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SOME COMPLEMENTED FUNCTION SPACES IN $C(X)$

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et X and Z be compact Hausdorff spaces, and let P be a linear subspace of $C(X)$ which is isometrically isomorphic to $C(Z)$. In this paper conditions, some necessary and some sufficient, are presented which insure that P is complemented in $C(X)$. For example if X is metrizable, P contains a strictly positive function, and the decomposition induced on X by P is lower semi-continuous then P is complemented in $C(X)$.

D. Amir has shown that not all such spaces P are complemented when X is metrizable ([1], see also R. Arens, [4]). However, R. Arens [4] has constructed a class of subspaces of $C(X)$ which are complemented. In § 2 we present classes of complemented subspaces which extend the class exhibited by R. Arens [Theorem 4, Lemma 5, Theorem 8]. A comparison of these results precedes Theorem 8.

Suppose that X is the Stone-Čech compactification of a locally compact completely regular space Y, Z is a compactification of Y which has first countable remainder, and P is the natural embedding of $C(Z)$ in $C(X)$. In § 3 we show that if P is complemented in $C(X)$, then Y is pseudo-compact. This theorem was proved by J. Conway [6] for the case in which Z is the one point compactification of Y.

By introducing the concept of weakly separating in § 2, we are paralleling the concept of a Choquet boundary. Related results and definitions are found in [22].

1. If A and B are subsets of a topological space, $\operatorname{cl} A$ will denote the closure of A, and $A\text{-}B$ will denote the set of points which are in A but not in B. If E is a normed linear space, $S(E)$ and E^* denote the unit ball in E and the dual of E respectively. If K is a convex subset of a topological vector space, $\operatorname{ext} K$ will represent the set of extreme points of K. If g and h are functions such that the range of g is contained in the domain of h, the composite of g and h will be written $h \circ g$. Finally, if X is a topological space and x is in X, the point evaluation functional associated with x is the linear functional x' defined on $C(X)$ by $x'(f) = f(x)$ for each f in $C(X)$. In this paper $C(X)$ will denote the Banach space of all bounded real-valued continuous functions on X normed with the supremum norm.

2. Let P be a subspace of a normed linear space E. We define $D(P) = \{b \text{ in } S(E^*)\colon b \text{ restricted to } P \text{ is in } \operatorname{ext} S(P^*)\}$. We say that P is *weakly separating* (with respect to E) if P separates the points

of $D(P)$ intersect ext $S(E^*)$, that is, if g and h are distinct points in this intersection, then there is a p in P such that $g(p) \neq h(p)$. Although we have stated the definition for an arbitrary normed linear space, we are mainly interested in the space $E = C(X)$, where X is a compact Hausdorff space. It follows readily from the definition that a subspace P of $C(X)$ is weakly separating if for any two distinct point evaluation functionals x' and y' whose restrictions to P have norm one, there is a p in P such that $|p(x)| \neq |p(y)|$. In particular, a subspace of $C(X)$ which contains the constants and separates the points of X, or a closed ideal in $C(X)$ is weakly separating.

LEMMA 1. *Let P be a subspace of E. The following are equivalent*:

(i) *P separates the members of $D(P)$*

(ii) *P separates the members of $D(P)$ intersect* ext $S(E^*)$

(iii) ext $S(E^*)$ *contains* $D(P)$.

Proof. (iii) *implies* (i). If P does not separate the elements of $D(P)$, then there must exist distinct elements g and h in $D(P)$ such that the restriction of $g - h$ to P is the zero functional. It follows that $b = (1/2)(g + h)$ agrees with g and h on P. Hence b is in $D(P)$ but not in ext $S(E^*)$.

