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MINIMAL INJECTIVE COGENERATORS FOR THE CLASS OF MODULES OF ZERO SINGULAR SUBMODULE

VASILY C. CATEFORIS

Let R be a ring (with 1) of zero singular right ideal and let Q be its maximal right quotient ring; let $\mathscr{N}$ be the class of all (unitary) right R-modules of zero singular submodule. An element M of $\mathscr{N}$ is said to be an injective cogenerator for $\mathscr{N}$ if M is an injective module and every element of $\mathscr{N}$ can be embedded in a direct product of copies of M; M is said to be a minimal injective cogenerator for $\mathscr{N}$ if M is the only direct summand of M, which is an injective cogenerator for $\mathscr{N}$. This paper deals with the question of existence and uniqueness of a minimal injective cogenerator for $\mathscr{N}$ (and in $\mathscr{N}$). If a minimal injective cogenerator for $\mathscr{N}$ exists, then it is (a) isomorphic to a minimal faithful direct summand of Q, (b) isomorphic to a direct summand of every injective cogenerator for $\mathscr{N}$ (and in $\mathscr{N}$) and (c) unique (up to isomorphism). Whether Q is (or is not) a prime ring, affects the structure, though not the existence, of a minimal injective cogenerator for $\mathscr{N}$: a minimal injective cogenerator for $\mathscr{N}$, if it exists, is (up to isomorphism) a faithful minimal right ideal of Q iff Q is a prime ring and so in this case Q is a minimal injective cogenerator for $\mathscr{N}$ iff Q is a division ring. On the other hand, if R_R is finite dimensional (Goldie) then a minimal injective cogenerator for $\mathscr{N}$ exists; it is Q iff Q is (ring) isomorphic to a finite product of division rings.

We begin with a list of conventions, assumptions and well known facts:

(a) By a ring R it is meant an associative ring R with 1, whose singular right ideal [2] is zero. R_R is used when R is considered as a right R-module.

(b) Q denotes the maximal right quotient ring [2] of a ring R and so Q is a Von Neumann regular ring, i.e. a ring every principal right ideal of which is a direct summand, and also the injective hull of R_R [2].

Now for the rest of the list, let R be a given ring.

(c) By a module M it is meant a unitary right R-module M; $Z(M)$ denotes the singular submodule of M [2] and a module of zero singular submodule is called (for short) nonsingular. $\mathscr{N}$ denotes the class of all nonsingular R-modules.

(d) For each module M, $E(M)$ denotes the injective hull [2] of M. If M and N are modules such that $M \subseteq N$ we write $M \subseteq' N$ to denote the fact that M is essential in N (N is an essential extension of M

[2]).

(e) A module C is said to be M-torsionless, for a given module M, if C can be embedded in a direct product of copies of M, or, equivalently, if $\cap \ker f = (0)$ where f ranges over $\mathrm{Hom}_R(C, M)$.

(f) A module M is said to be a cogenerator for a class $\mathscr{A}$ of modules, if every module in $\mathscr{A}$ is M-torsionless, and an injective cogenerator if, also, M is injective.

(g) Whenever a cogenerator M for $\mathscr{N}$ is considered it is assumed that M is also in $\mathscr{N}$. As a corollary to Gentile's [3, p. 427, Prop. 1] we have:

PROPOSITION 0.1. *Q is an injective cogenerator for $\mathscr{N}$.*

(h) Perhaps the most crucially, certainly the most often used result is the following consequence of [6, p. 119, Remark] and [7, p. 226, Lemma 2.3]:

LEMMA 0.2. *If A is an injective module and C is a nonsingular module, then any homomorphism $f: A \to C$ splits* (i.e. $\ker f$ is a direct summand of A).

The following will also be of frequent use:

LEMMA 0.3. *If I is a right ideal of R, then $E(I) = eQ$ for some idempotent in Q.*

(i) For a nonempty subset S of a module M, $r.\ \mathrm{ann}_R S = \{r \in R / sr = 0$, for all $s \in S\}$ and thus a module M is faithful if $r.\ \mathrm{ann}_R M = (0)$; a module M is said to be minimal faithful if M is faithful and no proper ($\neq M$) direct summand of M is (faithful).

