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NOTE ON SOME SPECTRAL INEQUALITIES OF C. R. PUTNAM

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It is shown that if A is any operator in Hilbert space and $\lambda = re^{i\theta}$ is in the approximate point spectrum of A, then

$$\min A^*A \leqq (\max J_\theta)^2$$

and

$$|r - \max J_\theta| \leqq [(\max J_\theta)^2 - \min A^*A]^{1/2} ,$$

where

$$J_\theta = (1/2)(Ae^{-i\theta} + A^*e^{i\theta}) .$$

Several corollaries are deduced for arbitrary operators, generalizing results of C. R. Putnam on semi-normal operators.

We employ the notations in Putnam's paper [3]. In particular if A is any operator (bounded linear, in a Hilbert space) and θ is a real number, $J_\theta = \mathrm{Re}\,(Ae^{-i\theta}) = (1/2)(Ae^{-i\theta} + A^*e^{i\theta})$. We write $\sigma(A)$ and $\pi(A)$ for the spectrum and approximate point spectrum of A, and $(x \mid y)$ for the inner product of vectors.

The following result extracts the essentials of the proof of Theorem 1 in Putnam's paper:

THEOREM. *If A is any operator and $\lambda \in \pi(A)$, $\lambda = re^{i\theta}$ $(r \geqq 0)$, then*

$$\max J_\theta \geqq r \geqq (\min A^*A)^{1/2} , \tag{1}$$

$$\max J_\theta - r \leqq [(\max J_\theta)^2 - \min A^*A]^{1/2} . \tag{2}$$

Proof. Let x_n be a sequence of unit vectors with $(A - \lambda I)x_n \to 0$. Clearly $(Ax_n \mid x_n) \to \lambda$, $(x_n \mid Ax_n) \to \bar{\lambda}$; it follows that $(J_\theta x_n \mid x_n) \to r$ and therefore $\max J_\theta \geqq r$. Since $\| Ax_n \|$ is bounded,

$$0 = \lim ((A - \lambda I)x_n \mid Ax_n) = \lim \{(A^*Ax_n \mid x_n) - \lambda(x_n \mid Ax_n)\} ,$$

thus $(A^*Ax_n \mid x_n) \to \lambda\bar{\lambda} = r^2$ and therefore $\min A^*A \leqq r^2$. Thus (1) is proved. Since $(A - \lambda I)^*(A - \lambda I) = A^*A - 2rJ_\theta + r^2I$, one has

$$\| (A - \lambda I)x_n \|^2 = (A^*Ax_n \mid x_n) - 2r(J_\theta x_n \mid x_n) + r^2 ,$$

hence

$$\begin{aligned} \min A^*A \leqq (A^*Ax_n \mid x_n) &= \| (A - \lambda I)x_n \|^2 + 2r(J_\theta x_n \mid x_n) - r^2 \\ &\leqq \| (A - \lambda I)x_n \|^2 + 2r \max J_\theta - r^2 ; \end{aligned}$$

letting $n \to \infty$,

$$\min A^*A \leqq 2r \max J_\theta - r^2 .$$

Thus $\min A^*A \leqq (\max J_\theta)^2 - (\max J_\theta - r)^2$, which proves (2).

Incidentally, if $\lambda = 0 \in \pi(A)$ then obviously $\min A^*A = 0$ and the theorem yields no information other than $\max J_\theta \geqq 0$ for all θ.

If the dependence of J_θ on A is indicated by writing $J_\theta = J_\theta(A)$, evidently $J_{-\theta}(A^*) = J_\theta(A)$. One has $\pi(A^*) \subset \sigma(A^*) = (\sigma(A))^*$, thus $(\pi(A^*))^* \subset \sigma(A)$; if $\lambda = re^{i\theta} \in (\pi(A^*))^*$ then $re^{-i\theta} \in \pi(A^*)$ and application of the theorem to A^* yields the following:

COROLLARY 1. *If A is any operator and $\lambda \in (\pi(A^*))^*$, $\lambda = re^{i\theta}$, then*

$$\max J_\theta \geqq r \geqq (\min AA^*)^{1/2} , \tag{3}$$

$$\max J_\theta - r \leqq [(\max J_\theta)^2 - \min AA^*]^{1/2} . \tag{4}$$

If A is hyponormal ($AA^* \leqq A^*A$) then $\pi(A^*) = \sigma(A^*) = (\sigma(A))^*$ [cf. 1, p. 1175] and Corollary 1 yields:

