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EXTENDING HOMEOMORPHISMS TO HILBERT CUBE MANIFOLDS

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In this paper conditions are obtained under which two closed homotopic embeddings of a locally compact separable metric space into a separable metric manifold modeled on the Hilbert cube are ambient isotopic.

1. Introduction. A *Fréchet manifold* (or *F-manifold*) is a separable metric space admitting an open cover by sets homeomorphic to s, the countable infinite product of open intervals $(-1, 1)$. A *Hilbert cube manifold* (or *Q-manifold*) is a separable metric space admitting an open cover by sets homeomorphic to open subsets of the Hilbert cube I^∞, which we represent as the countable infinite product of closed intervals $[-1, 1]$.

A subset K of a space X is a *Z-set* provided that K is closed and for every nonnull homotopically trivial (i.e., all homotopy groups are trivial) open set U in X, $U \backslash K$ is nonnull and homotopically trivial. In [1] Anderson introduced Z-sets and proved that if K_1, K_2 are Z-sets in $X = I^\infty$ or s and h is a homeomorphism of K_1 onto K_2, then h can be extended to a homeomorphism of X onto itself.

Z-sets occur quite naturally in I^∞; indeed it is known that any compact subset of s is a Z-set in s or I^∞, any compact subset of $I^\infty \backslash s$ is a Z-set in I^∞, and any relatively closed subset of s is a Z-set in s if and only if its closure in I^∞ is a Z-set in I^∞. More generally it is known that any compact subset of any F-manifold is a Z-set and any closed subset of a complete metric space is a Z-set if it is a countable union of Z-sets. In [2] it is shown that a closed subset K of an F-manifold X is a Z-set if and only if K is *metrically negligible*, i.e., for each $\varepsilon > 0$ there is a homeomorphism $h: X \to X \backslash K$ which satisfies $d(h, id_x) < \varepsilon$. (id_x is the identity mapping on X and all homeomorphisms are onto.) Also Z-sets in I^∞ have been widely used in constructing homeomorphisms on manifolds, and there are other instances in which Z-sets have been used to extend homeomorphisms. Let (X, X') be a pair of F-manifolds or Q-manifolds with X' a closed collared subset of X (collared in the sense of Brown). Then (X, X') may be considered as a manifold-with-boundary, X' being the boundary. It is known and easy to prove that X' is a Z-set in X. Thus a study of Z-sets in manifolds includes a study of boundaries of manifolds. Homeomorphism extension theorems for Z-sets imply homeomorphism extension theorems for manifolds-with-boundary.

In [3] the homeomorphism extension result of [1] was generalized to F-manifolds by proving that if X is an F-manifold, K_1 and K_2 are Z-sets in X, and h is a homeomorphism of K_1 onto K_2 which is homotopic in X to the identity on K_1, then there exists an ambient isotopy G of X onto itself such that G_0 is the identity on X and $G_1 \mid K_1 = h_1$. (By ambient isotopy we mean that each level is onto). It is obvious that an isotopic homeomorphism extension property for F-manifolds requires the property that the homeomorphism be homotopic to the identity.

An isotopy $F: X \times I \to Y$ is said to be an *invertible ambient isotopy* provided that F is ambient and $F^*: X \times I \to Y \times I$, defined for each t by $F^*(x, t) = (F(x, t), t)$, is a homeomorphism. In the extension theorem of [3] cited above, it was not asserted that G be invertible, but it is obvious that the argument given there does allow one to conclude that it is invertible.

It is the purpose of this paper to obtain an analogue for Q-manifolds of the result cited above as established in [3]. A continuous function $f: X \to Y$ is *proper* provided that the inverse image of each compact subset of Y is compact. A homotopy $F: X \times I \to Y$ is *proper* provided that the map F is proper. The most general homeomorphism extension theorem we obtain for Q-manifolds is given in Theorem 6.1 of this paper, but we state a weaker version of it below.

THEOREM. *Let X be a Q-manifold, A be a locally compact separable metric space, and let f and g be embeddings of A into X such that $f(A)$ and $g(A)$ are Z-sets. Then there exists an invertible ambient isotopy G of X onto itself such that $G_0 = id$ and $G_1 \circ f = g$ if and only if f and g are properly homotopic.*

We remark that if A is compact, then we can drop the requirement that the homotopy be proper.

COROLLARY 1. *Let X be a Q-manifold, A be a compact metric space, and let f and g be embeddings of A into X such that $f(A)$ and $g(A)$ are Z-sets. Then there exists an invertible ambient isotopy G of X onto itself and that $G_0 = id$ and $G_1 \circ f = g$ if and only if f and g are homotopic.*

We also show that for manifolds which admit half-open interval factors $[0, 1)$, we can also drop the requirement that the homotopy be proper.

COROLLARY 2. *Let X be a Q-manifold, A be a locally compact* *separable metric space, and let f and g be embeddings of A into $X \times [0, 1)$ such that $f(A)$ and $G(A)$ are Z-sets. Then there exists an invertible ambient isotopy G of $X \times [0, 1)$ onto itself such that $G_0 = id$ and $G_1 \circ f = g$ if and only if f and g are homotopic.*

In § 5 we give an example to show the distinction between the use of homotopies and proper homotopies for (locally-compact) Q-manifolds. However, the restriction to proper homotopies is not a serious philosophical restriction, because for locally compact spaces the maps one constructs in natural ways are proper. Indeed, proper maps and proper homotopies seem to provide a suitable framework for studying locally compact (but not compact) spaces.

