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THE EVALUATION MAP AND *EHP* SEQUENCES

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Let $L(\Sigma B, X)$ be the space of maps from ΣB (the reduced suspension of B) to X with the compact-open topology, let $\ell\colon \Sigma B \to X$ and $L(\Sigma B, X; \ell)$ the path component of $L(\Sigma B, X)$ containing ℓ. For nice spaces the evaluation map ω: $L(\Sigma B, X, \ell) \to X$ defined by $\omega(f) = f(*)$ is a fibration and gives rise to a long exact sequence in homotopy. The purpose of this paper is to show that the boundary map in that long exact sequence can be given by a generalized Whitehead product and that the sequence generalizes the *EHP* sequence of G. W. Whitehead.

1. Preliminary definitions. All spaces are assumed to be CW complexes with base point at a vertex. Maps are base point preserving. The cartesian product $A \times B$ is assumed to be based at (a_0, b_0), the unit inverval, I, is based at 0, and quotient spaces are based at the image of the base point under the natural quotient map. Where the space is clear * will denote the base point as well as the constant map with image at the base point.

We use the following notations. $L(A, B)$ will denote the space of maps from A to B with the compact-open topology and $L(A, B; \ell)$ the path component of $L(A, B)$ containing $\ell\colon A \to B$. $L_0(A, B)$ and $L_0(A, B; \ell)$ will denote the space of base point preserving maps in $L(A, B)$ and $L(A, B; \ell)$ respectively. Let $A \vee B$ and $A \# B$ denote the one point union and smash product respectively.

Since spaces are assumed to be CW complexes the smash product can be taken as $A \times B$ with $A \vee B$ identified with (a_0, b_0). $q\colon A \times B \to A \# B$ will denote the quotient map. Note that $S^{p+q} = \Sigma^p S^q = S^p \# S^q$, $\Sigma^p A = S^p \# A$, and $\Sigma(A \vee B) = \Sigma A \vee \Sigma B$.

Let $p_1, p_2\colon A \times B \to A \vee B$ be defined by $p_1(a, b) = a \vee b_0$ and $p_2(a, b) = a_0 \vee b$. Define $k\colon \Sigma(A \times B) \to \Sigma A \vee \Sigma B$ by $k = \Sigma p_1 + \Sigma p_2 - \Sigma p_1 - \Sigma p_2$. Since $k \mid \Sigma(A \vee B)$ homotopically trivial, by the homotopy extension property there is a map $k'\colon \Sigma(A \times B) \to \Sigma A \vee \Sigma B$, homotopic to k, such that $k' \mid \Sigma(A \vee B) = *$. k' then induces a map $\tilde{k}\colon \Sigma(A \# B) \to \Sigma A \vee \Sigma B$. Arkowitz [1] has shown that $[\tilde{k}]$ is uniquely determined by the requirement $k \cong \tilde{k} \circ \Sigma q$. The following definition is due to Arkowitz [1].

DEFINITION 1.1. For $\alpha = [f] \in [\Sigma A, X]$ and $\beta = [g] \in [\Sigma B, X]$, the *generalized Whiteheal product* $[\alpha, \beta]$ is defined by $[\alpha, \beta] = [(f \vee g) \circ \tilde{k}] \in [\Sigma(A \# B) X]$.

Hardie shows (Theorem 2.3 in [2]) that the map $\Sigma p_1 + \Sigma p_2 + \Sigma q\colon \Sigma(A \times B) \to \Sigma A \vee \Sigma B \vee \Sigma(A \# B)$ is a homotopy equivalence for A and B CW complexes with a single vertex. Then there is a map $\phi\colon \Sigma(A \# B) \to \Sigma(A \times B)$ such that $\Sigma q \circ \phi \cong 1_{\Sigma(A \# B)}$.

DEFINITION 1.2. If $f\colon A \times B \to X$, where A and B have a single vertex, the element obtained from f by the *generalized Hopf construction* is defined to be the map $\Sigma f \circ \phi\colon \Sigma(A \# B) \to \Sigma X$.

Hardie shows in [2] that if A and B are spheres, Definition 1.2 reduces to the classical definition of the Hopf construction.