1. Minimal injective cogenerators for $\mathscr{N}$. Let R be a ring. We start with a generalization of a theorem of Armendariz [1, p. 568, Theorem 3].

THEOREM 1.1. *For a nonsingular module M, the following statements are equivalent:*

(a) *M is a cogenerator for $\mathscr{N}$.*

(b) *M contains a faithful submodule D such that D contains the injective hull of every one of its finitely generated submodules.*

Proof. (a) *implies* (b). By hypothesis $\mathrm{Hom}_R(Q, M) \neq 0$ and so, by Lemma 0.2, M contains nonzero injective submodules. Let D be the sum of all injective submodules of M; if $N_1, \cdots, N_k$ are injective submodules of M (finitely many) then $N_1 + \cdots + N_k$ is also injective as it is a homomorphic image of the injective module $N_1 \times \cdots \times N_k$ (Lemma 0.2 again). It follows that D contains the injective hull of each of its finitely generated submodules. Now observe that $\mathrm{Hom}_R(Q, M) = \mathrm{Hom}_R(Q, D)$ and thus Q is D-torsionless. It follows that D is faithful; in fact we have shown that D is a cogenerator for $\mathscr{N}$ as Q is (Proposition 0.1).

(b) *implies* (a). If a is a nonzero element of Q, then for some $r \in R$, ar is a nonzero element of R. Since $D\ ar \neq 0$, there exists $d \in D$ such that $dar \neq 0$, and thus a module map $f: arR \rightarrow daR$ such that $f(ar) \neq 0$. Since $E(\mathrm{Im}\ f) \subseteqq D$, the map f has an extension $f': Q \rightarrow D$, and $f'(a) \neq 0$. Thus Q is D-torsionless and hence M-torsionless; by Proposition 0.1 M is a cogenerator for $\mathscr{N}$.

The following quite obvious corollaries to the above theorem are singled out for later usage.

COROLLARY 1.1.1. *A right ideal A of Q is a cogenerator for $\mathscr{N}$ if and only if A is a faithful $R-$ (or $Q-$) module.*

COROLLARY 1.1.2. *An injective nonsingular module M is a cogenerator for $\mathscr{N}$ if and only if M is faithful.*

The rest of this section is devoted to results about the existence and uniqueness of a minimal injective cogenerator for $\mathscr{N}$, this concept defined in the obvious manner as follows:

DEFINITION 1.2. A nonsingular module M is said to be a minimal injective cogenerator for $\mathscr{N}$ if (a) M is an injective cogenerator for $\mathscr{N}$ and (b) no direct summand of M different from M is a cogenerator for $\mathscr{N}$.

We have as a corollary to Theorem 1.1:

PROPOSITION 1.3. *An injective nonsingular module M is a minimal injective cogerator for $\mathscr{N}$ if and only if M is a minimal faithful module.*

The quotient field K of a commutative integral domain R is isomorphic to a submodule of every nonzero torsion-free injective R-module M. A similar result about a nonzero nonsingular injective module M and Q does not in general hold, even when M is assumed faithful. However some theory relating Q and the nonzero injective elements of $\mathscr{N}$ is possible and essentially a consequence of the following result

(Theorem Z) contained in a theorem of J. Zelmamowitz [8, Theorem 2]:

THEOREM Z. *If M is a nonsingular module then there exists a collection of right ideals of R, $\{I_\alpha: \alpha \in A\}$, such that M is an essential extension of a submodule isomorphic to $\bigoplus I_\alpha$* (external direct sum).

In view of Lemma 0.3 we have immediately:

COROLLARY Z.1. *A nonsingular injective module M is* (up to isomorphism) *the injective hull of a direct sum $\bigoplus e_\alpha Q$, where $\{e_\alpha: \alpha \in A\}$ is a set of* (not necessarily orthogonal) *idempotents of Q.*

We now take a closer look at the direct summands $eQ(e^2 = e)$ of Q on the way to establishing results on the existence and uniqueness of a minimal injective cogenerator for $\mathscr{N}$.

