PACIFIC JOURNAL OF MATHEMATICS
Vol. 24, No. 3, 1968

A NOTE ON CERTAIN BIORTHOGONAL POLYNOMIALS

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Konhauser has introduced two polynomial sets $\{Y_n^c(x; k)\}$, $\{Z_n^c(x; k)\}$ that are biorthogonal with respect to the weight function $e^{-x}x^c$ over the interval $(0, \infty)$. An explicit expression was obtained for $Z_n^c(x; k)$ but not for $Y_n^c(x; k)$. An explicit polynomial expression for $Y_n^c(x; k)$ is given in the present paper.

1. Konhauser [2] has discussed two sets of polynomials $Y_n^c(x; k)$, $Z_n^c(x; k)$, $n = 0, 1, \cdots, k = 1, 2, 3, \cdots, c > -1$; $Y_n^c(x; k)$ is a polynomial in x while $Z_n^c(x; k)$ is a polynomial in x^k. Moreover

$$(1) \qquad \int_0^\infty e^{-x}x^c Y_n^c(x; k)x^{ki}dx = \begin{cases} 0 & (0 \leqq i < n) \\ \neq 0 & (i = n) \end{cases}$$

and

$$(2) \qquad \int_0^\infty e^{-x}x^c Z_n^c(x; k)x^{i}dx = \begin{cases} 0 & (0 \leqq i < n) \\ \neq 0 & (i = n)\ . \end{cases}$$

For $k = 1$, conditions (1) and (2) reduce to the orthogonality conditions satisfied by the Laguerre polynomials $L_n^c(x)$.

It follows from (1) and (2) that

$$(3) \qquad \int_0^\infty e^{-x}x^c Y_i^c(x; k)Z_j^c(x; k)dx = \begin{cases} 0 & (i \neq j) \\ \neq 0 & (i = j)\ . \end{cases}$$

The polynomial sets $\{Y_n^c(x; k)\}, \{Z_n^c(x; k)\}$ are accordingly said to be biorthogonal with respect to the weight function $e^{-x}x^c$ over the interval $(0, \infty)$.

Konhauser showed that

$$(4) \qquad Z_n^c(x; k) = \frac{\Gamma(kn + c + 1)}{n!} \sum_{j=0}^n (-1)^j \binom{n}{j} \frac{x^{kj}}{\Gamma(kj + c + 1)}$$

As for $Y_n^c(x; k)$, he showed that

$$(5) \qquad \begin{aligned} Y_n^c(x; k) &= \frac{k}{2i} \int_C \frac{e^{-xt}(t + 1)^{c+kn}}{[(t + 1)^k - 1]^{n+1}}\, dt \\ &= \frac{k}{n!} \frac{\partial^n}{\partial t^n} \left\{ \frac{e^{-xt}(t + 1)^{c+kn}t^{n+1}}{[(t + 1)^{k+1} - 1]^{n+1}} \right\}_{t=0} . \end{aligned}$$

In the integral in (5), C may be taken as a small circle about the origin in the t-plane.

In the present note we give a generating function and an explicit polynomial expression for the polynomial $Y_n^c(x; k)$. Moreover we show that $Y_n^c(x; k)$ can be identified with a polynomial studied recently by S. K. Chatterjea [1].

2. We apply the Lagrange expansion in the form [4, p. 125]

$$\frac{f(t)}{1 - w\phi'(t)} = \sum_{n=0}^{\infty} \frac{w^n}{n!} \left\{\frac{d^n}{dt^n} [f(t)(\phi(t))^n]\right\}_{t=0}, \tag{6}$$

where

$$w = \frac{t}{\phi(t)}, \qquad \phi(t) = a_0 + a_1 t + \cdots \qquad (a_0 \neq 0).$$

Take

$$f(t) = \frac{e^{-xt}(t+1)^c t}{(t+1)^k - 1}, \qquad \phi(t) = \frac{(t+1)^k t}{(t+1)^k - 1}.$$

Then we have

$$1 - w\phi'(t) = \frac{kt}{(t+1)(t+1)^k - 1},$$

so that

$$\frac{f(t)}{1 - w\phi'(t)} = e^{-xt}(t+1)^{c+1}.$$

Thus, by (5) and (6), we have

$$e^{-xt}(t+1)^{c+1} = \sum_{n=0}^{\infty} Y_n^c(x; k)\left(\frac{t}{\phi(t)}\right)^n. \tag{7}$$

If we put

$$w = \frac{t}{\phi(t)} = \frac{(t+1)^k - 1}{(t+1)^k} = 1 - (t+1)^{-k},$$

then

$$t = (1-w)^{-1/k} - 1$$

and (7) becomes

$$(1-w)^{-(c+1)/k} \exp\{-x[(1-w)^{-1/k} - 1]\} = \sum_{n=0}^{\infty} Y_n^c(x; k) w^n. \tag{8}$$

