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TESTING 3-MANIFOLDS FOR PROJECTIVE PLANES

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It is well known that a closed 3-manifold M contains a (piecewise linearly embedded) essential separating 2-sphere if and only if $\pi_1(M)$ is a nontrivial free product. In this paper necessary and sufficient conditions, in terms of $\pi_1(M)$, are given for the existence of a projective plane in M. If M is irreducible this condition is that $\pi_1(M)$ be an extension of Z or a nontrivial free product by Z_2. In particular this provides a criterion for deciding which irreducible closed 3-manifolds are not P^2-irreducible.

P^2-irreducible 3-manifolds have been studied in [2], [4]; if they are sufficiently large then their covering spaces are also P^2-irreducible. This property is not shared by irreducible but not P^2-irreducible manifolds; in [9] such manifolds are constructed having non prime covering spaces. This leads to the question as to which 3-manifolds are irreducible but not P^2-irreducible.

0. **Notation and definitions.** We work in the piecewise linear category. A 3-manifold M is a compact, connected 3-manifold. *A surface F in M* is a compact 2-manifold embedded in M.

We denote by $U(X)$ a small regular neighborhood of X in M.

$F \subset \mathrm{Int}\,(M)$ is *2-sided* in M if $U(F)$ is homeomorphic to $F \times I$. M is *irreducible* if every 2-sphere in M bounds a 3-cell in M. M is *P^2-irreducible* if M is irreducible and contains no 2-sided projective planes. M is *prime* if it is not the connected sum of two manifolds each different from the 3-sphere. (Here the connected sum $M_1 \# M_2$ is obtained by removing a 3-ball in the interior of M_1 and M_2 and identifying the boundary spheres under an orientation reversing homeomorphism.) F in M is *incompressible* if the following holds:

(a) if D is a disc in M such that $D \cap F = \partial D$, then ∂D bounds a disc in F, and

(b) if F is a 2-sphere, then S does not bound a 3-ball in M.

A *homotopy* N is a manifold that is homotopy equivalent to the manifold N.

Disjoint surfaces F and G in M are *pseudo parallel* if there exists an embedding of a homotopy $(F \times I)$ into M that has two boundary components, one of which is mapped onto F, the other one onto G. Finally, M is called π-trivial, if $\pi_1(M) = 1$.

REMARK. If the Poincaré conjecture is true, then pseudo parallel is the same as parallel.

1. Preliminaries. Let S^2, P^2 denote the 2-sphere and projective plane, resp.

LEMMA 1. *Let F be a closed surface, let M be an irreducible 3-manifold.*

(a) *If $F \neq S^2, P^2$ then M is a homotopy $(F \times I)$ if and only if M is homeomorphic to a line bundle over F.*

(b) *If M is nonorientable and $\pi_1(M) = \boldsymbol{Z}_2$, then ∂M consists of two projective planes and M is a homotopy $(P^2 \times I)$.*

(c) If $\pi_1(M) = \boldsymbol{Z} + \boldsymbol{Z}_2$, *then $\partial M = \varnothing$ and M is a homotopy $(P^2 \times S^1)$.*

Proof. Part (a) follows from [5, Proposition 1]. Part (b) follows from [1, Theorem 5.1]. Part (c) follows from [11]: We map M onto a circle such that the inverse image of a point is a projective plane P^2 in M. Then, by (b), $\text{cl}\,(M - U(P^2))$ has as boundary two copies of P^2 and is a homotopy $(P^2 \times I)$.

LEMMA 2. *If M is irreducible and contains a 1-sided projective plane, then M is P^3 (the 3-dim. projective space).*

Proof. $U(P^2)$ is the twisted line bundle over P^2, with boundary a 2-sphere. Since this 2-sphere bounds a 3-cell in M, the result follows.

The next lemma is due to J. Tollefson [13, Lemma 1]:

LEMMA 3. *A non-irreducible closed 3-manifold M admitting a fixed point free involution T contains a 2-sphere S not bounding a 3-cell in M such that either $T(S) = S$ or $T(S) \cap S = \varnothing$.*

We will also need the following generalization of Tollefson's lemma.

LEMMA 4. *Let M be a 3-manifold (with or without boundary) admitting a fixed point free involution T. Suppose there exists a 2-sphere in M that does not separate M into two components one of which is π-trivial. Then there exists a 2-sphere S in M having the same property and such that either $T(S) \cap S = \varnothing$ or $T(S) = S$.*

Proof. Take a 2-sphere S in M with the following properties: S does not separate M into two components one of which is π-trivial, $T(S) \cap S$ is a system of disjoint simple closed curves at which the intersection is transversal, and the number $n(T(S) \cap S)$ of components $T(S) \cap S$ is minimal. We show that either $n = 0$ or there exists an S' with the desired properties such that $T(S') = S'$.