Let $\phi_r\colon S^r \to S^r \vee S^r$ be the map which identifies the equator of S^r. G. W. Whitehead (Theorem 1.17 in [6]) shows for $n < p + q + \min(p, q) - 3$ that $\pi_n(S^p \vee S^q) = \pi_n(S^p) \oplus \pi_n(S^q) \oplus \pi_n(S^{p+q-1})$. Let $Q\colon \pi_n(S^p \vee S^q) \to \pi_n(S^{p+q-1})$ be the natural projection onto the direct summand $\pi_n(S^{p+q-1})$.

DEFINITION 1.3. For $n < 3r - 3$ the *generalized Hopf invariant* $\widetilde{H}\colon \pi_n(S^r) \to \pi_n(S^{2r-1})$ is defined by $\widetilde{H} = Q \circ \phi_{r*}$.

DEFINITION 1.4. For $\lambda = [\ell] \in [\Sigma B, X]$ the λ-Whitehead homomorphism $P_\lambda\colon [\Sigma A, X] \to [\Sigma(A \# B), X]$ is defined by $P_\lambda(\alpha) = [\alpha, \lambda]$.

DEFINITION 1.5. If $F\colon A \to L(B, X)$ the map $G\colon A \times B \to X$ given by $G(a, b) = F(a)(b)$ is said to be an associated map for F.

2. The λ-component EHP sequence. The purpose of this section is to show that the map P_λ of Definition 1.4 is embedded in a long exact sequence resulting from the fibration $\omega\colon L(\Sigma B, X; \ell) \to X$. Each $\lambda \in [\Sigma B, X]$ determines a path component of $L(\Sigma B, X)$ and ω restricted to each path component determine a fibration and a long exact sequence. In § 3 the relationship between these sequences and the James suspension sequence is explored and it is shown that G. W. Whitehead's EHP sequence [7] is a special case of an ι_n-component EHP sequence where $\iota_n = [1_{S^n}]$ in $\pi_n(S^n)$.

LEMMA 2.1. *For $\ell \in L_0(\Sigma B, X)$, $L_0(\Sigma B, X; *)$ is homotopy equivalent to $L_0(\Sigma B, X; \ell)$.*

Proof. Let $\hat{\ell}\colon L_0(\Sigma B, X; *) \to L_0(\Sigma B, X; \ell)$ be defined by $\hat{\ell}(g) = g + \ell$ and $\hat{\ell}^{-1}\colon L_0(\Sigma B, X; \ell) \to L_0(\Sigma B, X; *)$ by $\hat{\ell}^{-1}(g) = g - \ell$. Then it is clear that $\hat{\ell}^{-1}$ is a two sided homotopy inverse of $\hat{\ell}$.

In remaining parts of this section the map $\hat{\ell}$ will be taken to be given by

$$\hat{\ell}(g)(b, t) = \begin{cases} g\left(b, \frac{5}{4}t\right) & 0 \leqq t \leqq \frac{4}{5} . \\ \ell(b, 5t - 4) & \frac{4}{5} \leqq t \leqq 1 . \end{cases}$$

LEMMA 2.2. *$[\Sigma(A \# B), X]$ is isomorphic to $[A, L_0(\Sigma B, X; *)]$.*

This fact is well know. For the remainder of this section the isomorphism will be denoted by $\theta: [\Sigma(A \# B), X] \to [A, L_0(\Sigma B, X; *)]$ defined as follows. If $f: \Sigma(A \# B) \to X$, $\theta(f)(a)$ is the map taking (b, t) to $f((a, b), t)$ in X.

DEFINITION 2.3. $A @ B$ is defined as $A \times B$ with $A \times \{b_0\}$ identified with (a_0, b_0).

Let $m: A \times \Sigma B \to (A \# \Sigma B) \vee (A @ \Sigma B)$ be defined by

$$(m(a, (b,)) = \begin{cases} \left(a, \left(b, \frac{5}{4}t\right)\right) \vee * & 0 \leqq t \leqq \frac{4}{5} \\ * \vee (a, (b, 5t - 4)) & \frac{4}{5} \leqq t \leqq 1 . \end{cases}$$

Now let $G: A \# \Sigma B \to X$ be a map associated with $[g] \in [A, L_0(\Sigma B, X; *)]$, $\ell \in L_0(\Sigma B, X)$, and $p_2: A @ \Sigma B \to \Sigma B$ the natural projection.

The following lemma can be easily verified.