In the next place, we have

$$
\begin{aligned}
(1-w)^{-(c+1)/k} & \exp\{-x[(1-w)^{-1/k}-1]\} \\
&= (1-w)^{-(c+1)/k} \sum_{r=0}^{\infty} \frac{x^r}{r!} [(1-w)^{-1/k}-1]^r \\
&= \sum_{r=0}^{\infty} \frac{x^r}{r!} \sum_{s=0}^{r} (-1)^s \binom{r}{s} (1-w)^{-(s+c+1)/k} \\
&= \sum_{r=0}^{\infty} \frac{x^r}{r!} \sum_{s=0}^{r} (-1)^s \binom{r}{s} \sum_{n=0}^{\infty} \frac{((s+c+1)/k)_n}{n!} w^n \\
&= \sum_{n=0}^{\infty} \frac{w^n}{n!} \sum_{r=0}^{n} \frac{x^r}{r!} \sum_{s=0}^{r} (-1)^s \binom{r}{s} \left(\frac{s+c+1}{k}\right)_n ,
\end{aligned}
$$

where

$$(a)_n = a(a+1)\cdots(a+n-1), \qquad (a)_0 = 1 .$$

It therefore follows from (8) that

$$Y_n^c(x;k) = \frac{1}{n!} \sum_{r=0}^{n} \frac{x^r}{r!} \sum_{s=0}^{r} (-1)^s \binom{r}{s} \left(\frac{s+c+1}{k}\right)_n . \tag{9}$$

3. Chatterjea [1] has defined the polynomial

$$T_{k,n}^{(\alpha)}(x) - \frac{1}{n!} x^{-\alpha} e^{x^k} D^n(e^{\alpha+n} e^{-x^k}) \tag{10}$$

with $k = 1, 2, 3, \cdots$. The case $\alpha = 0$ had been discussed by Palas [3]. Chatterjea showed that (10) implies

$$T_{k,n}^{(\alpha)}(x) = \sum_{r=0}^{\infty} \frac{x^{kr}}{r!} \sum_{s=0}^{r} (-1)^s \binom{r}{s} \binom{\alpha+n+ks}{n} . \tag{11}$$

He also obtained operational formulas and a generating function for $T_{k,n}^{(\alpha)}(x)$. The assumption that k is a positive integer is not used in deriving (11).

If we replace k by k^{-1} and α by $k^{-1}\alpha$, (10) becomes

$$T_{k^{-1},k}^{(k^{-1}\alpha)}(x) = \sum_{r=0}^{n} \frac{x^{kr}}{r!} \sum_{s=0}^{r} (-1)^s \binom{r}{s} \binom{k^{-1}(\alpha+s)+n}{n} .$$

On the other hand, since

$$\frac{1}{n!}\left(\frac{s+c+1}{k}\right)_n = \binom{k^{-1}(s+c+1)+n-1}{n} ,$$

(9) gives

$$Y_n^{c+k-1}(x^k;k) = \sum_{r=0}^{n} \frac{x^{kr}}{r!} \sum_{s=0}^{r} (-1)^s \binom{r}{s} \binom{k^{-1}(s+c)+n}{n} .$$

It follows at once that

$$Y_n^{c+k-1}(x^k;k) = T_{k^{-1},n}^{(k^{-1}c)}(x) , \tag{12}$$

or, if we prefer,

$$Y_n^{k\alpha+k-1}(x^k;k) = T_{k-1,n}^{(\alpha-1)}(x)\,. \tag{13}$$

4. It may be of interest to point out that a formula equivalent to (9) can be obtained without the use of the Lagrange expansion. In the integral representation (5), put

$$t = (1+u)^{1/k} - 1\,.$$

Then (5) becomes

$$Y_n^c(x;k) = \frac{1}{2\pi i}\int_C \frac{\exp\{-x[(1-u)^{1/k}-1]\}(1+u)^{k^{-1}}(c+1)+n-1}{u^{n+1}}\,du\,,$$

where C denotes a small circle about the origin in the u-plane. The numerator of the integral is equal to

$$\begin{aligned}
&\sum_{r=0}^{\infty}\frac{x^r}{r!}\sum_{s=0}^{r}(-1)^s\binom{r}{s}(1+u)^{k^{-1}(c+s+1)+n-1}\\
&\qquad = \sum_{m=0}^{\infty}u^m\sum_{r=0}^{m}\frac{x^r}{r!}\sum_{s=0}^{r}(-1)^r\binom{r}{s}\binom{k^{-1}(c+s+1)+n-1}{m}\,.
\end{aligned}$$

Taking $m = n$, we therefore get

$$Y_n^c(x;k) = \sum_{r=0}^{n}\frac{x^r}{r!}\sum_{s=0}^{r}(-1)^r\binom{r}{s}\binom{k^{-1}(c+s+1)+n-1}{n}\,. \tag{14}$$

Since

$$\binom{c+n-1}{n} = \frac{(c)_n}{n!}\,,$$

it is evident that (14) is identical with (9).