(ii) *implies* (iii). Now suppose that P separates the elements of $D(P)$ intersect ext $S(E^*)$. Let b be a point in $D(P)$. We are to prove that b is in ext $S(E^*)$. Let $K = \{k \text{ in } S(E^*)\colon k \text{ agrees with } b \text{ on } P\}$. Clearly K is a convex set containing b. Also K is closed, and hence compact, in the weak* topology on E^*. By the Krien-Milman theorem, K has extreme points. We will show that ext K is contained in ext $S(E^*)$. Suppose $k = (1/2)(g + h)$ where k is in ext K and g and h are in $S(E^*)$. Thus for each p in P, $1/2h(p) + 1/2g(p) = k(p) = b(p)$. The restrictions of g and h to P both belong to $S(P^*)$, and the restriction of b is in ext $S(P^*)$. Therefore g and h agree with b on P and both must belong to K. Since k was assumed to be an extreme point of K, we have $g = h = k$. We conclude that ext $S(E^*)$ contains ext K. If b is the only point in K, then b must be in ext $S(E^*)$. Otherwise K must contain two distinct extreme points. Clearly P can not separate these two points of $D(P)$ intersect ext $S(E^*)$. This proves that (ii) implies (iii).

Since the fact that (i) implies (ii) is obvious, the proof is complete.

LEMMA 2. *If P is weakly separating in E, then the weak topology on $D(P)$ induced by P is equivalent to the weak topology induced by E.*

Proof. Clearly, the weak topology induced by P is coarser than the one induced by E. To prove the converse, suppose that g_i is a net of functionals in $D(P)$ which converge with respect to the weak topology induced by P to a functional g which is also in $D(P)$. If g_i does not converge to g with respect to the weak topology induced by E, there will exist a subnet which never intersects some neighborhood (in topology induced by E) of g. Since by Alaoglu's theorem $S(E^*)$ is compact, we may assume the existence of a further subset g_j which converges to a functional h distinct from g. Since g_j is a subset of g_i, h must agree with g on P. Since the norm of h is less than or equal to one, h is in $D(P)$. Since P does not distinguish between g and h, the previous lemma contradicts the hypothesis that P is weakly separating. The lemma is proved.

In the following let X be a compact Hausdorff space.

LEMMA 3. *Let P be a weakly separating subspace of $C(X)$. The following are equivalent*:

(i) *There is a projection of norm one of $C(X)$ onto P,*

(ii) *P is isometrically isomorphic to $C(Z)$ for some compact Hausdorff space Z,*

(iii) *There exist a closed subset Y of X such that P is isometrically isomorphic to $C(Y)$ via the restriction mapping.*

Furthermore, if P is weakly separating there can exist at most one projection of norm one of $C(X)$ onto P.

Proof. (i) *implies* (iii). Let L be a projection of norm one of $C(X)$ onto P. If x' is an evaluation functional in $D(P)$, then $x' \circ L$ is a functional in $S(C(X)^*)$ which agrees with x' on P. Since P is weakly separating in $C(X)$, $x' \circ L = x'$. Hence for each f in $C(X)$, Lf agrees with f on $\{x \text{ in } X: x' \text{ is in } D(P)\}$, and therefore on the closure Y of this set. With a simple application of the Tietze Extension Theorem, we see that the restriction map carries P onto $C(Y)$. Furthermore, this restriction mapping does not decrease the norm of points in P. For by Lemma 1 every functional in $D(P)$ can be expressed as either an evaluation functional of a point in Y or as the negative of such a functional, and for p in P, $\|p\| = \sup\{h(p): h \text{ in } D(P)\}$. We have shown that the restriction mapping is an isometric isomorphism of P onto $C(Y)$.

(ii) *implies* (i). Let Z be a compact Hausdorff space, and let L be an isometric isomorphism of P onto $C(Z)$. Let L' denote the adjoint of L. Since L is an isometric isomorphism, L' is an isometric isomorphism of $C(Z)^*$ onto P^*. Furthermore, L' restricted to $\operatorname{ext} S(C(Z)^*)$ is a homeomorphism onto $\operatorname{ext} S(P^*)$ with the weak topologies induced by $C(Z)$ and P respectively. Now for x in $\operatorname{ext} S(P^*)$, let

$H(x)$ be the unique element in ext $S(C(X)^*)$ which agrees with x on P. For z in Z let $E(z)$ denote the evaluation functional of z. Now for f in $C(X)$ consider the function $f \circ H \circ L' \circ E(\cdot)$ defined on Z. By Lemma 2 this function is continuous. The map Q which carries f in $C(X)$ onto $L^{-1}(f \circ H \circ L' \circ E(\cdot))$ is a mapping of norm one of $C(X)$ into P. Furthermore, if p is in P, then $p \circ H \circ L' \circ E(z) = Lp(z)$, for all z in Z. Thus $p \circ H \circ L' \circ E(\cdot) = Lp$, and Q is a projection of $C(X)$ onto P.