Suppose $n > 0$. Let D be an innermost disc on $T(S)$, with ∂D a component of $T(S) \cap S$, (that is, $\operatorname{int}(D) \cap S = \emptyset$). D separates S into two discs D_1, D_2. Let $S_1 = D \cup D_1$, $S_2 = D \cup D_2$. It is easy to see that at least one of S_1 or S_2 does not separate M into two components one of which is π-trivial. Suppose S_1 has this property. If $T(S_1) = S_1$, we are done. If $T(S_1) \neq S_1$, then a component S' of $\partial U(S_1)$ (U is small wrt T) has the same property as S_1, but $n(T(S') \cap S') < n(T(S) \cap S)$ (since the component ∂D has vanished), a contradiction.

LEMMA 5. *If M is closed and $\pi_1(M) \approx \boldsymbol{Z}$, then M is a connected sum of a homotopy 3-sphere and a S^2-bundle over S^1.*

Proof. Write $M \approx M_1 \# M_2$, where M_1 is prime and $\pi_1(M_1) \approx \boldsymbol{Z}$, $\pi_1(M_2) = 1$ (see §5). An irreducible manifold with fundamental group $\boldsymbol{Z}$ is bounded (see e.g. [11]). Hence M_1 is not irreducible. Therefore M_1 is an S^2-bundle over S^1 (see §5).

2. The closed case.

THEOREM 1. *A closed irreducible 3-manifold M contains a 2-sided projective plane if and only if $\pi_1(M)$ is an extension of $\boldsymbol{Z}$ or a nontrivial free product by $\boldsymbol{Z}_2$.*

Proof. Suppose M contains a 2-sided P^2. Thus M is nonorientable and we let $p\colon M' \to M$ be the 2-fold orientable covering of M. Then $P^2 \subset M$ lifts to an essential 2-sphere $S^2 \subset M'$. If S^2 separates M' into M_1, M_2 then $\pi_1(M') \cong \pi_1(M_1) * \pi_1(M_2)$, a nontrivial free product. (Otherwise, if $\pi_1(M_1) = 1$, say, from $\partial M_1 = S^2$ it would follow that S^2 is contractible in M_1). If S^2 does not separate M', let k be a simple closed curve that intersects S^2 in exactly one point and let $U = U(S^2 \cup k)$. Then $\pi_1(M') = \boldsymbol{Z} * \pi_1(\operatorname{cl}(M - U))$.

Conversely, assume $\pi_1(M)$ is an extension of $\boldsymbol{Z}$ or of a nontrivial free product G by $\boldsymbol{Z}_2$. Let $p\colon N \to M$ be the covering of M associated with $\boldsymbol{Z}$ or G, respectively, and let $T\colon N \to N$ be the covering transformation. By Lemma 5 and Kneser's conjecture [12] there exists an essential 2-sphere S^2 in N. Therefore, by Lemma 3 we can find a 2-sphere $S \subset N$ not bounding a 3-cell, such that either $T(S) \cap S = \emptyset$ or $T(S) = S$. The first case cannot occur, since M is irreducible. In the second case, $p(S)$ is a projective plane in M that is 2-sided, by Lemma 2.

3. The bounded case.

THEOREM 2. *Let M be an irreducible 3-manifold with (nonempty) incompressible boundary. M contains a 2-sided P^2 that is not pseudo parallel to a component of ∂M if and only if $\pi_1(M)$ is an extension of a nontrivial free product by $\boldsymbol{Z}_2$.*

Proof. Suppose M contains a 2-sided P^2 that is not pseudo parallel to a component of ∂M. Lift P^2 to S^2 in the 2-fold orientable cover M' of M, let $T: M' \to M'$ be the covering transformation. If S^2 separates M into M_1, M_2, we have that $T(M_1) = M_1$, $T(M_2) = M_2$, since P^2 is 2-sided in M. If $\pi_1(M_1) = 1$, say, then M_1 covers a submanifold M_{1*} having fundamental group $\boldsymbol{Z}_2$. By Lemma 1 (b), M_{1*} is a homotopy $(P^2 \times I)$, hence P^2 would be pseudo parallel to a component of ∂M, a contradiction. Therefore, in this case, $\pi_1(M') = \pi_1(M_1) * \pi_1(M_2)$, a nontrivial free product.

If S^2 does not separate M', then as in the proof of Theorem 1, $\pi_1(M') \cong \boldsymbol{Z} * \pi_1(\mathrm{cl}\,(M' - U))$. If $\pi_1(\mathrm{cl}\,(M' - U))$ would be trivial, then $\pi_1(M) = \boldsymbol{Z} + \boldsymbol{Z}_2$. By Lemma 1 (c), M would be closed, a contradiction.