5. Making use of the explicit formulas (4) and (9), we can give a rather brief proof of (3). Indeed we have

$$\begin{aligned}
J_{n,m} &= \int_0^\infty e^{-x}x^c Z_n^c(x;k)Y_m^c(x;k)dx\\
&= \frac{\Gamma(kn+c+1)}{n!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{1}{\Gamma(kj+c+1)}\\
&\quad\cdot\frac{1}{m!}\sum_{r=0}^{m}\frac{1}{r!}\sum_{s=0}^{r}(-1)^s\binom{r}{s}\left(\frac{s+c+1}{k}\right)_m\cdot\int_0^\infty e^{-x}x^{kj+c+r}dx\\
&= \frac{\Gamma(kn+c+1)}{n!\,m!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\\
&\quad\cdot\sum_{r=0}^{m}\sum_{s=0}^{r}(-1)^s\binom{r}{s}\left(\frac{s+c+1}{k}\right)_m\binom{kj+c+r}{r}\,.
\end{aligned}$$

If $f(x)$ is a polynomial of degree m, it is familiar that

$$f(x) = \sum_{r=0}^{m} \binom{x}{r} \Delta^r f(0) ,$$

where

$$\Delta^r f(0) = \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} f(s) .$$

In particular, for

$$f(x) = \left(\frac{x+c+1}{k}\right)_m ,$$

we have

$$\begin{aligned} \left(\frac{x+c+1}{k}\right)_m &= \sum_{r=0}^{m} \binom{x}{r} \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} \left(\frac{s+c+1}{k}\right)_m \\ &= \sum_{r=0}^{m} \binom{+x+r-1}{r} \sum_{s-r}^{n} (-1)^{s} \binom{r}{s} \left(\frac{s+c+1}{k}\right)_m . \end{aligned}$$

For $x = -kj - c - 1$ this reduces to

$$(-j)_m = \sum_{r=0}^{m} \binom{kj+c+r}{r} \sum_{s=0}^{r} (-1)^{s} \binom{r}{s} \left(\frac{s+c+1}{k}\right)_m .$$

Thus

$$\begin{aligned} J_{n,m} &= \frac{\Gamma(kn+c+1)}{n!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{(-j)_m}{m!} \\ &= (-1)^m \frac{\Gamma(kn+c+1)}{n!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \binom{j}{m} . \end{aligned}$$

Since

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} \binom{j}{m} = \binom{n}{m} \sum_{j=m}^{n} (-1)^j \binom{n-m}{j-m} = (-1)^m \binom{n}{m} (1-1)^{n-m}$$

it is evident that

$$J_{n,m} = \frac{\Gamma(kn+c+1)}{n!} \delta_{nm} \tag{15}$$

in agreement with (3). In particular

$$J_{n,n} = \frac{\Gamma(kn+c+1)}{n!}$$

as proved in [2].

A little more generally, we have

$$\begin{aligned}J'_{n,m} &= \int_0^\infty e^{-x}x_c Z_n^c(x;k)Y_n^{c'}(x;k)dx \\ &= \frac{\Gamma(kn+c+1)}{n!\,m!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\left(-j-\frac{c-c'}{k}\right)_m \\ &= (-1)^m\frac{\Gamma(kn+c+1)}{n!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\binom{j+a}{m},\end{aligned}$$

where $a = (c - c')/k$. It follows that

$$J'_{n,m} = \begin{cases} 0 & (n > m)\,, \\ (-1)^{n+m}\dfrac{\Gamma(kn+c+1)}{n!}\dbinom{a}{m-n} & (n \leqq m)\,. \end{cases} \tag{16}$$

Clearly (16) includes (15).

REFERENCES

1. S. K. Chatterjea, *A generalization of Laguerre polynomials*, Collectanea, Mathematica **15** (1963), 285-292.
2. J. D. E. Konhauser, *Biorthogonal polynomials suggested by the Laguerre polynomials*, Pacific J. Math. **21** (1967), 303-314.
3. F. J. Palas, *A Rodrigues formula*, Amer. Math. Monthly **66** (1959), 402-404.
4. G. Pólya and G. Szegö, *Aufgaben und Lehrsatze aus der Analysis*, Vol. 1, Berlin, 1925.

Received May 6, 1966. Supported in part by NSF Grant GP-5174.