It is evident that (iii) implies (ii).

To prove the second part of the lemma, suppose that H and L are two projections from $C(X)$ onto P, both of which have norm one. Let Y be the subset of X constructed in the proof that (i) implies (iii). For any f in $C(X)$, we have shown that Lf, Hf and f all agree on Y. It of course follows that $(H - L)(f)$ vanishes on Y. However, we have shown that the restriction mapping carries P isometrically onto $C(Y)$. Therefore, $(H - L)(f)$ must be the zero function, and $Hf = Lf$ for all f in $C(X)$. This completes the proof.

We will say that a subspace P of $C(X)$ has a *weakly separating quotient* if it has the property that for any two distinct points x and y in X such that $p(x) = -p(y)$ for every p in P, then the evaluation functional of x (or equivalently the evaluational functional of y) restricted to P is not an extreme point of $S(P^*)$.

REMARK. Each of the following properties on a subspace P of $C(X)$ imply that P has a weakly separating quotient:

(i) P is weakly separating in $C(X)$,

(ii) P contains a function which is strictly positive,

(iii) for each p in P, $|p|$ is also in P.

A proof for the above remark is straightforward. In particular, any closed ideal in $C(X)$, or any subspace of $C(X)$ which contains the constants has a weakly separating quotient.

In order to state the next theorem we make a few more definitions. Let X be a Hausdorff space and let M be a partition of X into closed subsets. For x in X let $M(x)$ denote the member of M which contains x. Corresponding to the standard definitions we say that M is *lower semi-continuous* if $\{x$ in X: $M(x)$ intersect U is non-empty$\}$ is an open set in X for every open set U in X.

If P is a linear space of bounded, continuous functions, then the *P-partition* of X is the partition associated with the following equivalence relation R. A couple (x, y) is in R if and only if $p(x) = p(y)$ for every p in P. Now let $K(P) = \cup$ $\{K$ contained in X: K is a member of the P-partition of X, and K contains more than one point of $X\}$. We will say that P has a *lower semi-continuous quotient* if the restriction of the P-partition to cl $K(P)$ is lower semicontinuous.

In the following let X denote a compact Hausdorff space, and let

P be a linear subspace of $(C(X)$ which has a weakly separating quotient.

THEOREM 4. *If there is a projection of norm one of $C(X)$ onto P, then P is isometrically isomorphic to $C(Z)$ for some compact Hausdorff space Z. Conversely, suppose that X is metrizable, and that P has a lower semi-continuous quotient. If P is isometrically isomorphic to $C(Z)$, for some compact Hausdorff space Z, then there is a projection of $C(X)$ onto P which has norm less than or equal three.*

Proof. Let M denote the P-partition of X. Let X/M have the quotient topology, and let $M(\cdot)$ denote the natural mapping of X onto X/M. We observe that X/M is a compact Hausdorff space. Now let Q denote the linear subspace of $C(X)$ consisting of all functions that are constant on each closed subset of X which is a member of M. One can verify that P is contained in Q, and that the mapping which carries q in Q onto the function $q \circ M^{-1}(\ \ )$ in $C(X/M)$ is an isometric isomorphism of Q onto $C(X/M)$. The image P' of P under this mapping is a weakly separating subspace of $C(X/M)$ since P has a weakly separating quotient. If there is a projection of norm one from $C(X)$ onto P, then there certainly is a projection of norm one from $C(X/M)$ onto P. By the preceding lemma, we conclude that P', and hence P, is isometrically isomorphic to $C(Z)$ for some compact Hausdorff space Z.

To prove the second part of the theorem, we assume that X is metrizable, P has a lower semi-continuous quotient, and that there is a compact Hausdorff space Z such that P is isometrically isomorphic to $C(Z)$. We maintain the same notation used directly above. Since P' is weakly separating in $C(X/M)$, and P is isometrically isomorphic to $C(Z)$, it follows from the preceding lemma that there is a projection of norm one from Q onto P. To complete the proof it will suffice to show that there is a projection from $C(X)$ onto Q which has norm less than or equal to three. We will prove a stronger result.

