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GEOMETRIC PROPERTIES OF SOBOLEV MAPPINGS

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If f is a mapping from an open k-cube in R^k into R^n, $2 \leqq k \leqq n$, whose coordinate functions belong to appropriate Sobolev spaces, then the area of f is the integral with respect to k dimensional Hausdorff measure over R^n of a nonnegative integer valued multiplicity function.

1. Introduction. If $f\colon Q \to R^n$, Q an open k-cube in R^k, $2 \leqq k \leqq n$, is a mapping whose coordinate functions belong to appropriate Sobolev classes, it was shown in [6] that f is $k-1$ continuous and that the area of f, as defined in [5], is equal to the classical Jacobian integral. The purpose of this paper is to investigate, using the theory of currents as in [2], the geometric-measure theoretic properties of such a surface and to show that the area is equal to the integral with respect to k dimensional Hausdorff measure in R^n of an integer valued multiplicity function.

2. Suppose k and n are integers with $2 \leqq k \leqq n$. Let

$$Q = R^k \cap \{x\colon 0 < x_i < 1 \text{ for } 1 \leqq i \leqq k\}$$

and let $\Lambda(k, n)$ denote the set of all k-tuples $\lambda = (\lambda_1, \cdots, \lambda_k)$ of integers such that $1 \leqq \lambda_1 < \cdots < \lambda_k \leqq n$. Suppose $f\colon Q \to R^n$, $f = (f^1, \cdots, f^n)$, $f^i \in W^1_{p_i}(Q)$, $p_i > k-1$, and $\sum_{j=1}^k 1/p_{\lambda_j} \leqq 1$ whenever $\lambda \in \Lambda(k, n)$. Here $W^1_p(Q)$ denotes those functions in $L^p(Q)$ whose distribution partial derivatives are functions in $L^p(Q)$.

Let $e_1, \cdots, e_n$ be the usual basis for R^n and let

$$e_\lambda = e_{\lambda_1} \wedge \cdots \wedge e_{\lambda_k} ,$$

$\lambda \in \Lambda(k, n)$, denote the corresponding basis for the space of k-vectors in R^n. For $\lambda \in \Lambda(k, n)$ let p^λ denote the orthogonal projection of R^n onto R^k defined by letting

$$p^\lambda(y) = (y_{\lambda_1}, \cdots, y_{\lambda_k}) \quad \text{for} \quad y = (y_1, \cdots, y_n) \in R^n .$$

For almost every (in the sense of k dimensional Lebesgue measure $\mathscr{L}_k$) $x \in Q$, let $Jf(x) = \sum_{\lambda \in \Lambda(k,n)} Jf^\lambda(x) e_\lambda$ where Jf^λ denotes the determinant of the matrix of distribution partial derivatives of $f^\lambda = p^\lambda \circ f$. In [6] it was shown that the area of f, as defined in [5] is equal to $\int_Q |Jf(x)|\, dx$ where $|Jf(x)|$ is the Euclidean norm of the k-vector $Jf(x)$.

Define a current valued measure T over Q by letting

$$T(B)(\phi) = \int_B \phi(f(x)) \cdot Jf(x)dx$$

whenever B is an $\mathscr{L}_k$ measurable subset of Q and ϕ is an infinitely differentiable k-form on R^n with compact support. Let σ denote the finite measure defined over R^n by letting

$$\sigma(Y) = \int_{f^{-1}(Y)} |Jf(x)|\, dx$$

whenever Y is a Borel subset of R^n.

It will be shown in part 3 that $T(B)$ is a locally rectifiable current whenever B is an $\mathscr{L}_k$ measurable subset of Q and this fact will be used to define a nonnegative integer valued function N on R^n which describes the multiplicity with which f assumes its values. The main results of the paper are summarized here.

THEOREM. *Let H_n^k denote k dimensional Hausdorff measure in R^n and let $\alpha(k)$ denote the $\mathscr{L}_k$ measure of the unit ball in R^k.*

1. *For H_n^k almost every $y \in R^n$*

$$N(y) = \lim_{r \to 0+} \frac{\sigma(B(y, r))}{\alpha(k)r^k} .$$

Here $B(y, r)$ denotes the open ball of radius r around y.