LEMMA 1.4. *If e and f are idempotents of Q other than 0 or 1, then* $\mathrm{Hom}\,(fQ, eQ) = (0)$ *if and only if eQ and fQ have no isomorphic nonzero direct summands.*

Proof. Clear. (Use Lemma 0.2 for the *if* part; use the injectivity of eQ, or fQ, for the *only if* part).

REMARK. If eQ and fQ have no isomorphic nonzero direct summands, then $eQfQ = 0$ and $fQeQ = 0$ because $\mathrm{Hom}_R(fQ, eQ) \cong eQf$ (as groups) and the fact that the condition on the direct summands is a symmetric one.

DEFINITION 1.5. We say that the modules M and N share a nonzero direct summand if M and N have isomorphic nonzero direct summands.

LEMMA 1.6. *If $\{f_\alpha: \alpha \in A\}$ is a set of idempotents in Q such that $N = E(\bigoplus^{f_\alpha} Q)$ is an injective cogenerator for $\mathscr{N}$, then for each nonzero idempotent e in Q, eQ shares a nonzero direct summand with some $f_\beta Q$.*

Proof. Let $f: eQ \to N$ be a homomorphism such that $f(e) \neq 0$. Since N is an essential extension of $\bigoplus f_\alpha Q$, there exists $r \in R$ such that $0 \neq f(e)r \in \bigoplus f_\alpha Q$ and so for some $\beta \in A$ $\pi_\beta f(e)r \neq 0$ where $\pi_\beta: \bigoplus f_\alpha Q \to f_\beta Q$ is the canonical projection. It follows now by Lemma 0.2 that $\pi_\beta f(er\ Q)$ is a nonzero direct summand shared by eQ and $f_\beta Q$.

COROLLARY. *If in Q there exist nonzero idempotents e_1 and e_2*

such that $e_1Qe_2Q = 0$ *then there also exist nonzero idempotents* f_1 *and* f_2 *such that* f_iQ, $i = 1, 2$, *is* (isomorphic to) *a submodule of* N *and* $f_1Qf_2Q = 0$ (N is an injective cogenerator for $\mathscr{N}$).

DEFINITION 1.7. (a) If e and f are idempotents in Q, the summands eQ and fQ of Q are said to be orthogonal if $eQfQ = (0)$.

(b) A nonzero right ideal B of Q is said to be only orthogonally decomposable if whenever $B = X \oplus Y$, for right ideals X and Y of Q, then $XY = (0)$.

It is easy to see that Q need not have any orthogonal summands different from (0) and Q; in fact we have:

LEMMA 1.8. *Q is a prime ring if and only if Q has no orthogonal summands other than* (0) *and* Q.

Proof. A prime ring is one in which, for example, the product of nonzero principal right ideals is nonzero. Every principal right ideal of Q is a direct summand of Q.

REMARK. If Q is not a prime ring and N is an injective cogenerator for $\mathscr{N}$, then N contains (isomorphic copies of) orthogonal nonzero summands of Q (Lemma 1.6).

Now we consider an existence theorem.

THEOREM 1.9. *The following statements are equivalent*:

(a) *There exists a minimal injective cogenerator M for* $\mathscr{N}$.

(b) *There exists a maximal set* $\{e_\alpha Q: \alpha \in A\}$ *of pairwise orthogonal summands of Q such that each $e_\alpha Q$ is only orthogonally decomposable.*

(c) *There exists a minimal faithful right ideal fQ for some idempotent f in Q.*

In particular if M is a minimal injective cogenerator for $\mathscr{N}$, then $M \cong fQ$ for some minimal faithful right ideal direct summand fQ of Q.