Let Y be a metric space. Let K be a partition of Y such that every member of K is a complete subset of Y. A member of K will be called a *plural set* if it contains two distinct points of Y. Let the restriction K' of K to the subset of Y,

$$B = \mathrm{cl} \cup \{A \text{ contained in } Y\colon A \text{ a plural set in } K\}$$

be lower semi-continuous. Assume also that B/K' is paracompact. Let Q denote the subspace of $C(Y)$ consisting of the functions which are constant on each member of K. We recall that by the notation we adopted, $C(Y)$ is the Banach space of all bounded continuous functions on Y. The following lemma establishes the theorem.

LEMMA 5. *There is a projection of $C(Y)$ onto Q which has norm less than or equal to three.*

Proof. In the usual manner we can embed B into the unit ball of $C(B)^*$. With the weak topology on $C(B)^*$ induced by $C(B)$, $C(B)^*$ is a locally convex space, B is embedded onto a homeomorphic image of itself, say B', and the closed convex hull of compact subsets of B' are again compact. Let s denote the composite of the quotient mapping of B onto B/K' with the homeomorphism, h, between B and B'.

We now show that s^{-1} is a lower semi-continuous function carrying points in B/K' onto closed subsets of B'. Let U be an open set in B'. Let

$$W = \{y \text{ in } B/K' \colon s^{-1}(y) \text{ intersect } U \text{ is not empty}\} .$$

To show that s^{-1} is lower semi-continuous we must show that W is open in B/K'. We note that $W = s(U)$. Now since K' is lower semi-continuous and $h^{-1} \circ s^{-1} \circ s \circ h(\cdot)$ carries a point b in B onto the member of K' which contains b, the set

$$V = \{b \text{ in } B \colon h^{-1} \circ s^{-1} \circ s \circ h(b) \text{ intersect } h^{-1}(U) \text{ is not empty}\}$$

is open in B. Hence $h(V) = \{b' \text{ in } B \colon h^{-1} \circ s^{-1} \circ s(b') \text{ intersect } h^{-1}(U)$ is not empty$\}$ is open in B'. Since this last set is $s^{-1} \circ s(U)$, $s^{-1} \circ s(U)$ is open. Since B/K' has the quotient topology induced by s, this implies that $s(U)$—and hence W—is open in B/K'. Therefore s^{-1} is lower semi-continuous.

Now since B/K' is paracompact, and since there is a metric on B' (which induces an equivalent topology for B') for which the set $s^{-1}(y)$ is complete for each y in B/K', we have satisfied the hypothesis for a selection theorem proved by E. Michael [20]. This theorem proves the existence of a continuous function t which carries B/K' into $C(B)^*$, and has property that $t(y)$ is contained in the closed convex hull of $s^{-1}(y)$ for each y in B/K'.

We now define a projection from $C(B)$ onto Q' the subspace of functions in $C(B)$ which are constant on members of K'. For f in $C(B)$, let Lf denote the function such that for each b in B,

$$(Lf)(b) = [t(s \circ h(b))](f) .$$

Since t is continuous on B/K', Lf is a continuous function. Since $t(s \circ h(b))$ is in the closed convex hull of $s^{-1} \circ s \circ h(b)$, the norm of $t(s \circ h(b))$ does not exceed one. Thus the maximum of Lf over B does not exceed the maximum of f over B. Finally, one can verify that if q is in Q', $Lq = q$, and that for each f in $C(B)$, Lf is in Q'. We have shown that L is a projection of norm one of $C(B)$ onto Q'.

Since Y is a metric space, there is an operator E of norm one from $C(B)$ into $C(Y)$ such that $R \circ Ef = f$ for every f in $C(B)$. Here R denotes the operator which assigns to each function in $C(Y)$ its restriction to B (R. Arens [3], also Dugundji [8]). Following a construction due to Arens [4], we define an operator J by $Jf = f + E(LRf - Rf)$. The proof of the lemma is completed by verifying that J is a projection of $C(Y)$ onto Q which has norm no greater than three.

In the following corollaries let X denote a compact Hausdorff space.