2. $$\int_{R^n} N(y)\, dH_n^k y = \int_Q |Jf(x)|\, dx .$$

3. *There exists a countable family F of k dimensional C^1 submanifolds of R^n such that for σ almost every $y \in R^n$ there is an $M \in F$ with $y \in M$,*

$$\lim_{r \to 0+} \frac{\sigma(B(y, r) - M)}{\alpha(k)r^k} = 0$$

and

$$\lim_{r \to 0+} \frac{\sigma(B(y, r) \cap M)}{\alpha(k)r^k} = N(y) .$$

3. Definition of the function N and proof of the theorem. We follow a plan analogous to that of [2: 2.1]. For $1 \leqq i \leqq k$ and $r \in I = \{s\colon 0 < s < 1\}$ let $P_i(r) = Q \cap \{x\colon x_i = r\}$. Let $\{f_j\}$ be a sequence of mollifiers of f as in [6] and let $\bar{f}$ denote the pointwise limit of $\{f_j\}$ wherever it exists. Then $\bar{f}$ is a representative of f and according to [6], [7: Chap. 3], and [8: part 3] there exists a collection

P of the sets $P_i(r)$ such that for each i, $P_i(r) \in P$ for almost all (in the sense of 1 dimensional Lebesgue measure) $r \in I$ and if q is any k-cube in Q whose $k-1$ faces all lie in elements of P then

(1) $f_j \mid \text{Bdry } q$ converges uniformly to $\bar{f} \mid \text{Bdry } q$,

(2) $H_n^k(\bar{f}(\text{Bdry } q)) = 0$

(3) $L_{k-1}(\bar{f} \mid \text{Bdry } q) = \underline{\lim}_{j\to\infty} L_{k-1}(f_j \mid \text{Bdry } q)$, where L_{k-1} denotes $k-1$ dimensional Lebesgue area.

Henceforth we will denote by f the pointwise limit of mollifiers $\{f_j\}$ as described above. A k-cube $q \subset Q$ whose $k-1$ faces all lie in elements of P will be called "special".

For the notation concerning currents we refer to [3].

LEMMA 1. *If f is bounded then $T(B)$ is a rectifiable current whenever B is an $\mathscr{L}_k$ measurable subset of Q.*

Proof. If $q \subset Q$ is a special k-cube then

$$\lim_{j\to\infty} \int_q |Jf_j(x) - Jf(x)| \, dx = 0$$

and hence the sequence $\{f_{j\#}(q)\}$ of currents converges weakly to $T(q)$. The supports of the $f_{j\#}(q)$ and $T(q)$ are all contained in a fixed compact set,

$$M(f_{j\#}(q)) \leqq \int_q |Jf_j(x)| \, dx \, ,$$

and

$$M(\partial f_{j\#}(q)) \leqq L_{k-1}(f_j \mid \text{Bdry } q)$$

where M denotes mass in the space of currents. Thus, by [4: 8.14, 8.13], $T(q)$ is an integral current whenever q is special.

Since it is clear that

$$M(T(A)) \leqq \int_A |Jf(x)| \, dx$$

whenever A is an $\mathscr{L}_k$ measurable subset of Q, the lemma follows.

Let $\|T\|$ denote the finite measure over Q defined by letting $\|T\|(A)$ denote the supremum of the numbers $\sum_{j=1}^{\infty} M(T(B_j))$ taken over all countable disjoint collections of $\mathscr{L}_k$ measurable subsets $B_j \subset A$ whenever A is an $\mathscr{L}_k$ measurable subset of Q. Clearly

$$\|T\|(A) \leqq \int_A |Jf(x)| \, dx$$

whenever A is an $\mathscr{L}_k$ measurable subset of Q.