Proof. (a) *implies* (b). Let $\{e_\alpha Q: \alpha \in A\}$ be a set of summands of Q such that $M = E(\bigoplus e_\alpha Q)$ (given by Corollary Z.1). We show at once that the summands $\{e_\alpha Q\}$ are pairwise orthogonal and each only orthogonally decomposable. To this end suppose $e_\alpha Q$ and $e_\beta Q (\alpha \neq \beta)$ share a direct summand and so there exist module decompositions $e_\alpha Q = A' \oplus A''$ and $e_\beta Q = B' \oplus B''$ with $A' \cong B'$ and both A' and B' nonzero. These (decompositions) induce a module decomposition $M = A' \oplus B' \oplus C$; now since $A' \cong B'$ and M is a cogenerator for $\mathscr{N}$, it follows by the definition of cogenerator that, for example, $B' \oplus C$ is also a cogenerator for $\mathscr{N}$, contrary to the minimality of M. It follows that $e_\alpha Q$ and $e_\beta Q$ are orthogonal whenever $\alpha \neq \beta$ and, by the same argument, that each $e_\alpha Q$ is only orthogonally decomposable. Finally, the set $\{e_\alpha Q: \alpha \in A\}$ is a maximal set of pairwise orthogonal summands of Q by Lemma 1.6.

(b) *implies* (c). Let $fQ = E(\bigoplus e_\alpha Q)$, where $\{e_\alpha Q: \alpha \in A\}$ is as given in (b) (and so in particular the direct sum $\bigoplus e_\alpha Q$ is internal). To show that fQ is faithful, assume that, on the contrary, there exists a nonzero idempotent e in Q such that $fQe = (0)$; it follows that $e_\alpha QeQ = (0)$ for each $\alpha \in A$ and so $\{e_\alpha Q: \alpha \in A\} \cup \{eQ\}$ is a set of pairwise orthogonal summands of Q, properly containing $\{e_\alpha Q: \alpha \in A\}$, contrary to the latter's maximality. To show that fQ is minimal faithful, suppose that $fQ = B \bigoplus C$ where B is faithful and so, by Corollary 1.1.2, an injective cogenerator for $\mathscr{N}$. It needs to be shown that $C = (0)$. If C is not zero, that there exists a nonzero idempotent e in Q such that $eQ \subset C$ and since $\bigoplus e_\alpha Q \subseteq' fQ$ it may be assumed that $eQ \subset \bigoplus e_\alpha Q$. Furthermore as eQ shares a direct summand with some $e_\alpha Q$ (Lemma 1.6.) and as the $e_\alpha Q$'s are orthogonal, it may be assumed that $eQ \subset e_\beta Q$, for some $\beta \in A$. Now since B is an injective cogenerator for $\mathscr{N}$, it follows from Lemma 1.6 (and Corollary Z.1) that eQ shares a direct summand with some summand $e'Q$ of B, for some nonzero idempotent e' in Q, and it may be assumed that $e'Q$ is isomorphic to a summand of eQ. As in the case of eQ, it may be assumed that $e'Q$ is contained in one of the summands $e_\alpha Q$ and as they are orthogonal it must be that $e'Q \subset e_\beta Q$. Now since $e'Q \subset B$ and $eQ \subset C$, it follows from $B \cap C = (0)$, that $e'Q \cap eQ = (0)$ and thus $e'QeQ = (0)$, as $e_\beta Q$ is only orthogonally decomposable; however $e'QeQ \neq (0)$ (Lemma 1.4) and thus the assumption $C \neq (0)$ has led to a contradiction.

(c) *implies* (a). Proposition 1.3.

REMARKS. (1) It should be clear from the preceding considerations, that if Q is a prime ring then a minimal injective cogenerator for $\mathscr{N}$ exists if and only if Q has nonzero socle. The case in which Q (or R) is prime (including this remark) will be considered in detail in the next section.

(2) Any injective cogenerator M for $\mathscr{N}$ contains a submodule isomorphic to a faithful right ideal fQ, for some nonzero idempotent f of Q. In fact if $\{f_j Q: j \in J\}$ is a maximal set of pairwise orthogonal summands of Q contained in M (such exist by the Corollary to Lemma 1.6 in case Q is not prime) then fQ can be chosen to be $E(\bigoplus f_j Q)$.

The following is a uniqueness theorem.

