$$\frac{\varphi'(x+c)}{\varphi'(x)} \geqq c\frac{\varphi'(x+c)}{\varphi(x+c)-\varphi(x)} \geqq c\frac{\varphi'(x+c)}{\varphi(x+c)} ,$$

whence

$$\frac{\varphi'(x+c)}{\varphi'(x)} \longrightarrow \infty , x \longrightarrow \infty , \tag{10}$$

in view of (iii). Since $X_2 > X_1$, there is, by (8), a $c > 0$ such that, for all sufficiently large $k$, $c + b_k + X_1 \leqq a_k + X_2$. We thus have, from (6), (7), (ii), and (10):

$$\begin{aligned} \varphi(b_k + X_2) - \varphi(a_k + X_2) &= 2\varepsilon\frac{\varphi(b_k + X_2) - \varphi(a_k + X_2)}{\varphi(b_k + X_1) - \varphi(a_k + X_1)} \\ &\geqq 2\varepsilon\frac{\varphi'(b_k + X_1 + c)}{\varphi'(b_k + X_1)} \longrightarrow \infty \end{aligned}$$

as $k \longrightarrow \infty$, since $b_k \longrightarrow \infty, k \longrightarrow \infty$. This is the desired result which implies, in particular, the existence, for all sufficiently large $k$, of an $x_k \in [a_k, b_k]$ such that

$$\varphi(x_k + X_2) \equiv \arg\frac{1}{A_2} \pmod{2\pi} .$$

For such $x_k$ we have $A_2 e^{i\varphi(x_k+X_2)} = |A_2|$ which, together with (9), yields (5) for the case $M = 2$, since $\varepsilon > 0$ is arbitrary.

REMARK. Suppose $\varphi(x)$ is *even*, and fulfills condition (i), (ii), and (iii) of the theorem. Besides this, let it be twice continuously differentiable, and be such that $\varphi''(x) \geqq C > 0$ (example: $\varphi(x) = e^{x^2}$). Then, if $f(x) = e^{i\varphi(x)}$, $e^{i\lambda x}$ is not, for *any* $\lambda \in sp\ f$, in the weak closure of any bounded subset of $V_f$ (notation as in the preceding paper). (This observation is due to P. Koosis.)

Indeed, the function $f(x)$ clearly has property 2, according to the above work. A glance at the proof of Theorem 1 in the preceding paper now shows that the desired result will certainly follow if we establish, for all real $\lambda$, that

$\frac{1}{T}\int_0^T f(x+X)e^{-i\lambda x}dx \longrightarrow 0$ uniformly in $X$ as $T \longrightarrow \infty$. But by a lemma of Van der Corput ([1], vol I, p. 197),

$$\left|\int_0^T e^{i[\varphi(x+X)-\lambda x]}dx\right| \leqq 12\cdot\left\{\inf_{0\leqq x\leqq T} \varphi''(x+X)\right\}^{-1/2} \leqq 12C^{-1/2}$$

for all $T$, which implies the desired statement.

## REFERENCES

1. P. Koosis, *On the spectral analysis of bounded functions*, Pacific J. Math. **15** (1965),
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