Conversely, suppose $\pi_1(M)$ is an extension of a nontrivial free product G by $\boldsymbol{Z}_2$. Again, let $N \xrightarrow{P} M$ be the covering of M corresponding to G and let T be the covering transformation. By Kneser's conjecture for bounded 3-manifolds [6] there exists a 2-sphere S^2 in N that separates N into N_1, N_2, both not π-trivial. By Lemma 4, there exists a 2-sphere S that does not separate N into two components one of which is π-trivial and such that $T(S) = S$ (the case $TS \cap S = \phi$ cannot occur). By Lemma 2, S covers a 2-sided P^2 in M. If P^2 were pseudo parallel to a component of ∂M, then lifting the corresponding homotopy $(P^2 \times I)$ we see that S would separate N into two components, one of which would be π-trivial, a contradiction.

PROPOSITION. *Let M be irreducible and suppose $\pi_1(M)$ is not $\boldsymbol{Z}_2$, and not an extension of $\boldsymbol{Z}$ or of a nontrivial free product by $\boldsymbol{Z}_2$. Then if ∂M contains no P^2 (in particular, if M is closed) it follows that M contains no P^2.*

Proof. If M is orientable and contains a P^2, then $M = P^3$, by Lemma 2. If M is nonorientable, let M' be the 2-fold orientable cover of M. If $\pi_2(M') \neq 0$, then the sphere theorem [14] gives us an essential 2-sphere in M' and as in the proof of the preceding theorems, we see that $\pi_1(M') = \boldsymbol{Z}$ or a nontrivial free product. Therefore, $\pi_2(M') = 0$ and hence $\pi_2(M) = 0$. (In fact, M is aspherical.) But any 2-sided $P^2 \subset M$ would be essential [1, Lemma 6.3].

REMARK. A 2-sided P^2 in M is incompressible in M. This follows from the loop theorem and Dehn's lemma [10]. In particular $\pi_1(P^2) \to \pi_1(M)$ is an injection.

4. A counterexample to Theorem 2 if M is not incompressible. Let K be a solid Kleinbottle, T a solid torus. Choose $n \geqq 1$ disjoint discs $D_1, \cdots, D_n$ on ∂K and a disc D on ∂T. Let M be the manifold obtained from K by attaching n copies of T to K at D_i and D ($i = 1, \cdots, n$). Then M is irreducible and does not contain 2-sided projective planes (otherwise by the preceding remark, $\pi_1(M)$ would have an element of order 2, but $\pi_1(M) \cong (n+1)\boldsymbol{Z}$). However, the two-fold orientable cover M' of M has fundamental group $\pi_1(M') \cong (2n+1)\boldsymbol{Z}$, the free product of $2n+1$ copies of $\boldsymbol{Z}$, and therefore $\pi_1(M)$ is an extension of the nontrivial free product $(2n+1)\boldsymbol{Z}$ by $\boldsymbol{Z}_2$.

5. The general case. Suppose M is a compact 3-manifold such that ∂M contains no 2-spheres. As in [8, Lemma 1] it follows that if M is prime but not irreducible then M is a S^2-bundle over S^1. If M is not prime, then there exists a decomposition of M into a finite number of prime manifolds

$$M \approx M_1 \# M_2 \# \cdots \# M_n \,, \tag{$\#$}$$

(if M is nonorientable or with boundary see e.g. [3]). If K denotes the nonorientable S^2-bundle over S^1 then since $K \# K \approx K \# (S^2 \times S^1)$, we say that the decomposition ($\#$) is in *normal form* if at most one $M_i \approx K$. Then Milnor's proof in [8] can be generalized to yield the following:

PROPOSITION. *Any compact 3-manifold M whose boundary contains no 2-spheres has a unique normal decomposition ($\#$) into prime manifolds. Each summand M_i is irreducible or $S^1 \times S^2$ and at most one $M_i \approx K$.*

In the decomposition ($\#$) let m denote the number of prime manifolds which are not π-trivial ($m \leqq n$).

THEOREM 3. *Let M be a closed 3-manifold.*

(a) *If M contains a 2-sided P^2, then $\pi_1(M)$ is an extension of a free product of $2m$ nontrivial factors or of a free product of $2m-1$ nontrivial factors one of which is $\boldsymbol{Z}$, by $\boldsymbol{Z}_2$.*

(b) *If $\pi_1(M)$ is an extension of a free product of $2m$ nontrivial factors by $\boldsymbol{Z}_2$ then M contains a 2-sided P^2.*

Proof. Consider the decomposition ($\#$). Let $S_i \subset M$ be the 2-sphere at which M_i and M_{i+1} are amalgamated and let M_i' be obtained from M_i by removing the interiors of the 3-balls which are used in the construction of the connected sum. We can assume that $M_i' \cap M_{i+1}' = S_i (i = 1, \cdots, n-1)$.