THEOREM 1.10. *If M is a minimal injective cogenerator for $\mathscr{N}$, then*

(a) *M is unique up to isomorphism, and*

(b) *M is isomorphic to a submodule of every injective cogenerator N for $\mathscr{N}$.*

Proof. In view of Theorem 1.9 and Remark (2) following it, for a proof of both (a) and (b) of this theorem, it is sufficient to show that if e and f are idempotents in Q such that eQ is minimal faithful and fQ is faithful, then eQ is isomorphic to a direct summand of fQ. We show this next:

If an ideal A of Q is such that $eQfQA = (0)$ then $fQA = (0)$ and so $A = (0)$ as both eQ and fQ are faithful. It follows that $eQfQ = eQ$, as $eQfQ$ is a faithful direct summand of eQ. Thus there exist elements p and q in Q such that $e = epfq$ and so the homomorphism $h: fQ \rightarrow eQ$ given by $h(fx) = epfx$ is an epimorphism. It follows from Lemma 0.2 that eQ is isomorphic to a direct summand of fQ.

We do not know whether, in general, the property of being isomorphic to a submodule of every injective cogenerator for $\mathscr{N}$, characterizes the minimal injective cogenerator, among the injective cogenerators for $\mathscr{N}$.

A simple example to put the results of this section in some concrete form is the case when R is a commutative ring. It is easy to show that then Q, also, is a commutative ring and Q is only orthogonally decomposable. Q is the unique minimal injective cogenerator for $\mathscr{N}$.

2. Nonsingular uniform modules; rings of finite Goldie dimension; prime rings. The assumption that R is an associative ring with 1, of zero singular right ideal and that Q is its maximal right quotient ring, continues in force.

A module M is said to be finite dimensional (in the sense of Goldie) [4, p. 202] if it contains no infinite direct sum of nonzero submodules and we call R a finite dimensional ring if R_R is a finite dimensional module. A module U is said to be uniform if $U \neq 0$ and U is an essential extension of every one of its nonzero submodules. A uniform right ideal of R is, then, a uniform submodule of R_R.

For each module M, Soc(M) denotes the socle of M.

In this section we are primarily interested in nonsingular uniform modules, as when they exist "in abundance" (e.g. when the sum of uniform right ideals of R is essential in R_R), then they determine the minimal injective cogenerator for $\mathscr{N}$ in a simple way; in fact (they determine it) quite in the manner in which the (nonisomorphic) simple modules and their injective hulls determine the minimal injective cogenerator of the category of all modules. Thus we proceed in the following with a sequence of facts about nonsingular uniform modules (when they exist), some of them, probably, well known.

LEMMA 2.1. *A homomorphism $f: U \rightarrow A$ where U and A are nonsingular modules and U is uniform is either the zero map or a monomorphism.*

Proof. $U/\ker f$ is a nonsingular module, since A is and so if $\ker f \neq (0)$ then it must be that $\ker f = U$, since $U/\ker f$ is, then, its own singular submodule as well.

DEFINITION 2.2. A uniform module U is said to be equivalent to a uniform module V, and then we write $U \sim V$, if $E(U) \cong E(V)$ or, equivalently, if there exists monomorphism $A \rightarrow V$ for some nonzero submodule A of U.

It is clear that this relation is an equivalence relation on uniform modules.

LEMMA 2.3. *The following statement about a uniform module U are true*:

(a) $Z(U) = U$ *or* $Z(U) = (0)$

(b) *If* $Z(U) = (0)$, *then* $E(U)$ *is isomorphic* (as an R-module) *to a minimal right ideal of* Q.

(c) $Z(U) = (0)$ *if and only if* U *is equivalent to a uniform right ideal of* R.

Proof. (a) follows from the fact that $Z(U/Z(U)) = (0)$ [5, p. 270, Proposition 2.3] and at the same time, $Z(U/Z(U)) = U/Z(U)$, if $Z(U) \neq (0)$.