Even though the statement of the extension theorem given in this paper for Q-manifolds is similar to the corresponding theorem established in [3] for F-manifolds, the proof is quite different. This is due in large part to the open embedding theorem of David W. Henderson [7], which says that every F-manifold can be embedded as an open subset of s.

The extension theorem of [3] was obtained by applying the open embedding theorem of [7], replacing the given homotopy by one in which all the fibers are parallel *in* one coordinate direction, and then using motions only in this one direction for the extension. Unfortunately not all Q-manifolds can be embedded as open subsets of I^∞, nor does there appear to be a weaker version of an open embedding theorem that would help.

The extension theorem of [3] has been done with an "estimation" on the extension, i.e., if the given homotopy is "close" to the identity, then the ambient isotopy can be constructed "close" to the identity, where "close" will be made precise in the next section. In Theorem 6.1 we also obtain an estimation on the extension.

In a forthcoming paper, *On the structure of Hilbert cube manifolds*, the second author will use the results of this paper to establish some representation and characterization theorems concerning Q-manifolds.

2. Preliminaries. For each integer $n > 0$ we let $W_n^+ = \{(x_i) \in I^\infty \mid x_n = 1\}$ and $W_n^- = \{(x_i) \in I^\infty \mid x_n = -1\}$, where we use the representation $I^\infty = \Pi_{i=1}^\infty I_i$, *with* $I_i = [-1, 1]$. We call W_n^+ and W_n^- *endslices* of I^∞. The *pseudo-boundary* of I^∞ is defined to be $B(I^\infty) = I^\infty \backslash s$. It follows routinely from the apparatus in [1] that any finite union of endslices of I^∞ is a Z-set in I^∞.

A subset U of I^∞ is said to be a *basic open subset* of I^∞ provided that $U = \Pi_{i=1}^\infty J_i$, where each J_i is a connected open subset of I_i and $J_i = I_i$, for all but finitely many values of i. It is then clear that we may regard $\mathrm{Cl}(U)$ as a Hilbert cube, hence $\mathrm{Bd}(U)$ is a finite union of endslices of $\mathrm{Cl}(U)$. (For any space X and $A \subset X$, $\mathrm{Cl}_X(A)$ denotes the closure of A and $\mathrm{Bd}_X(A)$ denotes the boundary of A. Whenever no ambiguity results we will suppress the subscript X).

Let $\mathscr{U}$ be an open cover for a space Y. Maps f and g of a space X into Y are said to be *$\mathscr{U}$-close* provided that for each $x \in X$, there is an element of $\mathscr{U}$ containing both $f(x)$ and $g(x)$. A homotopy $F: X \times I \to Y$ is said to be *limited by* $\mathscr{U}$ if for each $x \in X$, $F(\{x\} \times I)$ is contained in some member of $\mathscr{U}$. By $\mathrm{St}^n(\mathscr{U})$ we will mean the n^{th} *star* of the cover $\mathscr{U}$, defined inductively as follows: $\mathrm{St}^0(\mathscr{U}) = \mathscr{U}$ and $\mathrm{St}^{n+1}(\mathscr{U})$ is the collection of all sets of the form $G \cup (\bigcup\{U \mid U \in \mathscr{U}, U \cap G \neq \emptyset\})$, for $G \in \mathrm{St}^n(\mathscr{U})$.

We will have need to consider proper maps between locally compact separable metric spaces. In this case there are two results we will find useful in the sequel. The proofs are elementary.

LEMMA 2.1. *Let X and Y be locally compact separable metric spaces and let $f: X \to Y$ be proper. Then $f(X)$ is closed.*

LEMMA 2.2. *Let X and Y be locally compact separable metric spaces. Then there is an open cover $\mathscr{U}$ of Y such that if $f: X \to Y$ is a proper map and $g: X \to Y$ is a continuous function such that f and g are $\mathscr{U}$-close, then g is also proper.*

A subset K of I^∞ is said to have *infinite deficiency* provided that for each of infinitely many different coordinate directions, K projects onto a single interior point of that interval. It is shown in [4] that if X is any Q-manifold, then X and $X \times I^\infty$ are homeomorphic. This result was used in [6] to obtain the following characterization of Z-sets in Q-manifolds.

LEMMA 2.3. *Let X be a Q-manifold and let K be a closed subset of X. Then K is a Z-set if and only if there is a homeomorphism $h: X \to X \times I^\infty$ such that $\pi_{I^\infty}(h(K))$ has infinite deficiency.*

We can use Lemma 2.3 to obtain an easy proof of the somewhat stronger result.