We first note that M contains a 2-sided P^2 if and only if one of the M_i' contains a 2-sided P^2. For, by general position we can assume that $P^2 \cap \bigcup S_i$ is a system of simple closed curves. If $P^2 \cap S_i \neq \varnothing$ then an innermost intersection curve on S_i bounds a disk on P^2 (since P^2 is incompressible) and on S_i. Replacing the disk on P^2 by the disk on S_i and pushing it slightly off S_i, we reduce the number of intersection curves of $P^2 \cap \bigcup S_i$.

Second, we note that we can assume that in the decomposition ($\sharp$) no M_i has trivial fundamental group i.e. that $n = m$. For otherwise we consider the manifold M_* obtained from M by deleting all the homotopy spheres M_i which occur in ($\sharp$). Clearly, $\pi_1(M_*) = \pi_1(M)$ and M_* contains a 2-sided P^2 if and only if M does.

Now assume M contains a 2-sided P^2. Let $p: N \to M$ be the 2-fold orientable covering and let $N_i = p^{-1}(M_i')$. If N_i is connected then $\pi_1(N_i) \neq 1$, because otherwise $\pi_1(M_i') = \boldsymbol{Z}_2$, and since $\partial M_i'$ consists of 2-spheres only, M_i' is orientable (Lemma 1(b)). But then M_i' lifts to two copies, hence N_i would not be connected. Similarily, if N_i is not connected then no component of N_i is π-trivial, because otherwise M_i would be π-trivial. Now each $S_i \subset M$ lifts to two 2-spheres S_i', S_i'' in N, and N is obtained from the N_i by identifying N_i and N_{i+1} along S_i' and S_i'' $(i = 1, \cdots, m-1)$.

Construct a manifold N' as follows. If both N_1 and N_2 are connected, identify N_1 and N_2 along one 2-sphere only, say S_1'. Otherwise identify N_1 and N_2 along both S_1' and S_1''. The result is a manifold $N^{(1)}$. If N_3 is connected, identify $N^{(1)}$ and N_3 along S_2' only, otherwise identify along S_2' and S_2'', etc. In this way we obtain a maximal connected manifold N' such that N is obtained from N' by identifying pairs of 2-spheres in $\partial N'$. Then $\pi_1(N') = G_1 * \cdots * G_k$ $(0 \leqq k \leqq 2m-1)$, where each G_j is the fundamental group of a component of some N_i. We obtain N from N' by adding $(2m-1)-k$ handles $S^1 \times S^2$ or K, hence $\pi_1(N) = G_1 * \cdots * G_k * \boldsymbol{Z} * \cdots * \boldsymbol{Z}$ is a free product of $2m-1$ nontrivial factors.

Now $P^2 \subset M_j'$, say $(1 \leqq j \leqq m-1)$. Then M_j' is nonorientable and N_j is connected. Therefore by the above construction, $\pi_1(N_j)$, is one of the groups G_i in the above decomposition of $\pi_1(N)$. Closing the boundary spheres of N_j with 3-balls we get a 2-fold covering $\hat{N}_j \to M_j$, and it follows from the proof of Theorem 1 that $\pi_1(\hat{N}_j)$ and hence $\pi_1(N_j)$ is $\boldsymbol{Z}$ or a nontrivial free product. This proves part (a) of Theorem 3.

Now suppose $\pi_1(M)$ is an extension of a product G of $2m$ nontrivial groups by $\boldsymbol{Z}_2$. Let $p: \tilde{M} \to M$ be the covering associated to G. Then as above $\pi_1(\tilde{M}) = \pi_1(\tilde{M}_1) * \cdots * \pi_1(\tilde{M}_k) * Z * \cdots * Z$ is a product of $2m - 1$ groups, where each $\tilde{M}_i$ is a component of $p^{-1}(M'_j)$, for some j. (It is possible that some $\pi_1(\tilde{M}_i) = 1$.) It follows from Kurosh's Theorem [7] that at least one factor, $\pi_1(\tilde{M}_1)$ say, is a nontrivial free product. If $\tilde{M}_1$ covers M'_j, then either $\pi_1(M_j) \approx \pi_1(\tilde{M}_1)$ or $\pi_1(M_j)$ is an extension of $\pi_1(\tilde{M}_1)$ by Z_2. In the first case M_1 can not be a handle and by Kneser's conjecture can not be irreducible, therefore this case can not occur. In the second case we apply Theorem 1 to obtain a P^2 in M_1 and hence in M.

It should be noted that the hypothesis in case (a) of Theorem 3 can not be weakened: If $M = (P^2 \times S^1) \# (S^2 \times S^1)$, then $\pi_1(M)$ is not an extension of a free product of 4 factors by Z_2.

It is now easy to see how to obtain an analogous result for 3-manifolds with incompressible boundary.

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