COROLLARY 6. *Let P be a finite dimensional subspace of $C(X)$ which has a weakly separating quotient. There is a projection of norm one from $C(X)$ onto P if and only if P has a basis $\{p_i\}_{i=1}^n$ such that* $\| \sum_{i=1}^n c_i p_i \| = \max | c_i |$.

COROLLARY 7. *$C(X)$ contains a weakly separating subspace of co-dimension n which has a projection of norm one if and only if X contains n isolated points.*

Proof. To prove the necessity of the condition, let L be a projection of norm one of $C(X)$ onto a weakly separating subspace P of co-dimension n in $C(X)$. Define $Y = \mathrm{cl}\,\{x \text{ in } X\colon x' \circ L = x'\}$. We will show that $X - Y$ contains precisely n points. Since $X - Y$ is open, these points will be isolated. We observe that the range, Q, of $I - L$ has dimension n, and that if q is in Q, then q vanishes on Y. Since the functions in Q take all their nonzero values on $X - Y$, $X - Y$ must contain at least n points. If $X - Y$ contained $n + 1$ points, there would exist $n + 1$ open sets U_i in $X - Y$, and corresponding functions f_i of norm one which vanish off U_i. These functions span an $n + 1$ dimensional subspace of $C(X)$; hence there is a nonzero function f in this span that is also in P. But f vanishes on Y. By Lemma 3, the restriction map is an isometry of P onto $C(Y)$. Hence we arrive at the contradiction that f is the zero function.

If X contains n isolated points, the space of all functions in $C(X)$ which vanish on these n points is a weakly separating subspace of $C(X)$ (sinces this space is an ideal) of co-dimension n in $C(X)$. It is also clear there is a projection of norm one from $C(X)$ onto this subspace. The proof is completed.

REMARK. R. Arens [4] has constructed an example of two compact metric spaces X and Z such that $C(X)$ contains an isometric isomorphic-copy of $C(Z)$ which has a weakly separating quotient, but which is not complemented in $C(X)$. Hence the assumption that P has a lower semi-continuous quotient cannot be simply omitted from the theorem, (Also see Amir [1]).

The preceding theorem and lemma should be compared to Theorem 2.2 in (R. Arens [4]). Using the notation preceding the lemma, Professor Arens proved that under the following conditions there will exist a projection of norm less than or equal to three of $C(Y)$ onto Q:

(i) K is a partition of Y into closed subsets

(ii) Y and Y/K are metrizable

(iii) the quotient map of Y onto Y/K is upper semi-continuous[1]

(iv) if $\{x_i\}$ is a sequence in Y such that each x_i belongs to a distinct plural set in K, then a member of K which contains a limit point of $\{x_i\}$ is a singleton.

Apropos to property (ii), A. H. Stone has proved ([23]) that a metrizable space is paracompact. Property (iv) above implies that K' is lower semi-continuous. In the special case that Y is a complete metric space, the preceding lemma contains the above theorem of Arens. If Y is compact, the previous theorem includes both of these results.

In the following, let Y be a metrizable space, and K a partition of Y satisfying properties (i), (iii), and (iv) above. For each K_i in K let P_i be a complemented subspace of $C(K_i)$ which contains the constants. Let L_i denote a projection of $C(K_i)$ onto P_i. We assume that $m = \sup\{\| L_i \|\} < \infty$. Finally, let Q denote the subspace of $C(Y)$ consisting of all functions q such that the restriction of q to K_i is a function in P_i.

THEOREM 8. *There is a projection of $C(Y)$ onto Q which has norm less than or equal to $2 + m$.*

Proof. For a set Z let $B(Z)$ denote the space of bounded functions on Z. Let $D = \cup\{K_i$ contained in Y: K_i is a plural set in $K\}$. Let R and R_i denote the restriction map of $B(Y)$ onte $B(\mathrm{cl}\, D)$ and of $B(Y)$ onto $B(K_i)$ respectively (K_i in K). Let E denote a linear mapping of $C(\mathrm{cl}\, D)$ into $C(Y)$ such that E has norm one, and $R \circ E$ is the identity mapping on $C(\mathrm{cl}\, D)$. Let H be the linear mapping of $C(Y)$ into $B(\mathrm{cl}\, D)$ such that $R_i \circ H = L_i \circ R_i$ for all K_i in K. Let I denote the identity on $C(Y)$, and let $L = I + E \circ R(H - I)$. The proof consists of establishing that L is the desired projection. The variation of a function f defined on a set Z is $\mathrm{var}\,(f) = \max\limits_{z \text{ in } Z} f(z) - \min\limits_{z \text{ in } Z} f(z)$.