LEMMA 2.4. *Let X be a Q-manifold and let K be a Z-set in X. Then there is a homeomorphism $h: X \to X \times I^\infty$ such that $h(x) = (x, (0, 0, \cdots))$, for all $x \in K$.*

Proof. Using Lemma 2.3 there is a homeomorphism $f: X \to X \times I^\infty$ such that $f(K) \subset X \times \{(0, 0, \cdots)\}$. Let $g: I^\infty \to I^\infty \times I^\infty$ be a homeomorphism such that $g((0, 0, \cdots)) = ((0, 0, \cdots), (0, 0, \cdots))$. Then the composition of the following homeomorphisms gives h.

$$X \xrightarrow{f} X \times I^\infty \xrightarrow{id \times g} X \times I^\infty \times I^\infty \xrightarrow{f^{-1} \times id} X \times I^\infty$$

A cover $\mathscr{U}$ of a space X is said to be *star-finite* provided that the closure of each element of $\mathscr{U}$ intersects the closures of only finitely many other members of $\mathscr{U}$. There is a useful convergence procedure from [5] that we list next.

LEMMA 2.5. *Let X be any space and let $\mathscr{U}$ be any countable star-finite open cover of X. Then we may order $\mathscr{U}$ as $\{U_i\}_{i=1}^\infty$ such that if $\{h_i\}_{i=1}^\infty$ is any sequence of homeomorphisms of X onto itself satisfying $h_i \mid X \backslash U_i = id$, for all $i > 0$, then $\{h_i \circ h_{i-1} \circ \cdots h_1\}_{i=1}^\infty$ converges pointwise to a homeomorphism of X onto itself. (We write $L\Pi_{i=1}^\infty h_i$ for the pointwise limit of $\{h_i \circ \cdots \circ h_1\}_{i=1}^\infty$)*.

We shall need a mapping replacement theorem that has been established in [3]. We list it below.

LEMMA 2.6. *Let A be a topologically complete separable metric space, X be an F-manifold, B be a closed subset of A, and let $f: A \to X$ be a continuous function such that $f \mid B$ is a homeomorphism of B onto a Z-set in X. If $\mathscr{U}$ is any open cover of X, then there exists a homeomorphism h of A onto a Z-set in X such that h is $\mathscr{U}$-close to f and $h \mid B = f \mid B$.*

3. Replacing mappings with embeddings. We will need to replace mappings with homeomorphisms that are "close" to the mappings. These results are established in Theorem 3.1 and 3.2 below. An analogous result for F-manifolds is stated in Lemma 2.6 above.

THEOREM 3.1. *Let X be a Q-manifold, $\mathscr{U}$ be an open cover of X, A be a locally compact separable metric space, and let B be a closed subset of A. If $f: A \to X$ is any proper map such that $f \mid B$ is a homeomorphism of B onto a Z-set in X, then there is an embedding $g: A \to X$ such that $g(A)$ is a Z-set, $g \mid B = f \mid B$, and g is* St $(\mathscr{U})$*-close to f.*

Proof. Using Lemma 2.3 there is a homeomorphism $h: X \to X \times I^\infty$ such that $h(f(B)) \subset X \times \{0\}$, where $0 = (0, 0, \cdots) \in I^\infty$. Using Lemmas 2.1 and 2.2 we can clearly obtain an embedding $\varphi: X \times I^\infty \to X \times s$ such that $\varphi(X \times I^\infty)$ is closed in $X \times I^\infty$, φ is $h(\mathscr{U})$-close to $id_{X \times I^\infty}$, and $\varphi \mid X \times \{0\} = id$.

Thus $\varphi \circ h \circ f\colon A \to X \times I^\infty$ is a proper map, $\varphi \circ h \circ f(A)$ is $h(\mathscr{U})$-close to $h \circ f$, and $\varphi \circ h \circ f \mid B = h \circ f \mid B$. Now $\varphi \circ h \circ f(B)$ is a closed subset of $X \times s$ which is σ-compact. Since $X \times s$ is obviously an F-manifold and each compact subset of an F-manifold is a Z-set, we conclude that $\varphi \circ h \circ f(B)$ is a Z-set in $X \times s$.

Note that A is a topologically complete separable metric space. Thus for each open cover $\mathscr{V}$ of $X \times I^\infty$ there is, by Lemma 2.6, a homeomorphism θ of A onto a Z-set in $X \times s$ such that $\theta \mid B = h \circ f \mid B$ and θ is $\mathscr{V}$-close to $\varphi \circ h \circ f$. Since $\mathscr{V}$ is arbitrary we can require that $\theta(A)$ be closed in $X \times I^\infty$, hence a Z-set in $X \times I^\infty$, and θ be $h(\mathscr{U})$-close to $\varphi \circ h \circ f$. Clearly $g = h^{-1} \circ \theta\colon A \to X$ is a homeomorphism of A onto a Z-set in X such that $g \mid B = f \mid B$ and g is $\mathrm{St}(\mathscr{U})$-close to f.

It is easy to produce examples to show that the assumption that f be proper is necessary in the preceding theorem. For example let $A = (-1, 1)$, $B = \{0\}$, $X = I^\infty$, and let $f\colon A \to X$ be defined by $f(x) = 0$, for all $x \in A$. Clearly A cannot be embedded as a closed subset of X.

Even though there are examples which show the necessity of assuming f to be proper in Theorem 3.1, there are a large class of Q-manifolds for which this assumption is not necessary. We will first need a preliminary result.