For $\mathscr{L}_k$ almost every $x \in Q$ there is a k-covector ω in R^n with

$|\omega| = 1$ such that $\omega \cdot Jf(x) = |Jf(x)|$ and

$$\overline{\lim_{r\to 0+}} \frac{\|T\|(B(x, r))}{\alpha(k)r^k} \geqq \lim_{r\to 0+} \frac{T(B(x, r))(\omega)}{\alpha(k)r^k} = |Jf(x)| .$$

It follows that $\|T\|(A) = \int_A |Jf(x)|\,dx$ whenever A is an $\mathscr{L}_k$ measurable subset of Q.

For each positive integer N let $f_N = (f_N^1, \cdots, f_N^n)$ where

$$f_N^i(x) = \begin{cases} N & \text{if } f^i(x) \geqq N \\ f^i(x) & \text{if } |f^i(x)| < N \\ -N & \text{if } f^i(x) \leqq -N . \end{cases}$$

Then f_N is bounded and $f_N^i \in W_{p_i}^1(Q)$ for $1 \leqq i \leqq n$. For any measurable set $B \subset Q$ let

$$T_N(B)(\phi) = \int_B \phi(f_N(x)) \cdot Jf_N(x)dx$$

whenever ϕ is an infinitely differentiable k-form on R^n. Note that, if Y is a bounded Borel set in R^n, then, for sufficiently large N, $T_N(B) \llcorner Y = T(B) \llcorner Y$ whenever B is an $\mathscr{L}_k$ measurable subset of Q. Consequently, if Y is a bounded open subset of R^n then $T(B) \llcorner Y$ is rectifiable whenever B is a measurable subset of Q.

Analogous to [2: 2.1 part 3] we have

LEMMA 2. *There exists a countable collection F of k dimensional C^1 submanifolds of R^n such that $\sigma(R^n - \bigcup F) = 0$.*

Proof. Suppose r and ε are positive real numbers and let

$$B(0, r) = R^n \cap \{y\colon |y| < r\} .$$

Let $\{A_1, \cdots, A_m\}$ denote a finite collection of disjoint measurable subsets of $f^{-1}(B(0, r))$ such that $\mathscr{L}_k\ (f^{-1}(B(0, r)) - \bigcup_{j=1}^m A_j) = 0$ and $\sigma(B(0, r)) - \varepsilon < \sum_{j=1}^m M(T(A_j))$. Note that $T(A_j) = T(A_j) \llcorner B(0, r)$ is rectifiable for $j = 1, \cdots, m$. Thus, by [4: 8.16], there exists for each j a countable collection G_j of k-dimensional C^1 submanifolds of R^n such that $\|T(A_j)\|(R^n - \bigcup G_j) = 0$. Letting $G = \bigcup_{j=1}^m G_j$, we have

$$\begin{aligned} \varepsilon &\geqq \sigma(B(0, r)) - \sum_{j=1}^m M(T(A_j)) = \sum_{j=1}^m (\|T\|(A_j) - M(T(A_j))) \\ &\geqq \sum_{j=1}^m \|T\|(A_j \cap f^{-1}(R^n - \bigcup G_j)) \\ &\geqq \sum_{j=1}^m \|T\|(A_j \cap f^{-1}(R^n - \bigcup G)) = \sigma(B(0, r) - \bigcup G) \end{aligned}$$

and the lemma follows.

If μ is a measure over R^n, $y \in R^n$, and $A \subset R^n$ we let

$$\theta^k(\mu, A, y) = \lim_{r \to 0+} \frac{\mu(A \cap B(y, r))}{\alpha(k) r^k}$$

whenever the limit exists. In case $A = R^n$ we write $\theta^k(\mu, y)$.

Recall that, if S is a k dimensional rectifiable current in R^n and Y is a Borel set in R^n with $H^k_n(Y) = 0$, the $S \llcorner Y = 0$. Thus σ is absolutely continuous with respect to H^k_n. This fact together with Lemma 2 and the finiteness of σ allow us to conclude using [1: 3.1, 3.2] that:

1. $\theta^k(\sigma, y)$ exists for H^k_n almost every $y \in R^n$.
2. For σ almost every $y \in R^n$ there exists an $M \in F$ such that $y \in M$, $\theta^k(\sigma, y) < \infty$, and $\theta^k(\sigma, R^n - M, y) = 0$.
3. $\int_{R^n} \theta^k(\sigma, y) dH^k_n y \leqq \sigma(R^n)$.