(b) If $Z(U) = (0)$ then there exists embedding (of R-modules) $U \rightarrow Q$ (Proposition 0.1 and Lemma 2.1). We may thus assume that U is a uniform R-submodule of Q and further assume that $U = qR$ for some $0 \neq q \in Q$, since $qR \sim U$ for every $0 \neq q \in U$. Thus $E(U) = E(qR) = qQ = eQ$ for some idempotent e in Q. Now eQ is a uniform R-submodule of Q_R, as U is, and so eQ is a uniform ideal of Q. Since Q is Von Neumann regular, it follows that eQ is a minimal right ideal of Q.

(c) Since injective hulls of nonsingular modules are nonsingular modules, $Z(U) = (0)$ if U is equivalent to a uniform right ideal of R. On the other hand, using the notation of part (b) above if $Z(U) = (0)$, we have $I = eQ \cap R$, a uniform right ideal of R such that $E(I) = eQ = E(U)$.

REMARK. Nonsingular uniform modules need not exist: a ring R such that $\operatorname{Soc}(Q_Q) = (0)$ exists (see Example following the Corollary to Proposition 2.11).

DEFINITION 2.4 (Terminology). For each module M, $\mathscr{U}(M)$ denotes the (module) sum of all uniform submodules of M. If M has no uniform submodules, then we write: $\mathscr{U}(M) = (0)$.

Any finite dimensional ring R satisfies $\mathscr{U}(R_R) \subseteq' R_R$ [4, p. 202, Theorem 1.1]. If, on the other hand, R is an infinite direct product of fields, then $\mathscr{U}(R_R) \subseteq' R_R$ but R is not finite dimensional.

PROPOSITION 2.5. *For a ring R the following statements are equivalent*:

(a) $\mathscr{U}(R_R) \subseteq' R_R$

(b) $\mathscr{U}(M) \subseteq' M$ *for every nonzero nonsingular module* M

(c) $\operatorname{Soc}(Q_Q) \subseteq' Q_Q$.

Proof. (a) *implies* (b). It is sufficient to show that $\mathscr{U}(M) \neq (0)$ in case M is a nonzero, nonsingular cyclic module. If M is such, then there exists epimorphism $f: R_R \to M$; now it cannot happen that $f(U(R_R)) = (0)$ as $Z(M) = (0)$, and this would imply $f(R) = (0)$, though $M \neq 0$. Thus $f(U) \neq 0$ for some uniform right ideal of R and it follows from Lemma 2.1 that $U \cong f(U) \subset M$.

(b) *implies* (c). Condition (b) in particular implies that $\mathscr{U}(R_R) \subseteq' R_R$ and so (c) follows from part (b) of Lemma 2.3.

(c) *implies* (a). For each minimal right ideal A of Q, $A \cap R$ is a uniform right ideal of R and so if $\operatorname{Soc}(Q_Q) = \Sigma A_i$ where $\{A_i\}$ are the minimal right ideal of Q, then $\Sigma(A_i \cap R) \subseteq \mathscr{U}(R)$. On the other hand $\Sigma(A_i \cap R) \subseteq' (\Sigma A_i) \cap R \subseteq' R_R$.

REMARK. Finite dimensional rings R have been characterized by F. L. Sandomierski [6, p. 115, Theorem 1.6] as those for which $\operatorname{Soc}(Q_Q) = Q_Q$.

THEOREM 2.6. *Let R be a ring such that $\mathscr{U}(R_R) \subseteq' R_R$ and let $\{e_\alpha Q : \alpha \in A\}$ be a complete set of non-isomorphic minimal right ideals of Q, where $\{e_\alpha : \alpha \in A\}$ is an appropriate set of primitive, orthogonal idempotents of Q. The ideal $eQ = E(\bigoplus e_\alpha Q)$ is, then, a minimal injective cogenerator for $\mathscr{N}$.*

Proof. We apply Theorem 1.9 (b). The summands $\{e_\alpha Q\}$ are pairwise orthogonal because they are (pairwise) non-isomorphic minimal right ideals of Q and each $e_\alpha Q$ is, clearly, only orthogonally decomposable. It remains to show that the set $\{e_\alpha Q: \alpha \in A\}$ is a maximal set of pairwise orthogonal summands of Q; now if eQ is a nonzero summand of Q, then $\mathrm{Soc}\,(eQ) \neq (0)$, by Proposition 2.5 (c), and so eQ contains a nonzero summand isomorphic to some $e_\alpha Q$, as $\{e_\alpha Q\colon \alpha \in A\}$ is a complete set of non-isomorphic minimal right ideals of Q. Thus $e_\alpha QeQ \neq (0)$, for some $\alpha \in A$.