We proceed by proving four assertions, the last of which establishes the theorem.

Assertion 1. If x_i is in K_i, K_i is in K, y is not in D and x_i con-

[1] Professor Arens has communicated that the assumption that the quotient mapping be upper semi-continuous had been inadvertently omitted from the statement of his theorem.

verges to y, then $\operatorname{var}(R_i f)$ converges to zero for each f in $C(Y)$.

Assertion 2. $\|L_i \circ R_i f - R_i f\| \leqq 1/2(1 + m) \operatorname{var}(R_i f)$.

Assertion 3. If f is in $C(Y)$, Hf is in $C(\operatorname{cl} D)$.

Assertion 4. The operator L is a projection from $C(Y)$ onto Q of norm at most $2 + m$.

If Assertion 1 is false it will be possible be find points z_i in K_i and a function f in $C(Y)$ such that for some r greater than zero, $f(x_i) - f(z_i)$ is greater than r. Since f is continuous, we may assume that there is a neighborhood N of y such that z_i does not belong to N. Put $Z = \{z_i\}$. Since the quotient map q of Y onto Y/K is, by hypothesis, closed $q(\operatorname{cl} Z)$ is closed in Y/K. But $q(x_i) = q(z_i)$ is in $q(\operatorname{cl} Z)$, and $q(x_i)$ converges to $q(y)$ by the continuity of q. Thus $q(y) = \{y\}$ is in $q(\operatorname{cl} Z)$, and $\{y\} = q(z)$ for some z in $\operatorname{cl} Z$. But $\operatorname{cl} Z$ is contained in $Y - N$ so $z \neq y$. This contradicts the assumption that y is not in D.

To prove the second assertion, let $c = 1/2 \operatorname{var}(R_i f)$. Since 1 is in P_i, $L_i \circ R_i 1 = 1$. Hence

$$\|L_i \circ R_i f - R_i f\| = \|L_i \circ R_i(f - c) - R_i(f - c)\| \leqq \|L_i - I\| \cdot \|R_i(f - c)\| \leqq (m + 1)(1/2) \operatorname{var}(R_i f) .$$

To prove Assertion 3 let y be a point in $\operatorname{cl} D$. We distinguish two cases. Case 1, y is in D. Let y be in the plural set K_i of the partition K. From the assumption of property (iv) it follows that there is an open set U containing K_i which meets no other plural set in K. Now let f be in $C(Y)$ and let N be a neighborhood of $Hf(y)$. Let V be a neighborhood of y such that $(L_i \circ R_i f)(V \cap K_i)$ is contained in N. Put $W = V \cap U$ and let x be an arbitrary point in W intersect $\operatorname{cl} D$. Then x is in U, and x is in the closed set K_i. This shows that $W \cap \operatorname{cl} D$ is contained in $K_i \cap V$. Hence on $W \cap \operatorname{cl} D$, $Hf = L_i \circ R_i f$. Thus $Hf(W \cap \operatorname{cl} D)$ is contained in $L_i \circ R_i f(K_i \cap V)$ which in turn is contained in N.

Case 2, y is not in D. In this case $\{y\}$ is in K, and $Hf(y) = f(y)$, since each P_i contains the constant functions. Let x_i converge to y. Then

$$|Hf(x_i) - Hf(y)| \leqq |Hf(x_i) - f(x_i)| + |f(x_i) - f(y)| .$$

It is clear that $f(x_i)$ converges to $f(y)$. For the other term we use Assertions 1 and 2 above to write, with x_i in K_i (and K_i in K),

$$
\begin{aligned}
| Hf(x_i) - f(x_i) | &\leqq | L_i \circ R_i f(x_i) - R_i f(x_i) | \\
&\leqq (1/2)(m + 1) \operatorname{var} (R_i f) .
\end{aligned}
$$

Since this last term converges to zero, Hf is continuous at y.