LEMMA 2.4. *$(G \vee (\ell \circ p_2)) \circ m: A \times \Sigma B \to X$ is an associated map for $\hat{\ell}_*([g]) \in [A, L_0(\Sigma B, X; \ell)]$.*

Let $h_1: A \times SB \to \Sigma(A \times B)$ be defined by $h_1(a, (b, t)) = ((a, b), t)$, where SA is the unreduced suspension.

By the homotopy extension property the quotient map $q_1: SB \to \Sigma B$ is a homotopy equivalence. Its homotopy inverse will be denoted $q_1^{-1}: \Sigma B \to SB$.

LEMMA 2.5. *Let $\alpha = [f] \in [\Sigma A, X]$ and $\lambda = [\ell] \in [\Sigma B, X]$, then $(f \vee \ell) \circ (\Sigma p_1 + \Sigma p_2 - \Sigma p_1 - \Sigma p_2 + \Sigma p_2) \circ h_1 \circ (1_A \times q_1^{-1}): A \times \Sigma B \to X$ is a map associated with $\hat{\ell}_* \circ \theta([\alpha, \lambda]) \in [A, L_0(\Sigma B, X; \ell)]$.*

Proof. Let $m_1: \Sigma(A \times B) \to \Sigma(A \times B) \vee \Sigma(A \times B)$ be given by

$$m_1((a, b), t) = \begin{cases} \left((a, b), \frac{5}{4}t\right) \vee * & 0 \leqq t \leqq \frac{4}{5} \\ * \vee ((a, b), 5t - 4) & \frac{4}{5} \leqq t \leqq 1 . \end{cases}$$

Consider the following diagram:

$$\begin{tikzcd}
 & & & & & X \\
 & \Sigma(A\times B) \arrow[rrr, "\Sigma p_1+\Sigma p_2-\Sigma p_1-\Sigma p_2+\Sigma p_2"] \arrow[dr, "m_1"] & & & \Sigma A\vee\Sigma B \arrow[ur, "f\vee\ell"] & \\
 & & \Sigma(A\times B)\vee\Sigma(A\times B) \arrow[urr, "\Sigma p_2"'] \arrow[drr, "\Sigma q"] & & & \\
 & & & & \Sigma(A\# B) \arrow[uu, "\tilde{k}"] & \\
A\times\Sigma B \arrow[r, "1_A\times q_1^{-1}"] & A\times SB \arrow[uuu, "h_1"] \arrow[rr, "1_A\times q_1"] & & A\times\Sigma B \arrow[r, "m"] & A\#\Sigma B\vee A@\Sigma B \arrow[u, "q'"] \arrow[uuuur, "\ell\circ p_2"'] &
\end{tikzcd}$$

$q'\colon A\#\Sigma B\to\Sigma(A\# B)$ is the homomorphism defined by $q'(a,(b,t))=((a,b),t)$ and k is as in Definition 1.1. It is easiest to check the homotopy commutativity of this diagram by looking first at the lower four fifths of the t coordinate in SB and then at the upper fifth.

Part 1. $\Sigma p_1+\Sigma p_2-\Sigma p_1-\Sigma p_2+\Sigma p_2\cong((\tilde{k}\circ\Sigma q)\vee\Sigma p_2)\circ m_1$.

The lower four fifths of $\Sigma(A\times B)$ is mapped in one case by $\Sigma p_1+\Sigma p_2-\Sigma p_1-\Sigma p_2$ and in the other by $\tilde{k}\circ\Sigma q$. But these are homotopic by the definition of $\tilde{k}$. The upper fifth is mapped by Σp_2 in either case.

Part 2. $(f\vee\ell)\circ(\tilde{k}\circ\Sigma q\vee\Sigma p_2)\circ m_1\circ h_1=(((f\vee\ell)\circ\tilde{k}\circ q')\vee\ell\circ p_2)\circ m\circ(1_A\times q_1)$.

On the lower four fifths the maps "meet" at $\Sigma(A\# B)$. In either case the point $(a,(b,t))\in A\times SB$ is mapped to $((a,b),(5/4)t)\in\Sigma(A\# B)$. On the upper fifth both maps are given by taking $(a,(b,t))$ to $\ell((b,5t-4)$ in X.