LEMMA 3.1. *Let X be a Q-manifold, $F \subset X \times [0, 1)$ be a Z-set, and let A, B be disjoint closed subsets of F. Then there exist a Z-set K in $X \times [0, 1)$ such that $A \subset K$, $K \cap (F \backslash A) = \varnothing$, and a homeomorphism $f\colon (X \times [0, 1)) \backslash K \to X \times [0, 1)$ such that $f \mid B = id$.*

Proof. It follows from Lemma 2.4 that there is a homeomorphism $h\colon X \times [0, 1) \to X \times [0, 1) \times [0, 1]$ such that $h(x) = (x, 1/2)$, for all $x \in F$. Let $\alpha\colon [0, 1) \times [0, 1] \to [0, 1) \times [0, 1)$ be a homeomorphism such that $\alpha \mid [0, 1) \times \{1/2\} = id$. Then $id \times \alpha\colon X \times [0, 1) \times [0, 1] \to X \times [0, 1) \times [0, 1)$ is a homeomorphism satisfying $(id \times \alpha) \circ h(x) = (x, 1/2)$, for all $x \in F$.

Put $K' = A \times [1/2, 1)$, which is a Z-set in $X \times [0, 1) \times [0, 1)$ containing $A \times \{1/2\}$ and missing $(F \backslash A) \times \{1/2\}$. It is clear that there is a homeomorphism $f'\colon (X \times [0, 1) \times [0, 1)) \backslash K' \to X \times [0, 1) \times [0, 1)$ such that $f' \mid B \times \{1/2\} = id$. Then put $f = ((id \times \alpha) \circ h)^{-1} \circ f' \circ (id \times \alpha) \circ h$ and $K = ((id \times \alpha) \circ h)^{-1}(K')$ to fulfill our requirements.

THEOREM 3.2. *Let X be a Q-manifold, A be a locally compact separable metric space, and let $B \subset A$ be a closed set. If $f\colon A \to X \times [0, 1)$ is any continuous function such that $f \mid B$ is a homeomorphism of B onto a Z-set in $X \times [0, 1)$, then there is an embedding $g\colon A \to X \times [0, 1)$*

such that $g(A)$ is a Z-set and $g \mid B = f \mid B$.

Proof. Using the idea contained in the proof of Theorem 3.1 let $h: X \times [0, 1) \to X \times [0, 1) \times I^\infty$ be a homeomorphism such that $h(f(B)) \subset X \times [0, 1) \times \{0\}$ and let $\varphi: X \times [0, 1) \times I^\infty \to X \times [0, 1) \times s$ be an embedding such that $\varphi \mid X \times [0, 1) \times \{0\} = id$. Then $\varphi \circ h \circ f: A \to X \times [0, 1) \times s$ is a continuous function such that $\varphi \circ h \circ f \mid B$ is a homeomorphism of B onto a Z-set in $X \times [0, 1) \times s$. Using Lemma 2.6 there is a homeomorphism θ of A onto a Z-set in $X \times [0, 1) \times s$ such that $\theta \mid B = h \circ f \mid B$.

Since A is locally compact we can prove that $F = Cl(\theta(A))\backslash\theta(A)$ is closed in $X \times [0, 1) \times I^\infty$, where closure is taken in $X \times [0, 1) \times I^\infty$. It is also easy to show that F is a Z-set in $X \times [0, 1) \times I^\infty$.

Using Lemma 3.1 there exist a Z-set $K \subset X \times [0, 1) \times I^\infty$ such that $F \subset K$, $K \cap (Cl(\theta(A))\backslash F) = \varnothing$, and a homeomorphism

$$\alpha: (X \times [0, 1) \times I^\infty)\backslash K \to X \times [0, 1) \times I^\infty$$

such that $\alpha \mid h \circ f(B) = id$. Then $g = h^{-1} \circ \alpha \circ \theta$ fulfills our requirements.

4. A preliminary homeomorphism extension theorem. In this section we obtain a homeomorphism extension theorem (Theorem 4.1) for Q-manifolds which is applicable only in special cases. We will use this in the next section for a more general version.

If f and g are embeddings of a space A into a space X, then by the *induced homeomorphism* of $f(A)$ onto $g(A)$ we mean the homeomorphism $\varphi: f(A) \to g(A)$ defined by $\varphi = g \circ f^{-1}$.

We will need the following preliminary result.

LEMMA 4.1. *Let X be a Q-manifold and let $f: I \to X$ be an embedding such that $f(I)$ is a Z-set. Then there is an embedding $h: I^\infty \to X$ such that $h(I^\infty\backslash W_1^+)$ is an open subset of X containing $f(I)$.*

Proof. We first show that there is a $\delta \in (0, 1)$ such that for each $t \in [0, \delta]$, there is a homeomorphism of X onto itself which takes $f(I)$ onto $f([t, 1])$.

It is clear that there is a basic open subset U of I^∞ and an embedding $g': \mathrm{Cl}\,(U) \to X$ such that $g'(U)$ is an open subset of X containing $f(0)$. We may treat $\mathrm{Cl}\,(U)$ as a Hilbert cube and regard $\mathrm{Bd}\,(U)$ as a finite union of endslices of $\mathrm{Cl}\,(U)$. By choosing a subset of U, if necessary, we can, without loss of generality, assume further that $\mathrm{Bd}\,(U)$ is homeomorphic to I^∞. Thus there is an embedding $g: I^\infty \to X$ such that $g(I^\infty\backslash W_1^+)$ is an open subset of X containing $f(0)$.