To prove Assertion 4, we first observe that linearity and bound for L are obvious. If f is in $C(Y)$ we must show that Lf is in Q. Indeed,

$$R \circ L = R + R \circ H - R = R \circ H .$$

Hence

$$R_i \circ L = R_i \circ R \circ L = R_i \circ R \circ H = L_i \circ R_i$$

for each plural set K_i in K. Thus $R_i \circ Lf$ is in P_i for each plural set K_i in K. If K_i is a member of K which is not a plural set then, $R_i \circ Lf$ is in P_i trivially since P_i contains the constants.

Now we must show that if f is in Q then $Lf = f$. Since $R_i f$ is in P_i for all K_i in K, $R_i \circ Hf = L_i \circ R_i f = R_i f$. Thus $R \circ Hf = Rf$, and $Lf = f + E(Rf - Rf) = f$. This completes the proof of the theorem

REMARK. The assumption that Y is metrizable was used only to guarantee the existence of the linear mapping E. If we drop the hypothesis that Y is metrizable and assume outright the existence of a bounded linear mapping E from $C(\mathrm{cl}\, D)$ into $C(Y)$ such that $R \circ E$ is the identity on $C(\mathrm{cl}\, D)$, then the same proof establishes the existence of a projection from $C(Y)$ onto Q which has norm less than or equal to $1 + (m + 1) \| E \|$.

COROLLARY 9. *Let Y, K, K_i, P_i, and Q be as in the theorem. If each P_i has dimension less than n, then there is a projection of norm at most $n + 1$ from $C(Y)$ onto Q.*

3. Let X be a locally compact, Hausdorff space. A compactification of X is a compact Hausdorff space that contains X (a homeomorphic image of X) as a dense subspace. The Stone-Čech compactification of X will be denoted by βX, and the one-point compactification will be denoted by pX.

If K is an arbitrary compactification of X, the linear mapping which carries a function in $C(K)$ onto the unique function in $C(\beta X)$ which agrees with it on X, is an isometric isomorphism of $C(K)$ into $C(\beta X)$. We will therefore assume that $C(\beta X)$ contains $C(K)$.

If Y is a closed subset of a compact Hausdorff space K, I_Y will denote the ideal of functions in $C(K)$ which vanish on Y. Let N denote the non-negative integers with the discrete topology. If K is a compactification of X, the remainder of K (with respect to X) is the topological space $K - X$ equipped with the relative topology from K. In accordance with the usual terminology let $(m) = C(\beta N)$, $(c) = C(pN)$, and $(c_0) = I_{pN-N} = I_{\beta N-N}$, where the ideals are interpreted as subspaces of $C(pN)$ and $C(\beta N)$ respectively.

THEOREM 10. *Let K be a compactification of X which has a first countable remainder. If there is a bounded linear mapping of $C(\beta X)$ into $C(K)$ which acts as the identity on $I_{\beta X-X}$, then X is pseudocompact.*

We first will prove the following lemma.

LEMMA 11. *Let M be a compactification of N which has a first countable remainder. There does not exist a bounded linear mapping of (m) onto any subspace of $C(M)$ which contains (c_0).*

Proof of lemma. Since N is both locally compact and the union of a countable family of compact sets, $M - N$ is a compact set which is the intersection of a countable family U of open sets in M. Let x be a point in $M - N$. Let V be a countable family of open sets in M whose intersections with $M - N$ form a basis for the neighborhood system for x in $M - N$. Let W be the countable family of open sets in M of the form u intersect v, where u is in U and v is in V. It is easy to see that the intersection of the members of W is the singleton containing x. A compactness argument shows that W is in fact a basis for the neighborhood system for x in M. Since N is first countable we have established that M is first countable. Hence M is sequentially compact.

There is a sequence of points in N, say J, which converges to some point k in M. Now suppose B is a subspace of $C(M)$ which contains (c_0). The restriction of functions in B to J union $\{k\}$ carries B onto a Banach space which is either isometrically isomorphic to (c) or to (c_0). In the former case since (c_0) is complemented in (c), there is a bounded linear mapping of B onto (c_0). In either case if there is a bounded linear mapping of (m) onto B, there is a bounded linear mapping, L, of (m) onto (c_0). But no such mapping can exist. For since (c_0) is a separable Banach space and βN is extremally disconnected, L must be weakly compact (Grothendieck [14], p. 168, Cor. 1). Now an application of the open mapping theorem implies the false assertion that (c_0) is reflexive. This completes the proof of the lemma.