By Lemma 2.4 and the definition of $[\alpha,\lambda]$, $(((f\vee\ell)\circ\tilde{k}\circ q')\vee(\ell\circ p_2))\circ m$ is an associated map for $\ell_*\theta([\alpha,\lambda])$. This is the lower route in the above diagram. Since q_1 and q_1^{-1} are homotopy inverses

$$\begin{aligned}
&(((f\vee\ell)\circ\tilde{k}\circ q')\vee\ell\circ p_2)\circ m\\
&\quad\cong(((f\vee\ell)\circ\tilde{k}\circ q')\vee\ell\circ p_2)\circ m\circ(1_A\times q_1)\circ(1_A\times q_1^{-1})\\
&\quad\cong(f\vee\ell)\circ(\tilde{k}\circ\Sigma q\vee\Sigma p_2)\circ m_1\circ h_1\circ(1_A\times q_1^{-1})\\
&\quad\cong(f\vee\ell)\circ(\Sigma p_1+\Sigma p_2-\Sigma p_1-\Sigma p_2+\Sigma p_2)\circ h_1\circ(1_A\times q_1^{-1})\,.
\end{aligned}$$

The last two homotopies follow from Part 2 and Part 1 respectively. But the last map is the one claimed to be an associated map for

$\hat{\ell}_*\theta([\alpha, \lambda])$ and the lemma is proven.

For $\lambda = [\ell] \in [\Sigma B, X]$ the evaluation map $\omega\colon L(\Sigma B, X; \ell) \to X$ is a fibration with fiber $L_0(\Sigma B, X; \ell)$. Then there is a long exact sequence of homotopy groups

$$\begin{aligned} \cdots \longrightarrow [\Sigma^{r+1}A, X] &\xrightarrow{\partial} [\Sigma^r A, L_0(\Sigma B, X; \ell)] \xrightarrow{i_*} \\ [\Sigma^r A, L(\Sigma B, X; \ell)] &\xrightarrow{\omega_*} [\Sigma^r A, X] \xrightarrow{\partial} \cdots \longrightarrow [\Sigma A, X] \xrightarrow{\partial} \\ [A, L_0(\Sigma B, X; \ell)] &\xrightarrow{i_*} [A, L(\Sigma B, X; \ell)] \xrightarrow{\omega_*} [A, X]\,, \end{aligned}$$

where exactness at the last two stages is as pointed sets. Recall that Lemma 2.2 shows there is an isomorphism $\theta\colon [\Sigma(A \# B), X] \to [A, L(\Sigma B, X; *)]$.

THEOREM 2.6. *For* $\alpha \in [\Sigma^r A, X]$, $\partial(\alpha) = \hat{\ell}_* \circ \theta \circ P_\lambda(\alpha)$.

Proof. Let α be represented by a map $f\colon \Sigma^r A \to X$ and let $q_2\colon C(\Sigma^{r-1}A) \to \Sigma^r A$ be the natural quotient map from the cone to the suspension. Define $F\colon C(\Sigma^{r-1}A) \times SB \to \Sigma(\Sigma^{r-1}A) \vee \Sigma B$ by

$$F((a, r), (b, t)) = \begin{cases} q_2(a, r+3t) \vee * & 0 \leqq t \leqq \frac{1}{3} \quad r \leqq -3t+1 \\ * & 0 \leqq t \leqq \frac{1}{3} \quad r \geqq -3t+1 \\ * \vee q_1(b, 3t-1) & \frac{1}{3} \leqq t \leqq \frac{2}{3} \\ * & \frac{2}{3} \leqq t \leqq 1 \quad r \geqq 3t-2 \\ q_2(a, 3+r-3t) \vee * & \frac{2}{3} \leqq t \leqq 1 \quad r \leqq 3t-2 \end{cases}$$

where $(a, r) \in C(\Sigma^{r-1}A)$, r being the level on the cone and $(b, t) \in SB$, t being the level on the suspension. At $t = 1/3$ or $2/3$ and on the lines $r = -3t+1$ and $r = 3t-2$, the image of F is at $*$ and F is well defined and continuous at these points. Since F is independent of a at $r = 1$ and independent of b at $t = 1$ and $t = 0$, F is well defined. Let $\Sigma^{r-1}A \times SB \to C(\Sigma^{r-1}A) \times SB$ be induced by including $\Sigma^{r-1}A$ at the 0 level of $C(\Sigma^{r-1}A)$. Consider the following diagram:

$$\begin{tikzcd} C(\Sigma^{r-1}A) \vee SB \arrow[d] \arrow[dr, "q_2 \vee q_1"] & \\ C(\Sigma^{r-1}A) \times SB \arrow[r, "F"] & \Sigma(\Sigma^{r-1}A) \vee \Sigma B \arrow[d, "\Sigma p_1 + \Sigma p_2 - \Sigma p_1"] \\ \Sigma^{r-1}A \times SB \arrow[u] \arrow[r, "h_1"] & \Sigma(\Sigma^{r-1}A \times B)\,. \end{tikzcd}$$