This follows from the homeomorphism extension theorem of [1].

Choose $\delta \in (0, 1)$ so that $f([0, 2\delta]) \subset g(I^\infty \backslash W_1^+)$. We show that there is a homeomorphism α of X onto itself which takes $f([0, 1])$ onto $f([\delta, 1])$. It will be clear from the construction that given any $t \in [0, \delta]$, there is similarly a homeomorphism of X onto itself taking $f([0, 1])$ onto $f([t, 1])$.

Let $\varphi: f(I) \to f([\delta, 1])$ be a homeomorphism satisfying

$$f^{-1} \circ \varphi \circ f \mid [2\delta, 1] = id .$$

Then $\theta = g^{-1} \circ \varphi \circ g$ defines a homeomorphism of $g^{-1}(g(I^\infty) \cap f(I))$ onto $g^{-1}(g(I^\infty) \cap f([\delta, 1]))$ satisfying $\theta \mid g^{-1}(g(I^\infty) \cap f(I)) \cap W_1^+ = id$. But $g^{-1}(g(I^\infty) \cap f(I))$ and $g^{-1}(g(I^\infty) \cap f([\delta, 1]))$ are Z-sets in I^∞. Using the extension theorem of [1] we can extend θ to a homeomorphism $\tilde{\theta}: I^\infty \to I^\infty$ such that $\tilde{\theta} \mid W_1^+ = id$. Then $g \circ \tilde{\theta} \circ g^{-1}$ gives a homeomorphism of $g(I^\infty)$ onto itself such that $g \circ \tilde{\theta} \circ g^{-1} \mid g(W_1^+) = id$ and $g \circ \tilde{\theta} \circ g^{-1}$ extends φ. Extend this to a homeomorphism $\alpha: X \to X$ so that $\alpha \mid X \backslash g(I^\infty) = id$. We then have $\alpha(f(I)) = f([\delta, 1])$.

Now let A be the subset of $[0, 1]$ consisting of all $\delta \in [0, 1)$ such that for each $t \in [0, \delta)$, there is a homeomorphism of X onto itself taking $f(I)$ onto $f([t, 1])$. From what we have shown above it follows routinely that $A = [0, 1)$. Let $h': I^\infty \to X$ be an embedding such that $h'(I^\infty \backslash W_1^+)$ is an open subset of X containing $f(1)$. Choose ε in $(0, 1)$ so that $f([1 - \varepsilon, 1]) \subset h'(I^\infty \backslash W_1^+)$. Since $A = [0, 1)$ there is a homeomorphism β of X onto itself satisfying $\beta(f(I)) = f([1 - \varepsilon, 1])$. Then $h = \beta^{-1}h': I^\infty \to X$ is an embedding satisfying the required properties.

In the following theorem we show how to extend the induced homeomorphism between two homotopic embeddings of a locally compact separable metric space into a Q-manifold, where the given homotopy is an embedding onto a Z-set. The procedure will be to apply Lemma 4.1 to do the extension along the fibers of the homotopy.

THEOREM 4.1. *Let X be a Q-manifold and let A be a locally compact separable metric space. If $F: A \times I \to X$ is an embedding such that $F(A \times I)$ is a Z-set, then the induced homeomorphism of $F_0(A)$ onto $F_1(A)$ can be extended to a manifold homeomorphism.*

Proof. Write $A = \bigcup_{n=1}^\infty A_n$, where each A_n is compact and $A_n \subset \mathrm{Int}\,(A_{n+1})$ (with $\mathrm{Int}\,(A_{n+1})$ denoting the interior of A_{n+1}). Also write $X = \bigcup_{n=1}^\infty X_n$, where each X_n is compact and $X_n \subset \mathrm{Int}\,(X_{n+1})$. It is clear that for each $n > 0$ there are integers $i_n, j_n \geqq 0$ such that

$$F(A_n \times I) \backslash F(A_{n-1} \times I) \subset (\mathrm{Int}\,(X_{i_n})) \backslash X_{j_n} ,$$

where $A_0 = X_0 = \emptyset$, $i_1 < i_2 < \cdots$, $j_1 \leqq j_2 \leqq \cdots$, and $\lim_{n\to\infty} j_n = \infty$.

For each $x \in A_n \backslash A_{n-1}$ (where $n > 0$) we let $h_x: I^\infty \to (\mathrm{Int}\,(X_{i_n}))\backslash X_{j_n}$ be an embedding such that $h_x(I^\infty \backslash W_1^+)$ is an open set containing $F(\{x\} \times I)$. For each $x \in A_n \backslash A_{n-1}$ let U_x be an open subset of $(\mathrm{Int}\,(A_{n+1}))\backslash A_{n-1}$ containing x such that $F(U_x \times I) \subset h_x(I^\infty \backslash W_1^+)$. Choose a subset $\{U_i\}_{i=1}^\infty$ of $\{U_x \mid x \in A\}$ which covers A and for which $\{i \mid U_i \cap A_n \neq \emptyset\}$ is finite, for all $n > 0$. If $\{h_i\}_{i=1}^\infty$ is the corresponding subset of $\{h_x \mid x \in A\}$, then $\{h_i(I^\infty \backslash W_1^+)\}$ is a star-finite open cover of $F(A \times I)$. We may therefore assume that $\{h_i(I^\infty \backslash W_1^+)\}_{i=1}^\infty$ is ordered as in Lemma 2.5. Also choose a closed cover $\{C_i\}_{i=1}^\infty$ of A such that $C_i \subset U_i$, for all i.