In view of Theorem 1.10 (b) and Lemma 2.3 (b), the following corollary to Theorem 2.6 is immediate.

COROLLARY. *If R is a ring such that $\mathscr{Z}(R_R) \subseteq' R_R$ and M is a nonsingular module, then M is a cogenerator for $\mathscr{N}$ if, and only if M contains a copy of the injective hull of every uniform nonsingular module.*

THEOREM 2.7. *If R is finite dimensional, then the following statements are equivalent:*

(a) *Q is a minimal injective cogenerator for $\mathscr{N}$.*

(b) Q (as a ring) *is isomorphic to a finite direct product $\Delta_i \oplus \cdots \oplus \Delta_n$ of division rings Δ_i.*

Proof. (a) implies (b). Q is artinian semi-simple in this case [6, p. 115, Theorem 1.6] and so there exist primitive orthogonal idempotents $e_1, \cdots, e_n$ such that $Q = e_1Q \oplus \cdots \oplus e_nQ$. Condition (a) now implies that the minimal ideals e_iQ are (pairwise) non-isomorphic. It follows now from the structure theory of artinian semi-simple rings that each e_iQ is a division ring and the sum $e_1Q \oplus \cdots \oplus e_nQ$ is a ring direct sum.

(b) implies (a). Each Δ_i is a minimal Q-ideal and they (the ideals Δ_i) are non-isomorphic. Now use Theorem 2.6.

Now we turn our attention to the case when R is a prime ring.

LEMMA 2.8. *For a ring R the following statements are equivalent:*

(a) *R is a prime ring.*

(b) *Every nonzero, nonsingular module M is a faithful module.*

Proof. (a) implies (b). Over a prime ring R a two-sided ideal of R is either an essential submodule of R_R or it is zero (e.g. [1, p. 570, Lemma 3]). Since $Z(M) = 0$, it follows that $r.\,\mathrm{ann}_R(M) = (0)$.

(b) implies (a). A nonzero right ideal A of R is a nonzero, nonsingular module and so $r.\,\mathrm{ann}_R A = (0)$.

PROPOSITION 2.9. *Let R be a prime ring and let M be a nonzero, nonsingular injective module. The following statements about M are, then, equivalent:*

(a) *M is a minimal injective cogenerator for $\mathscr{N}$.*

(b) *A nonzero homomorphism* $f: M \to C$, *where* C *is nonsingular, is a monomorphism.*

(c) M *is a uniform module.*

Proof. (a) *implies* (b). It follows from Lemma 0.2 that $M \to \operatorname{Im} f$ splits and so $\ker f$ is a direct summand of M. Now if $\ker f \neq 0$ then $\ker f$ is faithful (Lemma 2.8) contrary to minimality of M (Proposition 1.3). Thus it must hold that $\ker f = 0$.

(b) *implies* (c). If M is not uniform then, since M is injective, it is possible to find nonzero submodules A and B of M such that $M = A \oplus B$. Such a decomposition, however, gives rise to a homomorphism, e.g. the projection $M \to B$, of the kind which is forbidden by condition (b).

(c) *implies* (a). M is faithful, by Lemma 2.8, and so M is an injective cogenerator by Corollary 1.1.2. Condition (c) implies that M has no direct summands other than (0) and M and so (a) follows.