The map $(\Sigma p_1 + \Sigma p_2 - \Sigma p_1)\circ h_1$ is given by

$$(a, (b, t)) \longrightarrow \begin{cases} (a, 3t) \vee * & 0 \leqq t \leqq \dfrac{1}{3} \\ * \vee q_1(b, 3t-1) & \dfrac{1}{3} \leqq t \leqq \dfrac{2}{3} \\ (a, 3-3t) \vee * & \dfrac{2}{3} \leqq t \leqq 1 . \end{cases}$$

But this is the same as $F((a, 0), (b, t))$, that is $F \mid \Sigma^{r-1}A \times SB$. Therefore the lower square commutes. In the upper triangle, when $t = 0$ (the base point of SB), $F((a, r), (b, 0)) = q_2(a, r)$ by definition. At the base point of $C(\Sigma^{r-1}A)$ consider

$$F((a_0, 0), (b, t)) = \begin{cases} q_2(a_0, 3t) \vee * = * & 0 \leqq t \leqq \dfrac{1}{3} \\ * \vee q_1(b, 3t-1) & \dfrac{1}{3} \leqq t \leqq \dfrac{2}{3} \\ q_2(a_0, 3-3t) \vee * = * & \dfrac{2}{3} \leqq t \leqq 1 . \end{cases}$$

But this is clearly homotopic to $* \vee q_1$, thus the upper triangle commutes up to homotopy. Now consider the map $\tilde{F}\colon C(\Sigma^{r-1}A) \times \Sigma B \rightarrow X$ given by $\tilde{F} = (f \vee \ell)\circ F \circ (1_{C(\Sigma^{r-1}A)} \times q_1^{-1})$. $\tilde{F}$ is then an associated map for an element of $[(C(\Sigma^{r-1}A), \Sigma^{r-1}A), (L(\Sigma B, X; \ell), L_0(\Sigma B, X; \ell))]$. Since $\tilde{F} \mid C(\Sigma^{r-1}A) \times *$ is given by $f \circ q_2$, $\tilde{F}$ is associated to the class $[f] \in [\Sigma^r A, X]$ under the bijection (see p. 104 in [5]) $\omega_*\colon [(C(\Sigma^{r-1}A), \Sigma^{r-1}A), (L(\Sigma B, X; \ell), L_0(\Sigma B, X; \ell))] \rightarrow [\Sigma^r A, X]$. Then by definition of the boundary homomorphism, $\partial([f]) = \partial(\alpha)$ has associated map $\tilde{F} \mid \Sigma^{r-1}A \times \Sigma B$. But by commutativity of the above diagram $\tilde{F} \mid \Sigma^{r-1}A \times \Sigma B \cong (f \vee \ell)\circ(\Sigma p_1 + \Sigma p_2 - \Sigma p_1)\circ h_1 \circ (1_{\Sigma^{r-1}A} \times q_1^{-1})$ and by Lemma 2.5 this is an associated map for $\hat{\ell}_* \circ \theta \circ ([\alpha, \lambda]) = \hat{\ell}_* \circ \theta \circ P_\lambda(\alpha)$.

The existence of the λ-component *EHP* sequence now can be shown. Let $i'_*\colon [\Sigma(\Sigma^{r-1}A \# B), X] \rightarrow [\Sigma^{r-1}A, L(\Sigma B, X; \ell)]$ be given by $i'_* = i_* \circ \hat{\ell}_* \circ \theta$.

THEOREM 2.7. *There is a long exact sequence*

$$\cdots \longrightarrow [\Sigma^r A, L(\Sigma B, X; \ell)] \xrightarrow{\omega_*} [\Sigma^r A, X] \xrightarrow{P_\lambda}$$
$$[\Sigma(\Sigma^{r-1}A \# B); X] \xrightarrow{i'_*} [\Sigma^{r-1}A, L(\Sigma B, X; \ell)] \longrightarrow \cdots$$

Proof. Since $\hat{\ell}_*$ and θ are isomorphisms, the exactness of this sequence is immediate from the exactness of the homotopy exact sequence of the fibration $\omega_*\colon L(\Sigma B, X; \ell) \rightarrow X$ and Theorem 2.6.