Let $\varphi_1: A \to [0, 1]$ be a continuous function satisfying $\varphi_1(x) = 1$, for $x \in C_1$, and $\varphi_1(x) = 0$, for $x \in A \backslash U_1$. Then we let $B_1 = F(A \times I)$ and $D_1 = \bigcup\{F(\{x\} \times [\varphi_1(x) \cdot 1/2, 1]) \mid x \in A\}$. Define $g_1: B_1 \to D_1$ to be the homeomorphism which satisfies the property that for each $x \in A$, $F^{-1} \circ g_1 \circ F$ takes $\{x\} \times I$ linearly onto $\{x\} \times [\varphi_1(x) \cdot 1/2, 1]$.

We note that $B_1' = h_1^{-1}(B_1 \cap h_1(I^\infty))$ and $D_1' = h_1^{-1}(D_1 \cap h_1(I^\infty))$ are Z-sets in I^∞. It is then clear that there is a homeomorphism $g_1': B_1' \cup W_1^+ \to D_1' \cup W_1^+$ such that $g_1' \mid W_1^+ = id$ and $h_1^{-1} \circ g_1 \circ h_1 \mid B_1' = g_1' \mid B_1'$. Using the extension theorem of [1] we can extend g_1' to a homeomorphism $\tilde{g}_1': I^\infty \to I^\infty$. Thus there is a homeomorphism $f_1: X \to X$ such that $f_1 \mid X \backslash h_1(I^\infty) = id$ and $f_1 \mid h_1(I^\infty) = h_1 \circ \tilde{g}_1' \circ h_1^{-1}$; hence f_1 extends g_1.

Let $\varphi_2' \mid (A \backslash U_2) \cup C_1 \cup C_2 \to [0, 1]$ be the continuous function defined by $\varphi_2' \mid C_1 \cup (A \backslash U_2) = \varphi_1 \mid C_1 \cup (A \backslash U_2)$ and $\varphi_2(x) = 1$, for $x \in C_2$. Then we use the Tietze Extension Theorem to get a continuous function $\varphi_2: A \to [0, 1]$ such that $\varphi_2 \mid C_1 \cup (A \backslash U_2) = \varphi_1 \mid C_1 \cup (A \backslash U_2)$ and $\varphi_2(x) = 1$, for $x \in C_2$.

Define $B_2 = D_1$ and $D_2 = \bigcup\{F(\{x\} \times [\varphi_2(x) \cdot 1/2, 1]) \mid x \in A\}$. Let $g_2: B_2 \to D_2$ be the homeomorphism such that for each $x \in A$, $F^{-1} \circ g_2 \circ F$ takes $\{x\} \times [\varphi_1(x) \cdot 1/2, 1]$ linearly onto $\{x\} \times [\varphi_2(x) \cdot 1/2, 1]$. We note that $g_2 \mid B_2 \cap \mathrm{Bd}\,(h_2(I^\infty)) = id$. Thus using the techniques employed in the construction of f_1 we can construct a homeomorphism $f_2: X \to X$ which extends g_2 and satisfies $f_2 \mid X \backslash h_2(I^\infty) = id$. It is clear that if we continue this process and put $f = L\Pi_{i=1}^\infty f_i$, then f is a homeomorphism of X onto itself which extends the induced homeomorphism of $F_0(A)$ onto $F_{1/2}(A)$.

Using this result we can obtain a homeomorphism $\tilde{f}$ of X onto itself which extends the induced homeomorphism of $F_1(A)$ onto $F_{1/2}(A)$. Then $\tilde{f}^{-1} \circ f$ is a homeomorphism of X onto itself which extends the induced homeomorphism of $F_0(A)$ onto $F_1(A)$.

5. A general homeomorphism extension theorem. In this section we generalize Theorem 4.1, showing that we may weaken the condition that $F(A \times I)$ be embedded as a Z-set. We first establish a lemma which will be used here and in the next section.