COROLLARY 2. *If A is hyponormal then* (3) *and* (4) *hold for every $\lambda \in \sigma(A)$, $\lambda = re^{i\theta}$.*

Another way of fulfilling (3) and (4) is via the relation

$$\partial\sigma(A) \subset \pi(A) \cap (\pi(A^*))^* .$$

If $\lambda = re^{i\theta} \in \partial\sigma(A)$, the boundary of $\sigma(A)$, then $\lambda \in \pi(A)$ [cf. 2, p. 39] hence (1) and (2) hold by the theorem. Moreover, $\bar{\lambda} \in (\partial\sigma(A))^* = \partial(\sigma(A))^* = \partial\sigma(A^*) \subset \pi(A^*)$, i.e., $\lambda \in (\pi(A^*))^*$ and so (3) and (4) hold by Corollary 1. Thus:

COROLLARY 3. *If A is any operator and $\lambda = re^{i\theta}$ is a boundary point of $\sigma(A)$, then* (1), (2), (3), (4) *hold.*

Corollary 3 is stated in [3, Th. 1; 4, p. 44, Th. 3.3.1] assuming $AA^* \geqq A^*A$ (i.e., A^* hyponormal).

It follows readily from Corollary 3, as in [3], that the spectrum of a nonunitary isometry is the entire closed unit disc. The proof is similar to, and simpler than, the proof of the following corollary, which extends a result in [3, Corollary 2; 4, p. 44, Corollary 1] (the formulation there is inaccurate):

COROLLARY 4. *If A is an operator such that* $\min A^*A > 0$ *and* $0 \in \sigma(A)$, *then, for each real* θ, $\sigma(A)$ *contains the segment*

$$S_\theta = \{se^{i\theta}: 0 \leqq s \leqq R_\theta\} ,$$

where

$$R_\theta = \max J_\theta - [(\max J_\theta)^2 - \min A^*A]^{1/2} > 0 .$$

Moreover, $\min_\theta R_\theta > 0$, *thus* $\sigma(A)$ *contains the disc* $\{\lambda: |\lambda| \leqq \min_\theta R_\theta\}$.

Proof. The condition $\min A^*A > 0$ means that $0 \notin \pi(A)$ and therefore $0 \notin \partial\sigma(A)$, thus 0 is an interior point of $\sigma(A)$. (Incidentally, $\pi(A) \neq \sigma(A)$, so A is nonnormal; indeed, A^* cannot be hyponormal.)

Fix θ and let L be the ray from 0 at angle θ. If $\lambda = re^{i\theta}$ is a boundary point of $\sigma(A)$ on L, then (Corollary 3) by (1) one has $(\max J_\theta)^2 \geqq \min A^*A > 0$; since $\max J_\theta$ is nonnegative (indeed $\geqq r$) it follows that $R_\theta > 0$. Moreover, by (2) one has $|\lambda| = r \geqq R_\theta$.

To show that $S_\theta \subset \sigma(A)$, suppose $\mu = se^{i\theta}$, $0 < s \leqq R_\theta$. For any s_1, $0 \leqq s_1 < s$, the segment $\{te^{i\theta}: s_1 \leqq t \leqq s\}$ must contain a point of $\sigma(A)$ since otherwise some internal point λ of S_θ would belong to $\partial\sigma(A)$, contrary to the preceding paragraph; thus μ is adherent to, and therefore in, $\sigma(A)$.

Finally, since J_θ and therefore R_θ is a continuous function of θ ($0 \leqq \theta \leqq 2\pi$, 0 and 2π identified) one has $\min_\theta R_\theta > 0$.

In view of the symmetry in Corollary 3, the proof of Corollary 4 also shows: If $\min AA^* > 0$ and $0 \in \sigma(A)$, then, for each real θ, $\sigma(A)$ contains the segment $\{se^{i\theta}: 0 \leqq s \leqq R'_\theta\}$, where

$$R'_\theta = \max J_\theta - [(\max J_\theta)^2 - \min AA^*]^{1/2} > 0 ;$$

if, in addition, A^* is hyponormal, then $R'_\theta \geqq R_\theta$, which strengthens the conclusion of Corollary 4 [cf. 3, Corollary 2].

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