3. The Whitehead and James sequences. The purpose of this section is to compare the λ-component *EHP* sequence with the classical *EHP* sequence of George W. Whitehead [7] and the suspension sequence of I. M. James [4]. The spaces A and B will be assumed to be CW complexes with a single vertex. For $\alpha \in [A, L(\Sigma B, X; \ell)]$ the element $H(\alpha) \in [\Sigma(A \# \Sigma B), \Sigma X]$ is defined by the element obtained from a map associated with α by the Hopf construction of Definition 1.2. The homomorphism $E\colon [A \# \Sigma B, X] \to [\Sigma(A \# \Sigma B), \Sigma X]$ is defined by $E([f]) = [\Sigma f]$.

LEMMA 3.1. *The following diagram commutes*:

$$\begin{tikzcd}
{[A \# \Sigma B, X]} \arrow[r, "E"] \arrow[d, "\theta", "\cong"'] & {[\Sigma(A \# \Sigma B), \Sigma X]} \arrow[dd, "H"] \\
{[A, L_0(\Sigma B, X; *)]} \arrow[d, "\hat{\ell}_*", "\cong"'] & \\
{[A, L_0(\Sigma B, X; \ell)]} \arrow[r, "i_*"] & {[A, L(\Sigma B, X; \ell)]}\,.
\end{tikzcd}$$

Proof. Let $f\colon A \# \Sigma B \to X$ represent an element of $[A \# \Sigma B, X]$. Then $\hat{\ell}_* \circ \theta([f])$ has an associated map $F\colon A \times \Sigma B \to X$ given by

$$F(a, (b, t)) = \begin{cases} f(a, (b, 2t) & 0 \leqq t \leqq \dfrac{1}{2} \\ \ell(b, 2t - 1) & \dfrac{1}{2} \leqq t \leqq 1\,. \end{cases}$$

Let $\phi\colon \Sigma(A \# \Sigma B) \to \Sigma(A \times \Sigma B)$ and $q\colon A \times \Sigma B \to A \# \Sigma B$ be as in the comments preceding Difinition 1.2. Consider the following diagram:

$$\begin{tikzcd}
 & & \Sigma X \\
\Sigma(A \times \Sigma B) \arrow[urr, "\Sigma F"] \arrow[rr, "\Sigma q + \Sigma p_2"'] & & \Sigma(A \# \Sigma B) \vee \Sigma\Sigma B \arrow[u, "\Sigma f \vee \Sigma l"'] \\
\Sigma(A \# \Sigma B) \arrow[u, "\phi + *"] \arrow[urr, "i_1"'] & &
\end{tikzcd}$$

where i_1 is the inclusion of $\Sigma(A \# \Sigma B)$ in $\Sigma(A \# \Sigma B) \vee \Sigma\Sigma B$. The homotopy commutativity of this diagram will establish the result since $\Sigma F \circ (\phi + *) \cong \Sigma F \circ \phi$ which by definition is the element obtained from

$\hat{\ell}_* \circ \theta([f])$ by the Hopf construction and $(\Sigma f \vee \Sigma \ell) \circ i_1 = \Sigma f$, a representative of $E([f])$.

In the lower triangle of the diagram $(\Sigma q + \Sigma p_2) \circ (\phi + *) \cong \Sigma q \circ \phi$ which is homotopic to i_1 by the definition of ϕ.

In the upper triangle

$$\Sigma F(a, (b, t), r) = \begin{cases} (f(a, (b, 2t)), r) & 0 \leqq t \leqq \dfrac{1}{2} \\ (\ell(b, 2t - 1), r) & \dfrac{1}{2} \leqq t \leqq 1 \end{cases}$$

and

$$(\Sigma f \vee \Sigma \ell) \circ (\Sigma q + \Sigma p_2)(a, (b, t), r) = \begin{cases} (f(a, (b, t)), 2r) & 0 \leqq r \leqq \dfrac{1}{2} \\ (\ell(b, t), 2r - 1) & \dfrac{1}{2} \leqq r \leqq 1 . \end{cases}$$

The usual homotopy to interchange the roles of t and r for homotopy will work in this case since f is defined on $A \# \Sigma B$ and ℓ is independent of a. Thus the upper triangle is homotopy commutative and the lemma is established.