LEMMA 5.1. *Let X be a Q-manifold, A be a locally compact separable metric space, and let $F: A \times I \to X$ be a proper map such that F_0, F_1 are embeddings and $F_0(A)$, $F_1(A)$ are Z-sets. If $\mathscr{U}$ is any open cover of X such that F is limited by $\mathscr{U}$, then there is an invertible ambient isotopy $G: X \times I \to X$ such that $G_0 = id$, G is limited by $\mathscr{U}$, and $(G_1 \circ F_1(A)) \cap F_0(A) = \varnothing$. Moreover, there is an embedding $H: A \times I \to X$ such that $H(A \times I)$ is a Z-set, $H_0 = F_0$, $H_1 = G_1 \circ F_1$, and H is limited by $\mathrm{St}^4(\mathscr{U})$.*

Proof. Let $f: X \to X \times I^\infty$ be a homeomorphism such that $f(F_0(A) \cup F_1(A)) \subset X \times \{0\}$. By adjusting one of the coordinates of I^∞ we can easily obtain an invertible ambient isotopy $G': (X \times I^\infty) \times I \to X \times I^\infty$ such that $G'_0 = id$, $(G'_1 \circ f \circ F_1(A)) \cap f \circ F_0(A) = \varnothing$, and G' is limited by $f(\mathscr{U})$. Then $G: X \times I \to X$, defined by

$$G(x, t) = f^{-1} \circ G'(f(x), t) ,$$

is an invertible ambient isotopy which is limited by $\mathscr{U}$ and which satisfies $(G_1 \circ F_1(A)) \cap F_0(A) = \varnothing$.

Now define $F': A \times I \to X$ by

$$F'(x, t) = \begin{cases} F(x, t), & \text{for } 0 \leqq t \leqq \dfrac{1}{2} \\ G(F(x, t), 2t - 1), & \text{for } \dfrac{1}{2} \leqq t \leqq 1 \end{cases}$$

Then F' is a proper map and we have $F'_0 = F_0$, $F'_1 = G_1 \circ F_1$, and F' is limited by $\mathrm{St}(\mathscr{U})$.

It follows that $F' \mid (A \times \{0\} \cup (A \times \{1\})$ is an embedding of $(A \times \{0\}) \cup (A \times \{1\})$ onto a Z-set in X. Applying Theorem 3.1 there is an embedding $H: A \times I \to X$ such that $H \mid (A \times \{0\}) \cup (A \times \{1\}) = F' \mid (A \times \{0\}) \cup (A \times \{1\})$, $H(A \times I)$ is a Z-set, and H is $\mathrm{St}(\mathscr{U})$-close to F'. This means that H is limited by $\mathrm{St}^4(\mathscr{U})$.

We can obtain a version of Lemma 5.1 for Q-manifolds which admit half-open interval factors $[0, 1)$. The proof is similar to that of Lemma 5.1. In the proof of Corollary 5.1, Theorem 3.2 plays a role similar to the role that Theorem 3.1 played in the proof of Lemma 5.1.

COROLLARY 5.1. *Let X be a Q-manifold, A be a locally compact separable metric space, and let $F: A \times I \to X \times [0, 1)$ be a homotopy* *such that F_0, F_1 are embeddings and $F_0(A)$, $F_1(A)$ are Z-sets. Then there is an invertible ambient isotopy $G: X \times [0, 1) \times I \to X \times [0, 1)$ such that $G_0 = id$ and $(G_1 \circ F_1(A)) \cap F_0(A) = \emptyset$. Moreover, there is an embedding $H: A \times I \to X \times [0, 1)$ such that $H(A \times I)$ is a Z-set, $H_0 = F_0$, and $H_1 = G_1 \circ F_1$.*

THEOREM 5.1. *Let X be a Q-manifold, A be a locally compact separable metric space, and let $F: A \times I \to X$ be a proper homotopy so that F_0, F_1 are embeddings and $F_0(A)$, $F_1(A)$ are Z-sets. Then there is a homeomorphism of X onto itself which extends the induced homeomorphism of $F_0(A)$ onto $F_1(A)$.*

Proof. Using Lemma 5.1 there is a manifold homeomorphism h and an embedding $H: A \times I \to X$ such that $H_0 = F_0$, $H_1 = h \circ F_1$, and $H(A \times I)$ is a Z-set. Using Theorem 4.1 let g be a homeomorphism of X onto itself which extends the induced homeomorphism of $H_0(A)$ onto $H_1(A)$. Then $h^{-1} \circ g$ is a manifold homeomorphism which extends the induced homeomorphism of $F_0(A)$ onto $F_1(A)$.

We now present a simple example which shows that in Theorem 5.1 the condition that the homotopy be proper is necessary. Let $X = I^\infty \backslash \{p, q\}$, where p and q are distinct points of I^∞. Let $A = (0, 1)$ and let $f_1: A \to X$ be an embedding so that $f_1(A)$ is a Z-set and $\mathrm{Cl}_{I^\infty}(f_1(A))$ is an arc containing p and q as endpoints. Let $f_2: A \to X$ be an embedding so that $f_2(A)$ is a Z-set and $\mathrm{Cl}_{I^\infty}(f_2(A))$ is a simple closed curve containing p and missing q. Clearly there is a homotopy $F: A \times I \to X$ satisfying $F_0 = f_1$ and $F_1 = f_2$, but F is not proper.

If the induced homeomorphism of $f_1(A)$ onto $f_2(A)$ could be extended to a homeomorphism of X onto itself then $X \backslash f_1(A)$ would be homeomorphic to $X \backslash f_2(A)$. But the one-point compactification of $X \backslash f_1(A)$ is I^∞ and the one-point compactification of $X \backslash f_2(A)$ is not simply connected, giving us a contradiction.

We also obtain a version of Theorem 5.1 for Q-manifolds which admit half-open interval factors.