THEOREM 2.10. *If* R *is a prime ring, then a minimal injective cogenerator for* $\mathscr{N}$ *exists if and only if* $\mathscr{U}(R_R) \neq (0)$. *Furthermore if* $\mathscr{U}(R_R) \neq (0)$ *then* $\mathscr{U}(R_R) \subseteqq' R_R$ *and there is only one* (up to isomorphism) *nonsingular simple* Q*-module.*

Proof. If $\mathscr{U}(R_R) \neq (0)$ then there exists a uniform right ideal U of R and thus also a nonsingular injective uniform R-module, namely $E(U) \subset Q$. It follows from Proposition 2.9 that $E(U)$ is a minimal injective cogenerator for $\mathscr{N}$. On the other hand if M is a minimal injective cogenerator for $\mathscr{N}$ then M is uniform (Proposition 2.9) and so $\mathscr{U}(R_R) \neq (0)$ (Lemma 2.3 (c)). Now for the second part of the theorem, assume $\mathscr{U}(R_R) \neq (0)$. It follows (Lemma 2.3 (b)) that Q has a minimal right ideal fQ, where f is some (primitive) idempotent of Q; now if eQ is any nonzero summand of Q then $fQeQ \neq (0)$, as Q is prime, and so $\operatorname{Soc}(eQ) \neq (0)$ (by Lemma 1.4, for example). It follows that $\operatorname{Soc}(Q_Q) \subseteqq' Q_Q$ and so $\mathscr{U}(R_R) \subseteqq' R_R$. Finally if S is any nonsingular simple Q-module, then $S \cong gQ$ for some primitive idempotent g of Q (by Lemma 0.2) and, as before, $fQ \cong gQ$.

COROLLARY. *If* R *is a prime ring then* Q *is a minimal injective cogenerator for* $\mathscr{N}$ *if and only if* Q *is a division ring.*

Proof. If Q is a minimal injective cogenerator for $\mathscr{N}$ then Q is a uniform R-module (Proposition 2.9) and thus Q_Q is a minimal right Q-ideal. A ring with $1 \neq 0$ and no right ideals other than zero and itself is, of course, a division ring.

Now we obtain an example of a prime ring R such that $\mathscr{Z}(R) = (0)$.

By a simple ring R we mean a ring in which (0) and R are the only two-sided ideals.

PROPOSITION 2.11. *If R has no divisors of zero $\neq 0$ then Q is a simple ring.*

Proof. Let A be a nonzero two-sided ideal of Q and thus consider $a \in R \cap A$, $a \neq 0$ ($R \cap A \neq (0)$ as $R_R \subseteqq' Q_R$). Since R has no divisors of zero $\neq 0$ we have $r.\operatorname{ann}_R a = (0)$ and, hence, $r.\operatorname{ann}_Q a = (0)$. Now $Qa = Qe$ for some idempotent e in Q and so $a(1-e) = 0$. Thus $1 - e \in r.\operatorname{ann}_Q a = (0)$ and so $e = 1$; we have $Qa = Q$, for some $a \in A$, and so $Q = A$.

COROLLARY. *If R has no divisors of zero $\neq 0$ then a nonsingular uniform module exists if and only if Q is a division ring.*

Proof. If $\mathscr{Z}(R) \neq (0)$ then Soc $(Q) \neq (0)$ and as Soc (Q) is a two-sided ideal, it follows that Soc $(Q) = Q$ or that Q is simple semi-simple artinian. Now every element of $R - \{0\}$ is invertible in Q and so every element q of Q has the form ad^{-1} for appropriate elements a and d of R (i.e. Q is also the classical quotient ring [2] of R). It follows that if $q \neq 0$ then $q^{-1} = da^{-1}$ exists and Q is a division ring.

AN EXAMPLE. Since a finite dimensional ring R (of zero singular right ideal) has an artinian semi-simple maximal quotient ring Q [6, p. 115, Theorem 1.6] it follows from the proof of the above corollary that a ring R which has no divisors of zero $\neq 0$ can be one of only two Goldie dimensions (as a right R-module): either of dimension one (i.e. either R_R is a uniform module) or of infinite dimension (i.e. $\mathscr{Z}(R) = (0)$). A ring R of the latter kind is the ring $R = K[x, y]$ where K is a field and x, y are non-commuting indeterminates (but $\alpha x = x\alpha$ and $\alpha y = y\alpha$ for all $\alpha \in K$). Since $xR \cap yR = (0)$, R_R is not uniform and so $\mathscr{Z}(R_R) = (0)$.

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UNIVERSITY OF KENTUCKY