DEFINITION 3.2. The *classical EHP sequence* is given by:

$$\pi_{3n-2}(S^n) \xrightarrow{E} \pi_{3n-1}(S^{n+1}) \longrightarrow \cdots \longrightarrow \pi_{n+p}(S^n) \longrightarrow$$
$$\pi_{n+p+1}(S^{n+1}) \xrightarrow{H'} \pi_p(S^n) \xrightarrow{P} \pi_{p+n-1}(S^n) \xrightarrow{E} \cdots$$

where E is the suspension homomorphism, $H' = -E^{-n-1} \circ \tilde{H}$ where $\tilde{H}$ is the Hopf invariant of Definition 1.8, $P = P_{\iota_n}$, where $\iota_n = [1_{S^n}] \in \pi_n(S^n)$.

This sequence was shown exact in [7]; the form used in Definition 3.2 is that of P. J. Hilton and J. H. C. Whitehead in [3]. The classical EHP sequence can now be compared with the ι_n-component EHP sequence for the fibration ω: $L(S^n, S^n; 1_{S^n}) \to S^n$.

THEOREM 3.3. *For $q \leqq 3n - 2$ the following exact ladder is commutative and H is an isomorphism*:

$$\begin{tikzcd}
\longrightarrow \pi_q(S^n) \arrow[r, "E"] & \pi_{q+1}(S^{n+1}) \arrow[r, "H'"] & \pi_{q-n}(S^n) \arrow[r, "P"] & \pi_{q-1}(S^n) \longrightarrow \\
\longrightarrow \pi_q(S^n) \arrow[r, "i'_*"] \arrow[u, leftrightarrow, "1"] & \pi_{q-n}L(S^n, S^n; 1_{S^n})) \arrow[r, "\omega_*"] \arrow[u, "H"] & \pi_{q-n}(S^n) \arrow[r, "P_{\iota_n}"] \arrow[u, leftrightarrow, "1"] & \pi_{q-1}(S^n) \longrightarrow . \arrow[u, leftrightarrow, "1"]
\end{tikzcd}$$

Proof. The left square commutes by Lemma 3.1 since, by definition, $i'_* = i_* \circ \hat{\imath}_* \circ \theta$. The right square commutes by the definition of P. For the range $q \leqq 3n - 2$, G. W. Whitehead shows (Corollary 6-4 in [7]) that every element $\alpha \in \pi_{q+1}(S^{n+1})$ is obtainable from a map $F: S^{q-n} \times S^n \to S^n$ of type $(H'(\alpha), \iota_n)$ by a Hopf construction. Thus if F is considered as an associated map for an element $\beta \in \pi_{q-n}(L(S^n, S^n; 1_{S^n}))$, β has type $(\omega_*(\beta), \iota_n)$ and $H(\beta)$ is obtainable by a Hopf construction from a map of type $(\omega_*(\beta), \iota_n)$ as well as a map of type $(H'(\alpha), \iota_n)$. But then by 5.1 in [6], $\omega_*(\beta)*\iota_n = H'(\alpha)*\iota_n$, where $*$ is the join operation. Since ι_n is the homotopy class of 1_{S^n}, $E^{n+1}\omega_*(\beta) = \omega_*(\beta)*\iota_n = H'(\alpha)*\iota_n = E^{n+1}H'(\alpha)$. Now $q \leqq 3n - 2$ so $q - n \leqq 2n - 2$ and by the Freudenthal suspension theorem E^{n+1} is an isomorphism, thus $\omega^*(\beta) = H'(\alpha) = H'(H(\beta))$. This establishes the commutivity of the ladder. That H is an isomorphism follows from the five lemma.

Since the bottom line is the ι_n-component *EHP* sequence, the classical *EHP* sequence can be considered as the ι_n-component *EHP* sequence for spheres in the range $q \leqq 3n - 2$.

Some definitions will be required before describing the suspension sequence of James. Let D^n denote the solid n-ball. Then $\partial D^n = S^{n-1} = D_+^{n-1} \cup D_-^{n-1}$ where D_+^{n-1} and D_-^{n-1} are the northern and southern hemispheres of S^{n-1} respectively. Note that $D_+^{n-1} \cap D_-^{n-1} = S^{n-2}$.