COROLLARY 5.2. *Let X be a Q-manifold, A be a locally compact separable metric space, and let $F: A \times I \to X \times [0, 1)$ be a homotopy so that F_0, F_1 are embeddings and $F_0(A)$, $F_1(A)$ are Z-sets. Then there is a homeomorphism of $X \times [0, 1)$ onto itself which extends the induced homeomorphism of $F_0(A)$ onto $F_1(A)$.*

6. Converting homotopies to isotopies. In this section we establish an "estimated" version of Theorem 5.1, with the homotopy being approximated by an invertible ambient isotopy. We will first need the following result.

LEMMA 6.1 *Let X be a Q-manifold, A be a locally compact separable metric space, and let $F: A \times I \to X$ be an embedding so that $F(A \times I)$ is a Z-set. Then there is a homeomorphism $h: X \to X \times [-1, 1]$ and a subset B of X such that $h \circ F_0(A) = B \times \{0\}$, $h \circ F_1(A) = B \times \{1/2\}$, and for each $x \in A$ we have $h \circ f(\{x\} \times I) = \{y\} \times [0, 1/2]$, where $y = \pi_X \circ h \circ F(\{x\} \times I)$.*

Proof. Since $F(A \times I)$ is a Z-set there is a homeomorphism $f: X \to X \times [-1, 1]$ so that $f \circ F(A \times I) \subset X \times \{0\}$. Now let $B = \pi_X \circ f \circ F(A \times \{1\})$. It is clear that we may require f to be constructed so that $f(B \times [0, 1/2])$ is a Z-set in $X \times [-1, 1]$.

Define $G: (A \times I) \times I \to X \times [-1, 1]$ as follows:

$$G_s(x, t) = \begin{cases} f \circ F_{t+s}(x), & \text{for } 0 \leqq s \leqq 1 - t \\ \left(\pi_X \circ f \circ F_1(x),\ \dfrac{1}{2}(t + s - 1)\right), & \text{for } 1 - t \leqq s \leqq 1 \end{cases}$$

Then it can be verified that G is a proper homotopy such that $G_0 = f \circ F$ and $G_1(x, t) = (\pi_X \circ f \circ F_1(x), 1/2t)$; thus G_0 and G_1 are embeddings of $A \times I$ onto Z-sets. Using Theorem 5.1 there is a homeomorphism g of $X \times [-1, 1]$ onto itself which extends the induced homeomorphism of $G_0(A \times I)$ onto $G_1(A \times I)$. Thus $h = g \circ f$ is our required homeomorphism.

THEOREM 6.1. *Let X be a Q-manifold, A be a locally compact separable metric space, and let f and g be embeddings of A into X such that $f(A)$ and $g(A)$ are Z-sets. Then there exists an invertible ambient isotopy G of X onto itself such that $G_0 = id$ and $G_1 \circ f = g$ if and only if f and g are properly homotopic. Moreover, if F is a proper homotopy between f and G and $\mathscr{U}$ is an open cover of X for which F is limited by $\mathscr{U}$, then G may be chosen such that G is limited by $\mathrm{St}^5(\mathscr{U})$.*

Proof. The necessity is obvious. For the sufficiency there is an invertible ambient isotopy $H: X \times I \to X$ which is limited by $\mathscr{U}$ such that $H_0 = id$ and $H_1(g(A)) \cap f(A) = \varnothing$ (we are assuming that $F_0 = f$ and $F_1 = g$). Moreover there is an embedding $F': A \times I \to X$ such that $F'(A \times I)$ is a Z-set, $F'_0 = f$, $F'_1 = H_1 \circ g$, and F' is limited by $\mathrm{St}^4(\mathscr{U})$.

Now apply Lemma 6.1 to get a homeomorphism $h: X \to X \times [-1, 1]$ and a subset B of X such that $h \circ f(A) = B \times \{0\}$, $h \circ H_1 \circ g(A) = B \times \{1/2\}$, and for each $x \in A$ we have $h \circ F'(\{x\} \times I) = \{y\} \times [0, 1/2]$, where $\{y\} = \pi_X \circ h \circ F'(\{x\} \times I)$.

Using only motions in the $[-1, 1]$-direction we can, by a standard argument, find an invertible ambient isotopy G': $(X \times [-1, 1]) \times I \rightarrow X \times [-1, 1]$ such that $G'_0 = id$, G'_1 extends the induced homeomorphism of $h \circ F'_0(A)$ onto $h \circ F'_1(A)$ and G' is limited by $\mathrm{St}^4(h(\mathscr{U}))$. Now define $G: X \times I \rightarrow X$ by $G_t(x) = H_t^{-1} \circ h^{-1} \circ G'_t \circ h(x)$, for $x \in X$ and $t \in I$. It is clear that the required properties are satisfied.

Once more we can obtain a similar result for Q-manifolds admitting half-open interval factors.

COROLLARY 6.1. *Let X be a Q-manifold, A be a locally compact separable metric space, and let f and g be embeddings of A into $X \times [0, 1)$ such that $f(A)$ and $g(A)$ are Z-sets. Then there is an invertible ambient isotopy G of X onto itself such that $G_0 = id$ and $G_1 \circ f = g$ if and only if f and g are homotopic.*

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