A proof of the following statement concerning rectifiable currents can be found in [2: 2.1 part 7]: If S is a rectifiable k dimensional current in R^n, M is a k dimensional C^1 submanifold of R^n,

$$y \in M - \operatorname{spt} \partial S ,$$

$\theta^k(\| S \|, y) < \infty$, $\theta^k(\| S \|, R^n - M, y) = 0$, and P is an oriented k plane tangent to M at y, then there exists a unique integer m such that

$$\lim_{r \to 0+} \frac{1}{\alpha(k) r^k} F[S \llcorner B(y, r) - m(P \cap B(y, r))] = 0$$

where F denotes the flat norm [4: 3.2].

Now suppose q is a special k-cube in Q and $y \in R^n$. If there is an $M \in F$ with $y \in M - f(\mathrm{Bdry}\, q)$, $\theta^k(\sigma, y) < \infty$, and

$$\theta^k(\sigma, R^n - M, y) = 0 ,$$

let P denote an oriented k-plane tangent to M at y, let $m(q, y)$ denote the integer such that

$$\lim_{r \to 0+} \frac{1}{\alpha(k) r^k} F[T(q) \llcorner B(y, r) - m(q, y)(P \cap B(y, r))] = 0$$

and set $\alpha(q, y) = m(q, y)\, \zeta(y)$ where $\zeta(y)$ is the simple unit k-vector orienting P. Otherwise set $\alpha(q, y) = 0$.

Then, for H^k_n almost every $y \in R^n$,

$$\theta^k(T(q) \llcorner \phi, y) = \lim_{r \to 0} \frac{[T(q) \llcorner B(y, r)](\phi)}{\alpha(k) r^k} = \phi(y) \cdot \alpha(q, y)$$

whenever ϕ is an infinitely differentiable k-form in R^n. Consequently $T(q)(\phi) = \int_{R^n} \phi(y) \cdot \alpha(q, y) dH_n^k y$ whenever ϕ is an infinitely differentiable k-form and hence

$$M(T(q)) \leqq \int_{R^n} |\alpha(q, y)| \, dH_n^k y$$

whenever q is a special k-cube.

For $y \in R^n$ let $N(y)$ denote the supremum of the numbers $\sum_{q \in G} |\alpha(q, y)|$ taken over all finite collections G of nonoverlapping special k-cubes in Q.

Suppose $N(y) \neq 0$ and G is a finite collection of nonoverlapping special k-cubes such that $\alpha(q, y) \neq 0$ for $q \in G$. Let ω denote a k-covector with $|\omega| = 1$ and $\omega \cdot \zeta(y) = 1$. Then

$$\begin{aligned} \sum_{q \in G} |\alpha(q, y)| &= \sum_{q \in G} |\theta^k(T(q) \llcorner \omega, y)| \\ &= \lim_{r \to 0} \sum_{q \in G} \frac{|[T(q) \llcorner B(y, r)](\omega)|}{\alpha(k) r^k} \\ &\leqq \theta^k(\sigma, y) . \end{aligned}$$

Thus $N(y) \leqq \theta^k(\sigma, y)$ for H_n^k almost every $y \in R^n$.

On the other hand, if G is any finite collection of nonoverlapping special k-cubes,

$$\sum_{q \in G} M(T(q)) \leqq \int_{R^n} \sum_{q \in G} |\alpha(q, y)| \, dH_n^k y .$$

The supremum of the numbers $\sum_{q \in G} M(T(q))$ taken over all finite collections G of nonoverlapping special k-cubes is readily seen to be $\sigma(R^n)$ and hence

$$\sigma(R^n) \leqq \int_{R^n} N(y) dH_n^k y \leqq \int_{R^n} \theta^k(\sigma, y) dH_n^k y \leqq \sigma(R^n) .$$

Thus $N(y) = \theta^k(\sigma, y)$ H_n^k almost everywhere and

$$\int_{R^n} N(y) \, dH_n^k y = \int_Q |Jf(x)| \, dx .$$

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