DEFINITION 3.4. For A and B subspaces of X such that $A \cap B \neq \phi$ let $\pi_n(X; A, B)$ be the set of homotopy classes maps of $f: (D^n, D_+^{n-1}, D_-^{n-1}) \to (X, A, B)$.

There are natural boundary operators $\partial_1: \pi_n(X; A, B) \to \pi_{n-1}(A, A \cap B)$ and $\partial_2: \pi_{n-1}(A, A \cap B) \to \pi_{n-2}(A \cap B)$ defined by restriction to (D_+^{n-1}, S_-^{n-2}) and S^{n-2} respectively.

DEFINITION 3.5. The *repeated boundary operator* $\Delta: \pi_n(X; A, B) \to \pi_{n-2}(A \cap B)$ is defined by $\Delta = \partial_2 \circ \partial_1$.

The following result of James will be useful.

THEOREM 3.6. *There is a pairing* $\{\beta, \gamma\} \in \pi_{p+q+1}(\Sigma X; C_+X, C_-X)$ *for* $\beta \in \pi_p(X)$ *and* $\gamma \in \pi_q(X)$ *such that*

(i) $\Delta\{\beta, \gamma\} = [\beta, \gamma] \in \pi_{p+q+1}(X)$, *the usual Whitehead product and*

(ii) *If* $i_*: \pi_{p+q+1}(\Sigma X) \to \pi_{p+q+1}(\Sigma X; C_+X, C_-X)$ *is the natural inclusion, an element* $\alpha \in \pi_{p+q+1}(\Sigma X)$ *is obtainable by a Hopf construction of type* (β, γ) *iff* $i_*(\alpha) = \{\beta, \gamma\}$.

Proof. See § 4 and Theorem 2.17 in [4].

DEFINITION 3.7. *The James suspension sequence is*

$$\longrightarrow \pi_{p+q}(X) \xrightarrow{E} \pi_{p+q+1}(\Sigma X) \xrightarrow{i_*} \pi_{p+q+1}(\Sigma X; C_+X, C_-X) \xrightarrow{\Delta} \pi_{p+q-1}(X)$$

where E is the suspension homomorphism and Δ is the repeated boundary operator.

THEOREM 3.8. *The following exact ladder is commutative*:

$$\begin{tikzcd}
\arrow[r] & \pi_{p+q}(X) \arrow[r, "i'_*"] \arrow[d, "1"] & \pi_p(L(S^q, X; \ell)) \arrow[r, "\omega_*"] \arrow[d, "H"] & \pi_p(X) \arrow[r, "P_\lambda"] \arrow[d, "\tilde{P}"] & \pi_{p+q-1}(X) \arrow[r] \arrow[d, "1"] & {} \\
\arrow[r] & \pi_{p+q}(X) \arrow[r, "E"] & \pi_{p+q+1}(\Sigma X) \arrow[r, "i_*"] & \pi_{p+q+1}(\Sigma X; C_+X, C_-X) \arrow[r, "\Delta"] & \pi_{p+q-1}(X) \arrow[r] & {}
\end{tikzcd}$$

where H is as in Lemma 3.1, $\widetilde{P(\alpha)} = \{\alpha, \lambda\}$, *and* $\lambda = [\ell]$ *for any* $[\ell] \in \pi_q(X)$.

Proof. The left square commutes by Lemma 3.1 since $i'_* = i_* \circ \hat{\ell}_* \circ \theta$ by definition. If $\alpha \in \pi_p(L(S^q, X; \ell))$ then by definition of H, $H(\alpha)$ is obtainable by a Hopf construction of type $(\omega_*(\alpha), \lambda)$ and by Theorem 3.6, (ii), $i_*H(\alpha) = \{\omega_*(\alpha), \lambda\} = \tilde{P} \circ \omega_*(\alpha)$. Thus the middle square commutes. The right square commutes by Theorem 3.6, (i).

Theorem 3.8 clearly indicates the extent to which the map $i'_* = i_* \circ \hat{\ell}_* \circ \theta$ of the λ-component EHP sequence approximates the suspension homomorphism. Indeed, $E = H \circ i'_*$. While the James sequence contains the suspension homomorphism in a straight forward form, the λ-component EHP sequence contains the generalized Whitehead product in a more direct form.

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