Proof of theorem. If X is not pseudocompact there is countable family of disjoint open sets V_i in X such that $\mathrm{cl} \cup \{V_i\} = \cup \{\mathrm{cl}\, V_i\}$. For each i let U_i be an open set such that $\mathrm{cl}\, U_i \subseteqq V_i$, let u_i be in U_i, and let f_i be a continuous function which vanishes off U_i and attains its norm of one at u_i. For a bounded sequence $x = (x_i, x_2 \cdots)$ in (m), let Ax be the unique function in $C(\beta X)$ which agrees with $\sum_{i=1}^{\infty} x_i f_i$ on X. The mapping A is an isometric isomorphism of (m) onto the range of A. Let L be the hypothesized mapping of the theorem, and let J carry a function in $C(\beta X)$ onto its restriction to $\mathrm{cl}\{u_i\}$. Since $\mathrm{cl}\{u_i\} - \{u_i\}$ is contained in $K - X$, $\mathrm{cl}\{u_i\}$ is homeomorphic to a compactification M of N which has first countable remainder. Let G be the isometric isomorphism of $C(\mathrm{cl}\{u_i\})$ onto $C(M)$ induced by this homeomorphism. The proof is completed by verifying that $G \circ J \circ L \circ A$ is a bounded linear mapping of (m) onto a subspace of $C(M)$ which contains (c_0).

The case in which K is the one-point compactification of X was first proved by J. Conway ([6]). Examples to show that pseudocompactness of X is not sufficient to guarantee the existence of a projection from $C(\beta X)$ onto $I_{\beta X - X}$ have been constructed by J. Conway ([6]) and by A. Pełczýnski and V. N. Sudakov ([21]).

COROLLARY 12. *Let X be an extremally disconnected, compact, Hausdorff space, and let P be a subspace of $C(X)$ which contains the constants and separates the points of X. If P is isometrically isomorphic to $C(Z)$ for some compact Hausdorff space Z, then the Šilov boundary of P is an extremally disconnected subset of X which has a pseudo-compact complement.*

Proof. Under the hypothesis of the corollary, the Šilov boundary of P is the set Y of Lemma 3. To show that Y is extremally disconnected, we intend to apply a theorem due to Nachbin (Trans. AMS, 68 (1950), 28–46, 1950), Goodner ([13]), Kelley ([11]) and others. A Banach space B is called injective if every Banach space which contains an isometric isomorphic copy B' of B, admits a projection of norm one onto B'. The theorem we wish to apply states that a Banach space is injective if and only if it is isometrically isomorphic to $C(Z)$, for a compact, extremally disconnected, Hausdorff space Z. Now $C(X)$ is injective and from Lemma 3 there is a projection of norm one from $C(X)$ onto P. From this it can be shown that $C(Y)$ is injective, and hence Y is extremally disconnected.

From Lemma 3 it follows that I_Y is complemented in $C(X)$. Let $G = X - Y$. Since $\mathrm{cl}\, G$ is open in X, $I_{\mathrm{cl}\, G - G}$ is complemented in $C(\mathrm{cl}\, G)$. Since $\mathrm{cl}\, G$ is extremally disconnected, it is the Stone-Čech compactification of G ([10], p. 69, Prob. 6M2). By the theorem, G is pseudocompact (in this case K is the one-point compactification of G), and the corollary is proved.

COROLLARY 13. *If X is a locally compact space such that βX has a first countable remainder, then X is pseudocompact.*

REMARK. Relevant to the last corollary, we observe that if Z is any compact Hausdorff space, there is a pseudocompact, locally compact space X such that $\beta X - X$ is homeomorphic to Z. For let y be a nonisolated point in βN and let $X = (\beta N - \{y\}) \times Z$. From results in ([11]) and ([10], 6M3) we have that X is pseudocompact, and $\beta X = \beta N \times Z$.

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